Feynman-Kac formula for left continuous additive functionals

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Abstract. We establish a Feynman-Kac formula associated with measures charging no polar set and belonging to an extended Kato class. A main tool of this approach is the validity of a Khas'minskii Lemma for Stieltjes exponentials of positive left continuous additive functionals.

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Introduction

Let $\mathcal{L}$ be the infinitesimal operator of the strongly continuous sub-Markovian semigroup of contractions on $L^p(E, m)$, induced by a Borel right process X with state space E, where m is a fixed excessive measure and $p \in [1, \infty)$. In this frame one can consider measure perturbations of $\mathcal{L}$, namely the following Schrödinger type equation may be stated

$$(*) \qquad (q - \mathcal{L})u + \mu u = f$$

where μ is a signed measure on E, $\mu = \mu^+ - \mu^-$, and $f \in L^p(E, m)$. The problem is to find the convenient class of measures μ which ensure existence and uniqueness (in a weak sense) of the solutions for the equation $(*)$.

In the classical case (e.g. if $\mathcal{L} = \Delta$ and μ has a density g with respect to the Euclidean Lebesgue measure m) the appropriate class is the well known *Kato class* (see e.g. [ChZa 95]) and a probabilistic tool is the continuous additive functional $A_t = \int_0^t g \circ X_s \, ds$. In order to show that the Feynman-Kac semigroup

$$Q_t f(x) = E^x(\exp(-A_t) \, f \circ X_t), \quad x \in E,$$

has $\Delta - \mu$ as infinitesimal generator, a main argument is the so called "Khas'minskii Lemma" which gives evaluations for the mean of the exponential $\exp(A_t)$. Notice that a central result is the characterization of the Kato class (given by M. Aizenman and B. Simon [AiSi 82]) using the potential of the continuous additive functional $(A_t)_{t \geq 0}$: $U_A 1 = E^{\cdot} \displaystyle\int_0^\infty g \circ X_s \, ds$, $g \geq 0$. This technique has been extended by R.K. Getoor [Ge 99] to the frame given by a right process (see the references therein for many other contributions), the *extended Kato class* is replacing here the classical one. Recall that in particular these measures charge no m-semipolar set.

In this paper we establish the Feynman-Kac formula associated with the equation $(*)$ for the essentially larger class of measures charging no m-polar set, such a measure may be even carried by an m-semipolar set and therefore the methods based on continuous additive functionals fail. Our technique applies to the typical example given by the heat operator

$$\mathcal{L} = \Delta - \frac{\partial}{\partial t}$$

in $\mathbb{R}^{n+1}$, where Δ is the Laplacean in $\mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure on a horizontal hyperplane in $\mathbb{R}^{n+1}$, which is a semipolar set for the process in $\mathbb{R}^{n+1}$ having the generator $\Delta - \frac{\partial}{\partial t}$. Notice that this example is out of rich using the known Feynman-Kac formula methods.

The measures we are considering are precisely the Revuz measures of the *positive left (continuous) additive functionals* (abbreviated PLAFs); see [FiGe 03] and [BeBo 04].

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We show that the Feynman-Kac formula holds for PLAFs, however it is necessary to replace the usual exponential $\exp(A_t)$ by two types of "Stieltjes exponentials", denoted by $\mathrm{Exp}(A)_t$ and $\widetilde{\mathrm{Exp}}(A)_t$:

$$Q_t f(x) = E^x\Big(\frac{\widetilde{\mathrm{Exp}}(A^-)_{t-}}{\mathrm{Exp}(A^+)_t} f \circ X_t\Big),$$

where A^+ (resp. A^-) is the PLAF having μ^+ (resp. μ^-) as Revuz measure.

1 Preliminaries

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \zeta)$ be a Borel right process with state space E, a metrizable Lusin topological space. Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the resolvent of kernels associated with X, i.e.,

$$U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f \circ X_t dt,$$

for all $\alpha > 0$, $x \in E$ and $f \in p\mathcal{B}$; here $\mathcal{B}$ denotes the Borel σ-algebra on E and $p\mathcal{B}$ the set of all positive numerical $\mathcal{B}$-measurable functions on E.

Recall that a function $v \in p\mathcal{B}^u$ ($\mathcal{B}^u$ is the universally completion of $\mathcal{B}$) is named *$\mathcal{U}$-supermedian* if $\alpha U_\alpha v \le v$ for all $\alpha > 0$. A $\mathcal{U}$-supermedian function is named *$\mathcal{U}$-excessive* if in addition $\sup_\alpha \alpha U_\alpha v = v$. We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive functions on E. If v is $\mathcal{U}$-supermedian then the function $\widehat{v} := \sup_{\alpha>0} \alpha U_\alpha v$ is $\mathcal{U}$-excessive.

