Arithmetic subalgebras for Bost-Connes systems

Bora Yalkinoglu

Université Pierre et Marie Curie, Paris

April 26, 2011
The Bost-Connes problem ('95)

For a number field $K$ construct a $C^*$-dynamical system (QSM)

$$\mathcal{A}_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

**Analytic Properties**

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$
The Bost-Connes problem (’95)

For a number field $K$ construct a $C^*$-dynamical system (QSM)

$$A_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

**Analytic Properties**

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$
2) $\text{Gal}(K^{ab}/K)$ is acting by symmetries on $A_K$
The Bost-Connes problem (’95)

For a number field $K$ construct a $C^*$-dynamical system (QSM)

$$\mathcal{A}_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

**Analytic Properties**

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$
2) $Gal(K^{ab}/K)$ is acting by symmetries on $\mathcal{A}_K$
3) For $0 < \beta \leq 1$ there is a unique $\sigma$-$KMS_{\beta}$ state
The Bost-Connes problem (’95)

For a number field $K$ construct a $C^*$-dynamical system (QSM)

$$\mathcal{A}_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

**Analytic Properties**

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$

2) $Gal(K^{ab}/K)$ is acting by symmetries on $\mathcal{A}_K$

3) For $0 < \beta \leq 1$ there is a unique $\sigma$-$KMS_\beta$ state

4) For $\beta > 1$, $Gal(K^{ab}/K)$ acts free and transitively on the set of extremal $\sigma$-$KMS_\beta$ states
The Bost-Connes problem (’95)

For a number field $K$ construct a $C^*$-dynamical system (QSM)

$$\mathcal{A}_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

### Analytic Properties

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$
2) $\text{Gal}(K^{ab}/K)$ is acting by symmetries on $\mathcal{A}_K$
3) For $0 < \beta \leq 1$ there is a unique $\sigma$-$KMS_\beta$ state
4) For $\beta > 1$, $\text{Gal}(K^{ab}/K)$ acts free and transitively on the set of extremal $\sigma$-$KMS_\beta$ states

A system satisfying these properties is called **analytic BC system** for $K$. 
The Bost-Connes problem (’95)

Arithmetic Properties

5) \exists \text{ } K\text{-rational subalgebra } A^{arith} \subset A \text{ such that } \forall \ f \in A^{arith} \text{ and } \varphi \in \sigma\text{-KMS}_{\infty} \text{ we have } \\

\varphi(f) \in K^{ab}

Moreover, \( K^{ab} \) is generated in this way.
The Bost-Connes problem (’95)

Arithmetic Properties

5) \( \exists \) \( K \)-rational subalgebra \( A^{arith} \subset A \) such that \( \forall f \in A^{arith} \) and \( \varphi \in \sigma\text{-}KMS_{\infty} \) we have

\[ \varphi(f) \in K^{ab} \]

Moreover, \( K^{ab} \) is generated in this way.

6) For every \( \alpha \in \text{Gal}(K^{ab}/K) \) we have the following compatibility relation

\[ \alpha \varphi(f) = \alpha^{-1}(\varphi(f)) \in K^{ab} \]
The Bost-Connes problem (’95)

Arithmetic Properties

5) \( \exists K\)-rational subalgebra \( A^{arith} \subset A \) such that \( \forall f \in A^{arith} \) and \( \varphi \in \sigma\text{-}KMS_\infty \) we have

\[ \varphi(f) \in K^{ab} \]

Moreover, \( K^{ab} \) is generated in this way.

6) For every \( \alpha \in \text{Gal}(K^{ab}/K) \) we have the following compatibility relation

\[ \alpha \varphi(f) = \alpha^{-1}(\varphi(f)) \in K^{ab} \]

A system \( A_K \) satisfying these properties is called (full) BC system for \( K \) and the corresponding subagebra \( A^{arith} \) an arithmetic subalgebra.
What is known?

'95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

Remark

The difficulty of constructing arithmetic subalgebras does come from its connection to Hilbert's 12th problem ←− widely open for $K$ not $\mathbb{Q}$ or imaginary quadratic (or CM)!
What is known?

'95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

'05 Full BC system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)
What is known?

'95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

'05 Full BC system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)

'05 Analytic BC systems $A_K$ for $K$ arbitrary [property 1) & 2)] (Ha, Paugam)
What is known?

