The One Dimensional Free Poincaré Inequality

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joint work with Michel Ledoux

Classical Case

A classical Poincaré inequality on a bounded domain $D \subset \mathbb{R}^d$:

$$C \int_D \left(f - \fint f \right)^2 dx \leq \int_D |\nabla f|^2 dx$$

for any nice f. $\fint_D f = \frac{1}{|D|} \int_D f$.
dx rescaled on D, becomes a probability measure.

Classical Case

Poincaré's inequality for a probability measure μ on $\mathbb{R}^d$ with $\rho > 0$

$$2\rho \mathrm{Var}_\mu(f) \le \int |\nabla f|^2 d\mu, \text{ for all nice } f. \qquad (P(\rho))$$

$$\mathrm{Var}_\mu(f) = \int \left(f - \int f d\mu\right)^2 d\mu = \int f^2 d\mu - \left(\int f d\mu\right)^2.$$

Reinterpretations:

1. $$2\rho \Pi \le L \qquad (P(\rho))$$

   $$\mathrm{Var}_\mu(f) = \langle \Pi f, f\rangle_{L^2(\mu)} \quad \Pi f = f - \int f d\mu$$

   with Π the projection onto the orthogonal to constants and

   $$\langle L f, f\rangle_{L^2(\mu)} = \int |\nabla f|^2 d\mu \ge 0.$$

2. $spec(L) \subset \{0\} \cup [2\rho, \infty)$. There is a spectral gap of size 2ρ between 0 and the rest of $spec(L)$.

Example: The Gaussian

$\gamma(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

Hermite functions: $\phi_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$ $\quad n \geq 0$ form a basis of $L^2(\gamma)$.

$$\int |f'|^2 d\gamma = \langle Lf, f \rangle_{L^2(\gamma)}, \quad Lf(x) = -f''(x) + xf'(x), \quad L\phi_n = n\phi_n.$$

and for $f = \sum_{n \geq 0} \alpha_n \phi_n$,

$$\mathrm{Var}_\gamma(f) \leq \int |f'|^2 d\gamma$$

$$\langle \Pi f, f \rangle_{L^2(\gamma)} \leq \langle Lf, f \rangle_{L^2(\gamma)}$$

$$\sum_{n \geq 1} \alpha_n^2 \leq \sum_{n \geq 1} n\alpha_n^2.$$

Hence

$$\mathrm{Var}_\gamma(f) \leq \int (f')^2 d\gamma. \qquad (P(1/2))$$

Example: the Gaussian in $\mathbb{R}^d$

$$\gamma_d(dx) = \frac{1}{\sqrt{(2\pi)^d}} e^{-|x|^2/2} dx,\ x = (x_1, x_2, \dots, x_d).$$

Hermite functions: $\phi_n(x) = \phi_n(x_1)\phi_n(x_2)\dots\phi_n(x_d)$, $n \geq 0$ form a basis of $L^2(\gamma_d)$.

$$\langle Lf, f\rangle_{L^2(\gamma_d)} = \int |\nabla f|^2 d\gamma_d$$
$$Lf(x) = -\Delta f(x) + \langle x, \nabla f(x)\rangle$$
$$L\phi_n = n\phi_n.$$

$$\mathrm{Var}_{\gamma_d}(f) \leq \int |\nabla f|^2 d\gamma_d. \qquad (P(1/2))$$

Example: $\mu(dx) = e^{-V(x)}dx$

If $\mu(dx) = e^{-V(x)}dx$ with $HessV \geq \rho$, then

$$2\rho \mathrm{Var}_\mu(f) \leq \int |\nabla f|^2 d\mu. \qquad (P(\rho))$$

Other Members of the Family

The *Wasserstein distance* on $\mathbb{R}^n$ is defined as

$$W_2^2(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \iint |x-y|^2 \pi(dx,dy),$$

$\Pi(\mu,\nu)$: probability on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν.