A σ-finite measure ξ on $(E, \mathcal{B})$ is termed $\mathcal{U}$-excessive provided that $\xi \circ \alpha U_\alpha \le \xi$ for all $\alpha > 0$. We denote by $\mathrm{Exc}(\mathcal{U})$ the set of all $\mathcal{U}$-excessive measures on E. A $\mathcal{U}$-excessive measure of the form $\mu \circ U$ (where μ is a positive σ-finite measure) is called *potential*.

If $q > 0$ we consider the bounded sub-Markovian resolvent of kernels $\mathcal{U}_q = (\mathcal{U}_{q+\alpha})_{\alpha>0}$. Since $\mathcal{U}_q$ is the resolvent associated with a transient Borel right process with state space E (the q-subprocess of X) it follows that E is *semisaturated* with respect to $\mathcal{U}_q$, i.e., every $\mathcal{U}_q$-excessive measure dominated by a potential $\mathcal{U}_q$-excessive measure is also a potential.

For each $v \in \mathcal{E}(\mathcal{U}_q)$ and every subset M of E let

$$R_q^M v := \inf\{u \in \mathcal{E}(\mathcal{U}_q) / u \ge v \text{ on } M\}$$

be the *reduced function of v on M* (with respect to $\mathcal{E}(\mathcal{U}_q)$). It is known that if $M \in \mathcal{B}$ then $R_q^M v$ is a universally $\mathcal{B}$-measurable $\mathcal{U}_q$-supermedian function and we put $B_q^M v = \widehat{R_q^M v}$.

Let μ be a σ-finite measure on E. A subset M of E is called *μ-polar* if there exists $M_0 \in \mathcal{B}$, $M_0 \supset M$ such that $B_q^{M_0} 1 = 0$ μ-a.e. The set M is named *nearly Borel* if for every finite measure μ on E there exists a set $M_1 \subset M$, $M_1 \in \mathcal{B}$, such that the set $M \setminus M_1$is μ-polar and μ-negligible.

We denote by $\mathcal{B}^n$ the σ-algebra of all nearly Borel subsets of E. We have $\mathcal{B} \subset \mathcal{B}^n \subset \mathcal{B}^u$ and $\mathcal{E}(\mathcal{U}_q) \subset p\mathcal{B}^n$.

In the sequel m will be a fixed $\mathcal{U}$-excessive measure. Clearly the measure m is $\mathcal{U}_q$-excessive for all $q > 0$. We denote by $\mathcal{N}(m)$ the set of all nearly Borel sets which are m-polar.

Let σ_m be the set of all positive σ-finite measures charging no set from $\mathcal{N}(m)$.

A set $N \in \mathcal{B}^n$ is *m-inessential* if it belongs to $\mathcal{N}(m)$ and $R_q^N 1 = 0$ on $E \setminus N$. We remark that every m-polar set is the subset of a Borel measurable m-inessential set.

A property depending on $x \in E$ is said to hold *m-quasi everywhere* (abbreviated m-q.e.) if the set of all $x \in E$ for which it does not hold is m-polar.

Recall that the fine topology is the topology on E generated by all $\mathcal{U}_q$-excessive functions. A function $f \in p\mathcal{B}^n$ is called *m-finely continuous* if it is finely continuous outside a set from $\mathcal{N}(m)$. If $g \in p\mathcal{B}^n$ then a *m-fine version* of g is a function f which is m-finely continuous and $f = g$ m-a.e.

By Theorem 4.4.2 in [BeBo 04] it follows that if $\xi \in \mathrm{Exc}(\mathcal{U}_q)$ and $\xi \ll m$ then there exists a m-fine version of the Radon-Nikodym derivative $d\xi/dm$.

If $\mu \in \sigma_m$ and $q > 0$ then by Theorem 6.1.2 in [BeBo 04] there exists a kernel V_μ^q on $(E, \mathcal{B}^n)$ which is regular strongly supermedian with respect to $\mathcal{U}_q$ such that $\mu(f) = L_q(m, V_\mu^q f)$ for all $f \in p\mathcal{B}$, where L_q denotes the *energy functional* associated with $\mathcal{U}_q$; see (A1) in Appendix. The kernel V_μ^q is uniquely determined m-q.e. and for every $\xi \in \mathrm{Exc}(\mathcal{U}_q)$ such that $\xi \ll m$ the following *Revuz formula* holds:

$$L_q(\xi, V_\mu^q f) = L_q(m, V_\mu^q(\bar{\imath} f)), \text{ for all } f \in p\mathcal{B}^n,$$

where $\bar{t}$ is a m-fine version of the Radon-Nikodym derivative $d\xi/dm$. The map $\mu \longmapsto V^q_\mu$ is called the *Revuz correspondence.*

The following assertion follows by Proposition 2.2 and the proof of Theorem 3.1 in [BeBo 05].