'95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

'05 Full BC system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)

'05 Analytic BC systems $\mathcal{A}_K$ for $K$ arbitrary [property 1) & 2)] (Ha, Paugam)

'07 Complete classification of KMS states of $\mathcal{A}_K$ [property 3) & 4)] (Laca, Larsen, Neshveyev)
What is known?

’95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

’05 Full BC system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)

’05 Analytic BC systems $\mathcal{A}_K$ for $K$ arbitrary [property 1) & 2)] (Ha, Paugam)

’07 Complete classification of KMS states of $\mathcal{A}_K$ [property 3) & 4)] (Laca, Larsen, Neshveyev)

’10 Partial arithmetic subalgebras of $\mathcal{A}_K$ for $K$ a CM field (Y.)
What is known?

’95 Full BC system for $K = \mathbb{Q}$ (Bost, Connes)

’05 Full BC system for $K$ an imaginary quadratic number field (Connes, Marcolli, Ramachandran)

’05 Analytic BC systems $A_K$ for $K$ arbitrary [property 1) & 2]) (Ha, Paugam)

’07 Complete classification of KMS states of $A_K$ [property 3) & 4)] (Laca, Larsen, Neshveyev)

’10 Partial arithmetic subalgebras of $A_K$ for $K$ a CM field (Y.)

Remark

The difficulty of constructing arithmetic subalgebras does come from its connection to Hilbert’s 12th problem ← widely open for $K$ not $\mathbb{Q}$ or imaginary quadratic (or CM)!
Reminder: The analytic BC-system $\mathcal{A}_K$

The monoid of (non-zero) integral ideals $I(\mathcal{O}_K)$ of $\mathcal{O}_K$ is acting naturally on the topological space

$$Y_K = \hat{\mathcal{O}}_K \times \hat{\mathcal{O}}_K^\times \text{Gal}(K^{ab}/K)$$

The $C^*$-dynamical system $\mathcal{A}_K$ is defined by the crossed product

$$\mathcal{A}_K = (C(Y_K) \rtimes I(\mathcal{O}_K), \sigma_t)$$

with time evolution

$$\sigma_t(fu_s) = \mathcal{N}(s)^{it} fu_s$$
Aim of the talk: Explain the following results

Theorem 1 - Existence of arithmetic subalgebra (Y. ’10)
Let $K$ be an arbitrary number field. Then

Theorem 2 - Functoriality (Y. ’11)
There is a natural functor $K \mapsto E_K$ which recovers the functor $K \mapsto A_K$ recently constructed by Laca, Neshveyev and Trifkovic ('10).
Aim of the talk: Explain the following results

Theorem 1 - Existence of arithmetic subalgebra (Y. ’10)

Let $K$ be an arbitrary number field. Then

1) The system $A_K$ does come from an algebraic endomotive $E_K$.
Aim of the talk: Explain the following results

**Theorem 1 - Existence of arithmetic subalgebra (Y. ’10)**

Let $K$ be an arbitrary number field. Then

1) The system $\mathcal{A}_K$ does come from an algebraic endomotive $\mathcal{E}_K$.

2) $\mathcal{A}_K$ is in fact a **full BC system** with arithmetic subalgebra $\mathcal{E}_K$.

**Theorem 2 - Functoriality (Y. ’11)**

There is a natural functor $K \mapsto E_K$ which recovers the functor $K \mapsto A_K$ recently constructed by Laca, Neshveyev and Trifkovic ('10).
Aim of the talk: Explain the following results

Theorem 1 - Existence of arithmetic subalgebra (Y. ’10)

Let $K$ be an arbitrary number field. Then

1) The system $\mathcal{A}_K$ does come from an algebraic endomotive $\mathcal{E}_K$.

2) $\mathcal{A}_K$ is in fact a full BC system with arithmetic subalgebra $\mathcal{E}_K$.

Theorem 2 - Functoriality (Y. ’11)
Aim of the talk: Explain the following results

**Theorem 1 - Existence of arithmetic subalgebra (Y. ’10)**

Let $K$ be an *arbitrary* number field. Then

1) The system $\mathcal{A}_K$ does come from an algebraic endomotive $\mathcal{E}_K$.