1. This is a metric for the weak convergence on measure with finite second moment.
2. If ν does not give mass to “small” sets, then there is a unique map T such that $T_{\#}\nu = \mu$ and $\pi = (T, Id)_{\#}\nu$ is the optimal plan:

$$W_2^2(\mu,\nu) = \int |T(x)-x|^2 \nu(dx).$$

Talagrand, Log-Sobolev and HWI

1. A measure μ is said to satisfy $T(\rho)$ if for any probability ν

$$\rho W_2^2(\nu, \mu) \leq 2H(\nu|\mu).$$

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2. μ satisfies $LSI(\rho)$ if for any ν,

$$2\rho H(\nu|\mu) \leq I(\nu|\mu)$$

where $I(\nu|\mu) = 4 \int \left| \nabla \sqrt{\frac{d\nu}{d\mu}} \right|^2 d\mu$.

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3. μ satisfies $HWI(\rho)$ inequality if for any ν:

$$H(\nu|\mu) \leq \sqrt{I(\nu|\mu)}W_2(\nu,\mu) - \frac{\rho}{2}W_2^2(\nu,\mu).$$

Notice that if $\rho > 0$, then $HWI(\rho) \Longrightarrow LSI(\rho)$.

$T(\rho), LSI(\rho), HWI(\rho) \implies P(\rho)$ for $\rho > 0$

Example: $LSI(\rho) \implies P(\rho)$
$\mu(dx) = e^{-V}dx$ and $f \in C_0^1$, such that $\int f d\mu = 0$. Then for t small $\mu_t = (1+tf)\mu$ and $LSI(\rho)$ for μ_t, $(\frac{d\mu_t}{d\mu} = 1 + tf)$

$$2\rho H(\mu_t|\mu) \le I(\mu_t|\mu)$$

$$2\rho \int (1+tf)\log(1+tf)d\mu \le 4\int |\nabla \sqrt{1+tf}|^2 d\mu$$

$$2\rho \int (1+tf)(tf + O(t^2))d\mu \le 4\int |\nabla(1+\frac{t}{2}f) + O(t^2)|^2 d\mu$$

$$2\rho t^2 \int f^2 d\mu + o(t^2) \le t^2 \int |\nabla f|^2 d\mu + o(t^2).$$

Similarly $T(\rho)$ implies $P(\rho)$ in dual form.

V smooth such that $\lim_{|x|\to\infty} \frac{V(x)}{\log(1+|x|^2)} = \infty$.

$$E_V(\mu) = \int V(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

There is a unique probability measure μ_V such that

$$E_V(\mu_V) = \inf_{\mu\in\mathcal{P}(\mathbb{R})} E(\mu).$$

In addition, μ_V has compact support.

One Dimensional Free Entropy

V smooth such that $\lim_{|x|\to\infty} \frac{V(x)}{\log(1+|x|^2)} = \infty$.

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In addition, μ_V has compact support.
The variational characterization of μ_V:

$$V(x) \geq 2\int \log|x-y|\mu_V(dx) + C \quad \text{with equality for } x \in supp(\mu).$$

In particular, if the support of μ is a union of intervals, then for a.e. $x \in supp(\mu)$:

$$V'(x) = \int \frac{2}{x-y}\mu_V(dx).$$

The relative free entropy is defined as

$$E_V(\mu|\mu_V) = E_V(\mu) - E_V(\mu_V).$$

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If $V(x) = x^2/2$, then the minimizer of the free entropy μ_V is given by the semicircular law

$$\mu_V(dx) = \frac{1}{2\pi}\mathbb{1}_{[-2,2]}(x)\sqrt{4 - x^2}dx.$$

with $E(\mu_V) = 3/4$.

One Dimensional Free Information

$$I_V(\mu|\mu_V) = \int (H\mu(x) - V'(x))^2 \mu(dx)$$

where $H\mu(x) = \int \frac{2}{x-y} \mu(dx)$ in the principal value sense.
In the case $V(x) = x^2/2$,

$$I(\mu|\mu_V) = \int (H\mu(x) - x)^2 \mu(dx).$$

Free Talagrand and Log-Sobolev

Theorem (Biane & Speicher)

If $V(x) - \rho x^2$ is convex for $\rho > 0$, then

$$E_V(\mu|\mu_V) \leq \frac{1}{4\rho} I_V(\mu|\mu_V). \qquad (LSI(\rho))$$

Theorem (Biane & Voiculescu)

If $V(x) = x^2/2$ then

$$\frac{1}{2} W_2^2(\mu, \mu_V) \leq E_V(\mu|\mu_V). \qquad (T(1/2))$$

Theorem (Hiai & Ueda & Petz)

If $V(x) - \rho x^2$ is convex for $\rho > 0$, then

$$\rho W_2^2(\mu, \mu_V) \leq E_V(\mu|\mu_V). \qquad (T(\rho))$$

Proofs

One proof: Classical counterparts to random matrices of size n and let the dimension n grow to infinity.