(1.1) If $\mu \in \sigma_m$ then there exists a m-inessential set $N \in \mathcal{B}$ such that for all $q, q' > 0$, $q' > q$ we have on $E \backslash N : V^q_\mu = V^{q'}_\mu + (q' - q) U_q V^{q'}_\mu$.

Let $\mathcal{U}^* = (U^*_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B})$ such that $\sigma(p\mathcal{B} \cap \mathcal{E}(\mathcal{U}^*_q)) = \mathcal{B}$, $\mathcal{E}(\mathcal{U}^*_q)$ is min-stable, $1 \in \mathcal{E}(\mathcal{U}^*_q)$ (for one $q > 0$) and $\int f U_\alpha g dm = \int g U^*_\alpha f dm$ for all $f, g \in p\mathcal{B}$ and $\alpha > 0$; see Corollary 2.4 in [BeBoRö 06] for the existence of such a resolvent. Notice that if $g \in p\mathcal{B}$ is such that $U^*_q g < \infty$ m-a.e. then $U^*_q g \cdot m$ is a $\mathcal{U}_q$-excessive measure which is a potential, $U^*_q g \cdot m = (g \cdot m) \circ U_q$. Particularly $U^*_q g$ has a m-fine version denoted by $\overline{U^*_q g}$.

Lemma 1.1. *If $\mu \in \sigma_m$, $f, g \in p\mathcal{B}^n$ and $q > 0$ is such that $U^*_q g < \infty$ m-a.e. then*

$$\int g V^q_\mu f dm = \int f \overline{U^*_q} g d\mu.$$

Proof. The assertion follows by the Revuz formula,

$$\int f \overline{U^*_q} g d\mu = L_q(m, V^q_\mu(f \overline{U^*_q} g)) = L_q(U^*_q g \cdot m, V^q_\mu f) = L_q((g \cdot m) \circ U_q, V^q_\mu f) = \int g V^q_\mu f dm.$$

□

A measure $\mu \in \sigma_m$ is called *smooth* provided that there exists an increasing sequence $(A_k)_k \subset \mathcal{B}^n$ such that $\mu(A_k) < \infty$ for all k and $\inf_k R^{E \backslash A_k}_q U_q 1 = 0$ m-a.e. for one (and therefore for all) $q > 0$.

(1.2) By Theorem 6.3.1 in [BeBo 04] it follows that a measure $\mu \in \sigma_m$ is smooth if and only if there exists an increasing sequence $(A_k)_k \subset \mathcal{B}^n$ such that $\inf_k R^{E \backslash A_k}_q U_q 1 = 0$ m-a.e. and $V^q_\mu(1_{A_k}) < \infty$ m-a.e. for all k. In particular if $V^q_\mu 1 < \infty$ m-a.e. for one $q > 0$ then the measure μ is smooth.

Extended Kato class

For $\mu \in \sigma_m$ and $q > 0$ we define

$$\widehat{c}_q(\mu) := \inf\{\alpha > 0 / V^q_\mu 1 \leq \alpha \quad m\text{-q.e.}\}.$$

Clearly the function $q \longmapsto \widehat{c}_q(\mu)$ is decreasing and we put

$$\widehat{c}(\mu) := \inf_{q>0} \widehat{c}_q(\mu) = \lim_{q \to \infty} \widehat{c}_q(\mu).$$

For $p \in [1, \infty]$ we denote by $\| \cdot \|_p$ the norm in $L^p = L^p(E, m)$.

Proposition 1.2. *The following assertions hold for a measure $\mu \in \sigma_m$.*

i) If μ charges no m-semipolar set then for all $q > 0$ we have

$$\widehat{c}_q(\mu) = \|V^q_\mu 1\|_\infty = \sup\{\mu(\overline{U^*_q g}) / g \in p\mathcal{B}^n, \|g\|_1 \leq 1\}.$$

ii) We have $\widehat{c}(\mu) < \infty$ if and only if $\widehat{c}_q(\mu) < \infty$ for all $q > 0$. In this case μ will be a smooth measure.