2) $\mathcal{A}_K$ is in fact a **full BC system** with arithmetic subalgebra $\mathcal{E}_K$.

**Theorem 2 - Functoriality (Y. ’11)**

There is a natural functor

$$K \mapsto \mathcal{E}_K$$

which recovers the functor $K \mapsto \mathcal{A}_K$ recently constructed by Laca, Neshveyev and Trifkovic (’10).
Our methods and tools

There are two new main ingredients in our approach to the Bost-Connes problem.
Our methods and tools

There are two new main ingredients in our approach to the Bost-Connes problem.

1. The theory of **endomotives** by Connes, Consani and Marcolli ('05)
Our methods and tools

There are two new main ingredients in our approach to the Bost-Connes problem.

1. The theory of endomotives by Connes, Consani and Marcolli (’05)

2. the theory of $\Lambda$-rings due to Borger and de Smit (’08)
1. On endomotives

The notion endomotive has been introduced in ’05 by Connes, Consani and Marcolli.
1. On endomotives

The notion endomotive has been introduced in ’05 by Connes, Consani and Marcolli.

There are three types of endomotives (EM): algebraic EM → analytic EM → measured analytic EM → $K$-algebras → $C^*$-algebras → $C^*$-dynamical systems

Motivation

The classical BC system can be seen to come from an algebraic endomotive and we want to prove that in fact every BC-type system does come from an algebraic endomotive.
1. On endomotives

The notion endomotive has been introduced in ’05 by Connes, Consani and Marcolli.
There are three types of endomotives (EM)

\[
\text{algebraic EM} \rightarrow \text{analytic EM} \quad \cap \quad \text{measured analytic EM}
\]

\[
\cap \quad K\text{-algebras} \quad \rightarrow \quad C^*\text{-algebras} \quad \cap \quad C^*\text{-dynamical systems}
\]
1. On endomotives

The notion **endomotive** has been introduced in ’05 by Connes, Consani and Marcolli. There are **three** types of endomotives (EM)

\[
\text{algebraic EM } \rightarrow \text{ analytic EM } \quad \cap \quad \rightarrow \quad \cap \quad \text{measured analytic EM}
\]

\[
\cap \quad \rightarrow \quad \cap \quad \text{C*-algebras} \quad \rightarrow \quad \text{C*-dynamical systems}
\]

**Motivation**

The classical BC system can be seen to come from an algebraic endomotive and we want to prove that in fact every BC-type system does come from an algebraic endomotive.
Algebraic endo-motives

Reminder

\[
\text{Artin motives} \quad /_K \leftrightarrow^{1:1} \{\text{finite étale } K\text{-algebras}\}
\]
Algebraic endo-motives

Reminder

Artin motives $/K \overset{1:1}{\leftrightarrow} \{\text{finite étale } K\text{-algebras}\}$

Consider now the following data:

$(A_\alpha)$ inductive system of finite, étale $K$-algebras
$S$ abelian semigroup acting on $A = \lim_{\to} A_\alpha$ by endomorphisms (with some technical assumptions)

Definition

The algebraic endomotive $E$ associated with the above data is defined as the $K$-algebra $E = A \rtimes S$. Bora Yalkinoglu Université Pierre et Marie Curie, Paris ()

Arithmetic subalgebras for Bost-Connes systems

April 26, 2011 9 / 20
Algebraic endo-motives

Reminder

Artin motives $/\mathbb{K} \xleftrightarrow{1:1} \{\text{finite étale } \mathbb{K}\text{-algebras}\}$

Consider now the following data:

- $(A_\alpha)$ inductive system of finite, étale $\mathbb{K}$-algebras
Algebraic endo-motives

Reminder

Artin motives \( /_{K} \overset{1:1}{\leftrightarrow} \{ \text{finite étale } K\text{-algebras} \} \)

Consider now the following data:

- \((A_{\alpha})\) inductive system of finite, étale \(K\)-algebras
- \(S\) abelian semigroup acting on \(A = \lim A_{\alpha}\) by endomorphisms (with some technical assumptions)
Algebraic endo-motives

Reminder

Artin motives \( /_K \leftrightarrow^{1:1} \{ \text{finite étale } K\text{-algebras} \}

Consider now the following data:

- \((A_\alpha)\) inductive system of finite, étale \(K\)-algebras
- \(S\) abelian semigroup acting on \(A = \lim \rightarrow A_\alpha\) by endomorphisms (with some technical assumptions)

Definition

The algebraic endomotive \(E\) associated with the above data is defined as the \(K\)-algebra

\[ E = A \rtimes S \]
Generators and Relations

\[ \mathcal{E} = A \rtimes S \] has a presentation by adjoining to elements \( a \in A \) new generators \( U_\rho \) and \( U_\rho^* \), for \( \rho \in S \), satisfying the relations

\[
\begin{align*}
U_\rho^* U_\rho &= 1, & U_\rho U_\rho^* &= \rho(1), \\
U_{\rho_1 \rho_2} &= U_{\rho_1} U_{\rho_2}, & U_{\rho_1 \rho_2}^* &= U_{\rho_1}^* U_{\rho_2}^*, \\
U_\rho a &= \rho(a) U_\rho, & a U_\rho^* &= U_\rho^* \rho(a)
\end{align*}
\]
Analytic endomotives

\[ \mathcal{E} = A \rtimes S = \lim_{\to} A_\alpha \rtimes S \text{ algebraic endomotive.} \]
Analytic endomotives

\[ \mathcal{E} = A \rtimes S = \lim_{\to} A_\alpha \rtimes S \text{ algebraic endomotive.} \]

Define the profinite space

\[ X = \text{Hom}_{K\text{-alg}}(A, K) = \lim_{\leftarrow} \text{Hom}_{K\text{-alg}}(A_\alpha, K) \]

The analytic endomotive \( E \text{an} \) is the (semigroup) crossed product

\[ E \text{an} = C(X) \rtimes S \]

In particular

\[ E \subset E \text{an} \]
Analytic endomotives

\[ \mathcal{E} = A \rtimes S = \varprojlim A_\alpha \rtimes S \] algebraic endomotive.

Define the profinite space

\[ \mathcal{X} = \text{Hom}_{K\text{-alg}}(A, \overline{K}) = \varinjlim \text{Hom}_{K\text{-alg}}(A_\alpha, \overline{K}) \]

Then \( S \) acts naturally on the abelian \( C^* \)-algebra \( C(\mathcal{X}) \)
Analytic endomotives

\[ \mathcal{E} = A \rtimes S = \lim A_\alpha \rtimes S \] algebraic endomotive.

Define the profinite space

\[ \mathcal{X} = \text{Hom}_{K\text{-alg}}(A, \overline{K}) = \lim \text{Hom}_{K\text{-alg}}(A_\alpha, \overline{K}) \]

Then \( S \) acts naturally on the abelian \( C^* \)-algebra \( C(\mathcal{X}) \)

Definition

The analytic endomotive \( \mathcal{E}^{an} \) is the (semigroup) crossed product \( C^* \)-algebra

\[ \mathcal{E}^{an} = C(\mathcal{X}) \rtimes S \]
Analytic endomotives

\[ \mathcal{E} = A \rtimes S = \varprojlim A_\alpha \rtimes S \] algebraic endomotive.

Define the profinite space

\[ \mathcal{X} = \text{Hom}_{K\text{-alg}}(A, \overline{K}) = \varprojlim \text{Hom}_{K\text{-alg}}(A_\alpha, \overline{K}) \]

Then \( S \) acts naturally on the abelian \( C^* \)-algebra \( C(\mathcal{X}) \)

Definition

The **analytic endomotive** \( \mathcal{E}^{an} \) is the (semigroup) crossed product \( C^* \)-algebra

\[ \mathcal{E}^{an} = C(\mathcal{X}) \rtimes S \]

In particular

\[ \mathcal{E} \subset \mathcal{E}^{an} \]
Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $\mathcal{X}_\alpha = \text{Hom}(A_\alpha, \overline{K})$
### Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $\mathcal{X}_\alpha = \text{Hom}(A_\alpha, \overline{K})$

- Assume existence of limit probability measure $\mu = \lim \mu_\alpha$ on $\mathcal{X} = \lim \mathcal{X}_\alpha$ (i.e. compatibility with transition maps)
Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $\mathcal{X}_\alpha = \text{Hom}(A_\alpha, \overline{K})$