Second proof: Using mass transport tools (no random matrices).

A First version of free Poincaré

Theorem (Biane (2003))

If $\alpha(dx) = \mathbb{1}_{[-2,2]}(x)\frac{\sqrt{4-x^2}dx}{2\pi}$, *then*

$$\mathrm{Var}_\alpha(f) \le \iint \left(\frac{f(x)-f(y)}{x-y}\right)^2 \alpha(dx)\alpha(dy)$$

Essentially the same as the classical one with the derivative replaced by the non-commutative derivative.

The proof: The operator $\mathcal{M}$, whose Dirichlet form is given by the right had side is the counting number operator for the Chebyshev polynomials of the second kind.

A second version of free Poincaré

Theorem (M. Ledoux and I.P. (2009))

$$\int_{-2}^{2}\int_{-2}^{2}\left(\frac{f(x)-f(y)}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}} \leq \int (f')^2\, d\alpha \quad (*)$$

First proof: Poincaré+Fluctuations for random matrices.

Second Proof: If $\mathcal{N}$ is the counting number operator for the Chebyshev polynomials of the first kind, then (*) is the same as

$$\langle \mathcal{N}f, f\rangle_{L^2(\beta)} \leq \langle \mathcal{L}f, f\rangle_{L^2(\beta)}, \quad \text{with} \quad \beta(dx) = \mathbb{1}_{[-2,2]}(dx)\frac{dx}{\pi\sqrt{4-x^2}},$$

and $\mathcal{L}f = \mathcal{N}^2 f = -(4-x^2)f''(x) + xf'(x)$ a Jacobi type operator.

Which version is the true one

The judge: The other functional inequalities (transportation and Log-Sobolev).

Technically this requires handling the free relative entropy:

$$E_V(\mu|\mu_V) = E_V(\mu) - E_V(\mu_V).$$

$$E_V(\mu) = \int V(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

and μ_V is the minimizer of E_V over all probabilities of $\mathbb{R}$.

The One dimensional Free Poincaré

We say that the probability measure μ supported on $[-2, 2]$ satisfies $P(\rho)$, $\rho > 0$, if

$$2\rho \iint \left(\frac{f(x)-f(y)}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}} \le \int (f')^2\, d\mu \tag{$P(\rho)$}$$

If μ has $P(\rho)$, then necessarily its support is the whole $[-2, 2]$.

Theorem (M.Ledoux and I.P. (2011))

If μ_V is the equilibrium measure of E_V for a C^3 potential V such that μ_V has support $[-2, 2]$, then $T(\rho)$ and $LSI(\rho)$ imply $P(\rho)$ for μ_V.

Haagerup's Lemma

Lemma

For any real $x, y \in [-2, 2]$, $x \neq y$, then

$$\log|x-y| = -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right)$$

where the series here is convergent on $x \neq y$.
If $x > 2$ and $y \in [-2, 2]$, then

$$\log|x-y| = \log\left|\frac{x+\sqrt{x^2-4}}{2}\right| - \sum_{n=1}^{\infty} \frac{2}{n}\left(\frac{x-\sqrt{x^2-4}}{2}\right)^n T_n\left(\frac{y}{2}\right)$$

where the series is absolutely convergent.

Here T_n is the n^{th} Chebysev polynomial: $T_n(\cos x) = \cos(nx)$.
Orthogonal polynomials w.r.t. $\omega(dx) = \mathbb{1}_{[-2,2]}(x)\frac{dx}{\pi\sqrt{4-x^2}}$.