- Assume existence of limit probability measure $\mu = \lim_{\leftarrow} \mu_\alpha$ on $\mathcal{X} = \lim_{\leftarrow} \mathcal{X}_\alpha$ (i.e. compatibility with transition maps)

$\leadsto$ state $\varphi = \varphi_\mu : \mathcal{E}^{an} = C(\mathcal{X}) \rtimes S \to \mathbb{C}$ (by integration)
Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $\mathcal{X}_\alpha = \text{Hom}(A_\alpha, \overbar{K})$

- Assume existence of limit probability measure $\mu = \lim \mu_\alpha$ on $\mathcal{X} = \lim \mathcal{X}_\alpha$ (i.e. compatibility with transition maps)

$\leadsto$ state $\varphi = \varphi_\mu : \mathcal{E}^{an} = \mathcal{C}(\mathcal{X}) \rtimes S \to \mathbb{C}$ (by integration)

$\leadsto$ GNS construction $\pi : \mathcal{E}^{an} \to \mathcal{B}(\mathcal{H}_\varphi)$
Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $\mathcal{X}_\alpha = \text{Hom}(A_\alpha, \overline{K})$

- Assume existence of limit probability measure $\mu = \lim \mu_\alpha$ on $\mathcal{X} = \lim \mathcal{X}_\alpha$ (i.e. compatibility with transition maps)

$$\leadsto \text{state } \varphi = \varphi_\mu : \mathcal{E}^{an} = C(\mathcal{X}) \rtimes S \to \mathbb{C} \text{ (by integration)}$$

$$\leadsto \text{GNS construction } \pi : \mathcal{E}^{an} \to \mathbb{B}(\mathcal{H}_\varphi)$$

$$\leadsto \text{(under technical assumptions + Tomita-Takesaki)}$$

$$\exists \sigma : \mathbb{R} \to \text{Aut}(\mathcal{E}^{an})$$
Measured analytic endomotives

- $\mu_\alpha$ normalized counting measure on $X_\alpha = \text{Hom}(A_\alpha, K)$

- Assume existence of limit probability measure $\mu = \lim \mu_\alpha$ on $X = \lim X_\alpha$ (i.e. compatibility with transition maps)

$\rightsquigarrow$ state $\varphi = \varphi_\mu : \mathcal{E}^{an} = C(X) \rtimes S \rightarrow \mathbb{C}$ (by integration)

$\rightsquigarrow$ GNS construction $\pi : \mathcal{E}^{an} \rightarrow \mathcal{B}(\mathcal{H}_\varphi)$

$\rightsquigarrow$ (under technical assumptions + Tomita-Takesaki)

$\exists \sigma : \mathbb{R} \rightarrow \text{Aut}(\mathcal{E}^{an})$

Definition

The measured analytic endomotive $\mathcal{E}^{meas}$ (if it exists) is the $C^*$-dynamical system

$$\mathcal{E}^{meas} = (\mathcal{E}^{an}, (\sigma_t)_{t \in \mathbb{R}})$$
In good situations an algebraic endomotive $E^{alg} = A \rtimes S$ gives naturally rise to a $C^*$-dynamical system

$$(E^{an}, (\sigma_t)_{t \in \mathbb{R}})$$

with

$$E^{alg} \subset E^{an}$$

(plus other nice properties, e.g. a natural $Gal(\overline{K}/K)$-action!)
Example: The classical Bost-Connes system as endomotive

See blackboard. If time permits!
What’s next?

In order to construct interesting (algebraic) endomotives which are related to our Bost-Connes systems of HPLLN, we have to find the right class of finite, étale $K$-algebras!
2. On $\Lambda$-rings

**Definition**

An $\mathcal{O}_K$-algebra $\tilde{E}$ is a $\Lambda$-ring if there exists for each (non-zero) prime ideal $p$ of $\mathcal{O}_K$ an endomorphism $f_p$ such that

1) $f_p \circ f_q = f_q \circ f_p$

2) $f_p$ is a Frobenius lift
2. On Λ-rings

Definition

An $\mathcal{O}_K$-algebra $\tilde{E}$ is a $\Lambda$-ring if there exists for each (non-zero) prime ideal $p$ of $\mathcal{O}_K$ an endomorphism $f_p$ such that

1) $f_p \circ f_q = f_q \circ f_p$

$E_n = \mathbb{Q}[X]/(X^n - 1)$ has the integral $\Lambda$-structure $\mathbb{Z}[X]/(X^n - 1)$ with $f_p : X \mapsto X^p$.
2. On \( \Lambda \)-rings

**Definition**

An \( \mathcal{O}_K \)-algebra \( \tilde{E} \) is a \( \Lambda \)-ring if there exists for each (non-zero) prime ideal \( p \) of \( \mathcal{O}_K \) an endomorphism \( f_p \) such that

1) \( f_p \circ f_q = f_q \circ f_p \)

2) \( f_p \) is a Frobenius lift
2. On Λ-rings

Definition

An \( \mathcal{O}_K \)-algebra \( \tilde{E} \) is a \( \Lambda \)-ring if there exists for each (non-zero) prime ideal \( p \) of \( \mathcal{O}_K \) an endomorphism \( f_p \) such that

1) \( f_p \circ f_q = f_q \circ f_p \)

2) \( f_p \) is a Frobenius lift

A \( K \)-algebra \( E \) has an integral \( \Lambda \)-structure if it does come from a \( \Lambda \)-ring, i.e. \( E = \tilde{E} \otimes_{\mathcal{O}_K} K \).
2. On Λ-rings

**Definition**

An $\mathcal{O}_K$-algebra $\tilde{E}$ is a **Λ-ring** if there exists for each (non-zero) prime ideal $p$ of $\mathcal{O}_K$ an endomorphism $f_p$ such that

1. $f_p \circ f_q = f_q \circ f_p$

2. $f_p$ is a Frobenius lift

A $K$-algebra $E$ has an integral **Λ-structure** if it does come from a Λ-ring, i.e. $E = \tilde{E} \otimes_{\mathcal{O}_K} K$.

**Example**
2. On $\Lambda$-rings

Definition

An $\mathcal{O}_K$-algebra $\tilde{E}$ is a $\Lambda$-ring if there exists for each (non-zero) prime ideal $p$ of $\mathcal{O}_K$ an endomorphism $f_p$ such that

1) $f_p \circ f_q = f_q \circ f_p$

2) $f_p$ is a Frobenius lift

A $K$-algebra $E$ has an integral $\Lambda$-structure if it does come from a $\Lambda$-ring, i.e. $E = \tilde{E} \otimes_{\mathcal{O}_K} K$.

Example

Each $E_n = \mathbb{Q}[X]/(X^n - 1)$ has the integral $\Lambda$-structure
2. On \( \Lambda \)-rings

**Definition**

An \( \mathcal{O}_K \)-algebra \( \tilde{E} \) is a \( \Lambda \)-ring if there exists for each (non-zero) prime ideal \( p \) of \( \mathcal{O}_K \) an endomorphism \( f_p \) such that

1) \( f_p \circ f_q = f_q \circ f_p \)

2) \( f_p \) is a Frobenius lift

A \( K \)-algebra \( E \) has an integral \( \Lambda \)-structure if it does come from a \( \Lambda \)-ring, i.e. \( E = \tilde{E} \otimes_{\mathcal{O}_K} K \).

**Example**

Each \( E_n = \mathbb{Q}[X]/(X^n - 1) \) has the integral \( \Lambda \)-structure

\[ \mathbb{Z}[X]/(X^n - 1) \]
2. On $\Lambda$-rings

**Definition**

An $\mathcal{O}_K$-algebra $\tilde{E}$ is a $\Lambda$-ring if there exists for each (non-zero) prime ideal $p$ of $\mathcal{O}_K$ an endomorphism $f_p$ such that

1) $f_p \circ f_q = f_q \circ f_p$

2) $f_p$ is a Frobenius lift

A $K$-algebra $E$ has an integral $\Lambda$-structure if it does come from a $\Lambda$-ring, i.e. $E = \tilde{E} \otimes_{\mathcal{O}_K} K$.

**Example**

Each $E_n = \mathbb{Q}[X]/(X^n - 1)$ has the integral $\Lambda$-structure

$$\mathbb{Z}[X]/(X^n - 1)$$

with

$$f_p : X \mapsto X^p$$

$\sim$ classical BC system
The Deligne-Ribet monoid

The Deligne-Ribet monoid $DR_K$ is the profinite monoid

$$DR_K = \lim_{\leftarrow} DR_f$$

where for each $f \in I(O_K)$ we set $DR_f = I(O_K)/\sim_f$.