Proof of Haagerup's Formula

$x = 2\cos u$ and $y = 2\cos v$ with $u, v \in (0, \pi)$, $u \neq v$

$$x - y = 2(\cos u - \cos v) = 4\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\begin{aligned}
\log|x-y| &= \log\left|2\sin\left(\frac{u+v}{2}\right)\right| + \log\left|2\sin\left(\frac{u-v}{2}\right)\right| \\
&= \log|1 - e^{i(u+v)}| + \log|1 - e^{i(u-v)}| \\
&= Re\left(\log(1 - e^{i(u+v)}) + \log(1 - e^{i(u-v)})\right) \\
\overset{\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n}}{} &= -\sum_{n=1}^{\infty}\frac{1}{n}Re\left(e^{in(u+v)} + e^{in(u-v)}\right) \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos(n(u+v)) + \cos(n(u-v))\right) \\
&= -\sum_{n=1}^{\infty}\frac{2}{n}\cos(nu)\cos(nv) = -\sum_{n=1}^{\infty}\frac{2}{n}T_n\left(\frac{x}{2}\right)T_n\left(\frac{y}{2}\right).
\end{aligned}$$

Working with measures on $[-2, 2]$

Corollary

The logarithmic potential of a measure on $[-2, 2]$ *is given by*

$$\int \log |x - y| \mu(dx) = -\sum \frac{2}{n} T_n\left(\frac{x}{2}\right) \int T_n\left(\frac{y}{2}\right) \mu(dy)$$

where this series makes sense pointwise.

$$\iint \log |x - y| \mu(dx) \mu(dy) = -\sum_{n=1}^{\infty} \frac{2}{n} \left(\int T_n\left(\frac{x}{2}\right) \mu(dx) \right)^2 .$$

In particular $\iint \log |x - y| \mu(dx) \mu(dy)$ *is finite if and only if* $\sum_{n=1}^{\infty} \frac{2}{n} \left(\int T_n\left(\frac{x}{2}\right) \mu(dx) \right)^2$ *is finite.*

Working with measures on $[-2, 2]$

Corollary

If $\mu \in \mathcal{P}([-2, 2])$ *and* V *is a* C^3 *potential on* $[-2, 2]$*, then*

$$I_V(\mu) = \int V d\mu - \iint \log|x-y|\mu(dx)\mu(dy)$$
$$= \beta_0(V) + 2\sum_{n=1}^{\infty}\left(\beta_n(V)\alpha_n + \frac{\alpha_n^2}{n}\right)$$

where

$$\alpha_n = \int T_n\left(\frac{x}{2}\right)\mu(dx) \text{ and } \beta_n(V) = \int_{-2}^{2} \frac{V(x)T_n\left(\frac{x}{2}\right)dx}{\pi\sqrt{4-x^2}}.$$

Working with measures on $[-2, 2]$

$$
\begin{aligned}
I_V(\mu) &= \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)^2 \\
&\geq \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2
\end{aligned}
$$

with equality if and only if

$$
1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) \geq 0 \quad \text{for any} \quad x \in [-2, 2],
$$

in which case

$$
\mu(dx) = \left(1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right)\right) \frac{dx}{\pi\sqrt{4 - x^2}}.
$$

The formula for E_V

If V is C^3 and its equilibrium measure μ_V is $[-2, 2]$, then

$$E_V = \int_{-2}^{2} \frac{V(x)dx}{\pi\sqrt{4-x^2}} - \frac{1}{2}\iint \left(\frac{V(x)-V(y)}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}.$$

$$E_{V+tf} = \int_{-2}^{2} \frac{V(x)+tf(x)dx}{\pi\sqrt{4-x^2}} - \frac{1}{2}\iint \left(\frac{V(x)-V(y)+t(f(x)-f(y))}{x-y}\right)^2 \frac{(4-xy)dxdy}{4\pi^2\sqrt{(4-x^2)(4-y^2)}}.$$

The perturbation

Take ν_t the equilibrium measure of $V + tf$. Then $LSI(\rho)$,

$$4\rho(E_V(\nu_{V+tf}) - E_V) \leq I(\nu_t|\mu_V)$$

and the expansion to second order in t gives Poincaré's.

More?

The circle case

Multidimensional case