Proposition

There is a natural equivariant isomorphism of (topological) monoids $Y_K = \hat{O}_K \times \hat{O}_K \times \text{Gal}(K_{ab}/K) \sim DR_K$ w.r.t. the natural action of $I(O_K)$. 

Recall the analytic BC system $A_K = (C(Y_K) \rtimes I(O_K), \sigma_t)$.
The Deligne-Ribet monoid

The Deligne-Ribet monoid $DR_K$ is the profinite monoid

$$DR_K = \varprojlim DR_f$$

where for each $f \in I(O_K)$ we set $DR_f = I(O_K) / \sim_f$.

Proposition

There is a natural equivariant isomorphism of (topological) monoids

$$Y_K = \hat{O}_K \times \hat{O}_K^\times \text{Gal}(K^{ab}/K) \cong DR_K$$

w.r.t. the natural action of $I(O_K)$. 
The Deligne-Ribet monoid

The Deligne-Ribet monoid $DR_K$ is the profinite monoid

$$DR_K = \lim_{\leftarrow} DR_f$$

where for each $f \in I(\mathcal{O}_K)$ we set $DR_f = I(\mathcal{O}_K)/\sim_f$.

Proposition

There is a natural equivariant isomorphism of (topological) monoids

$$Y_K = \hat{\mathcal{O}}_K \times \hat{\mathcal{O}}_K^\times \text{Gal}(K^{ab}/K) \cong DR_K$$

w.r.t. the natural action of $I(\mathcal{O}_K)$.

Recall the analytic BC system

$$A_K = (C(Y_K) \rtimes I(\mathcal{O}_K), \sigma_t)$$
The Deligne-Ribet monoid as source for finite, étale alg.s

Theorem (Borger, de Smit)

The functor $E \mapsto \text{Hom}(E, K)$ induces an equivalence of categories

\{ finite, étale $K$-algebras with integral $\Lambda$-structure \} $\leftrightarrow$ \{ finite sets + $\preccurlyeq$ cont. DR $K$ \}

Key observation I

In particular $\text{DR}_f \sim = \text{Hom}(E_f, K)$ with finite, étale $K$-algebra $E_f \sim = \prod_{d | f} K_d$ with $K_d$ a strict ray class field of $K$, i.e. abelian over $K$.

Key observation II (due to $Y_K \sim = \text{DR}_K$)

$Y_K \sim = \text{Hom}(\lim_{\rightarrow} E_f, K)$

Bora Yalkinoglu Université Pierre et Marie
Arithmetic subalgebras for Bost-Connes system
April 26, 2011 18 / 20
The Deligne-Ribet monoid as source for finite, étale alg.s

**Theorem (Borger, de Smit)**

The functor $E \mapsto \text{Hom}(E, \overline{K})$ induces an equivalence of categories

\[
\left\{ \begin{array}{l}
\text{finite, étale } K\text{-algebras} \\
\text{with integral } \Lambda\text{-structure}
\end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l}
\text{finite sets} + \\
\circ_{\text{cont. }} DR_K
\end{array} \right\}
\]
The Deligne-Ribet monoid as source for finite, étale alg.s

**Theorem (Borger, de Smit)**

The functor $E \mapsto \text{Hom}(E, \overline{K})$ induces an equivalence of categories

\[
\left\{ \begin{array}{c}
\text{finite, étale } K\text{-algebras} \\
\text{with integral } \Lambda\text{-structure}
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
\text{finite sets} + \\
\text{cont. } DR_K
\end{array} \right\}
\]

**Key observation I**

In particular $DR_f \sim \text{Hom}(E_f, \overline{K})$ with finite, étale $K$-algebra $E_f \sim \prod_{d \mid f} K_d$ with $K_d$ a strict ray class field of $K$, i.e. abelian over $K$.

**Key observation II** (due to $Y_K \sim DR_K$)

$Y_K \sim \text{Hom}(\lim_{\rightarrow} - \to E_f, K)$
The Deligne-Ribet monoid as source for finite, étale alg.s

Theorem (Borger, de Smit)

The functor $E \mapsto Hom(E, \overline{K})$ induces an equivalence of categories

\[
\begin{aligned}
\{ \text{finite, étale } K\text{-algebras} \\
\text{with integral } \Lambda\text{-structure} \} & \quad \overset{1:1}{\iff} \quad \{ \text{finite sets + } \\
\text{cont. } \bigcup_{\text{cont. } DR_K} \}
\end{aligned}
\]

Key observation I

In particular

\[DR_f \cong Hom(E_f, \overline{K})\]

with finite, étale $K$-algebra $E_f \cong \prod_{\mathfrak{p}|f} K_\mathfrak{p}$ with $K_\mathfrak{p}$ a strict ray class field of $K$, i.e. abelian over $K$. 

Bora Yalkinoglu
Université Pierre et Marie Curie, Paris
Arithmetic subalgebras for Bost-Connes systems
April 26, 2011 18 / 20
The Deligne-Ribet monoid as source for finite, étale alg.s

**Theorem (Borger, de Smit)**

The functor $E \mapsto \text{Hom}(E, \overline{K})$ induces an equivalence of categories

$$\left\{ \text{finite, étale } K\text{-algebras with integral } \Lambda\text{-structure} \right\} \overset{1:1}{\iff} \left\{ \text{finite sets } + \left\langle\text{cont. } DR_K\right\rangle \right\}$$

**Key observation I**

In particular

$$DR_f \cong \text{Hom}(E_f, \overline{K})$$

with finite, étale $K$-algebra $E_f \cong \prod_{\mathfrak{p}|f} K_\mathfrak{p}$ with $K_\mathfrak{p}$ a strict ray class field of $K$, i.e. abelian over $K$.

**Key observation II (due to $Y_K \cong DR_K$)**
The Deligne-Ribet monoid as source for finite, étale alg.s

Theorem (Borger, de Smit)

The functor $E \mapsto \text{Hom}(E, \overline{K})$ induces an equivalence of categories

\[
\left\{ \text{finite, étale } K\text{-algebras} \right\} \xrightarrow{1:1} \left\{ \text{finite sets } + \bigcirc \text{cont. } DR_K \right\}
\]

Key observation I

In particular

$$DR_f \cong \text{Hom}(E_f, \overline{K})$$

with finite, étale $K$-algebra $E_f \cong \prod_{d|f} K_0$ with $K_0$ a strict ray class field of $K$, i.e. abelian over $K$.

Key observation II (due to $Y_K \cong DR_K$)

$$Y_K \cong \text{Hom}(\lim_{\rightarrow} E_f, \overline{K})$$
Finally: Our algebraic endomotive $\mathcal{E}_K$

Now we can define our desired algebraic endomotive

$$\mathcal{E}_K = ((E_f), I(O_K))$$
Finally: Our algebraic endomotive $\mathcal{E}_K$

Now we can define our desired algebraic endomotive

$$\mathcal{E}_K = ((E_f), I(O_K))$$

By construction we have

$$\mathcal{E}^{an} \cong C(Y_K) \rtimes I(O_K)$$
Finally: Our algebraic endomotive $\mathcal{E}_K$

Now we can define our desired algebraic endomotive

$$\mathcal{E}_K = ( (E_f), I(O_K) )$$

By construction we have

$$\mathcal{E}^{an} \cong C(Y_K) \rtimes I(O_K)$$

further one can show that in fact

$$\mathcal{E}^{meas} \cong \mathcal{A}_K = ( C(Y_K) \rtimes I(O_K), \sigma_t )$$
Finally: Our algebraic endomotive $\mathcal{E}_K$

Now we can define our desired algebraic endomotive

$$\mathcal{E}_K = ((E_f), I(\mathcal{O}_K))$$

By construction we have

$$\mathcal{E}^{an} \cong C(Y_K) \rtimes I(\mathcal{O}_K)$$

Further one can show that in fact

$$\mathcal{E}^{meas} \cong A_K = (C(Y_K) \rtimes I(\mathcal{O}_K), \sigma_t)$$

And, most importantly, one can identify

$$\mathcal{E}_K \subset C(Y_K) \rtimes I(\mathcal{O}_K)$$

as an arithmetic subalgebra of $A_K$. 
Mulțumesc!

Probably I have already overstepped my time! Therefore I will stop here and thank you very much for your attention!