Proof. *i)* If μ charges no m-semipolar set then the function $V^q_\mu 1$ is finely continuous and therefore $\widehat{c}_q(\mu) = \|V^q_\mu 1\|_\infty$. By Lemma 1.1 we have $\sup\{\int g V^q_\mu 1 dm / g \in p\mathcal{B}^n, \|g\|_1 \leq 1\} = \sup\{\mu(\overline{U^*_q g}) / g \in p\mathcal{B}^n, \|g\|_1 \leq 1\}$.

ii) Assume that $\widehat{c}_q(\mu)$ is finite for one $q > 0$, then by (1.1) it follows that it is finite for all $q > 0$. From $V^q_\mu 1 < \infty$ m-a.e. and by (1.2) we conclude that the measure μ is smooth. □

If $\mu \in \sigma_m$ and $q > 0$, following [Ge 99], we define dually

$$c_q(\mu) := \sup\{\mu(U_q g) / g \in p\mathcal{B}^n, \|g\|_1 \leq 1\}$$

and let

$$c(\mu) := \inf_{q>0} c_q(\mu) = \lim_{q\to\infty} c_q(\mu).$$

Analogously (as in Proposition 1.2) one can see that: $c(\mu) < \infty$ if and only if $c_q(\mu) < \infty$ for all $q > 0$. In this case μ will be a smooth measure.

Remark. *i*) Proposition 1.2 *i*) shows that our definition of $\widehat{c}_q(\mu)$ agrees with that one considered in [Ge 99] (see also [StVo 96]) in the particular case when the measure μ charges no m-semipolar set.

ii) The "extended Kato class" we shall consider in Section 4 will be that of all measures $\mu \in \sigma_m$ such that $\widehat{c}(\mu) < 1$ and $c(\mu) < \infty$; as in [Ge 99] two conditions are occurring, since we are not in the symmetric case. Notice that condition $\widehat{c}_q(\mu) < \infty$ is merely a boundedness property of the "potential" $V^q_\mu 1$, the classical Kato class being rather a boundedness and continuity property of $V^q_\mu 1$.

2 Stieltjes exponentials of a positive left additive functional

Throughout this section we assume that the given right process is transient, that is the kernel $U := \sup_{\alpha>0} U_\alpha$ is proper (i.e., there exists $f \in bp\mathcal{B}$, $f > 0$, such that $Uf \le 1$).

Let further m be a $\mathcal{U}$-excessive measure on E. Recall that a *positive left additive functional* (abbreviated *PLAF*) of the process X with respect to m is a family $A = (A_t)_{t\ge 0}$ of $\mathcal{F}_t$-measurable functions, $A_t : \Omega_A \to [0,\infty]$, where $\Omega_A \in \mathcal{F}$ (Ω_A is called *defining set* for A), and there exists a m-inessential set N_A (called *exceptional set* for A), such that the following assertions hold:

• $P^x(\Omega_A) = 1$ for all $x \in E\backslash N_A$ and $\theta_t(\Omega_A) \subset \Omega_A$ for all $t > 0$;

• For all $\omega \in \Omega_A$ the map $t \mapsto A_t(\omega)$ is increasing and left continuous on $[0,\infty]$, finite valued on $[0,\zeta(\omega))$, with $A_0(\omega) = 0$ and $A_t([\Delta]) = 0$ for all $t \ge 0$;

• There exists a function $a \in p\mathcal{B}^n$ such that for every $\omega \in \Omega_A$ we have $A_{0+}(\omega) = a(X_0(\omega))$;

• $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$ for all $\omega \in \Omega_A$ and $s,t \ge 0$.

The following assertion hold for a PLAF $A = (A_t)_{t\ge 0}$ of X with respect to m.

a) The exceptional set N_A may be replaced by a second one of the same type, which in addition belongs to $\mathcal{B}$.

b) If $A = (A_t)_{t\ge 0}$ is a PLAF such that $t \longmapsto A_t(\omega)$ is continuous on $[0,\zeta(\omega))$ for all $\omega \in \Omega_A$, then A is called *positive continuous additive functional* (abbreviated *PCAF*).

c) We denote by ${}^c\!A = ({}^c\!A)_{t\ge 0}$ (resp. ${}^d\!A = ({}^d\!A_t)_{t\ge 0}$) the continuous (resp. discontinuous) part of A, i.e.,

$${}^d\!A_t = \begin{cases} 0, & \text{if} \quad t = 0 \\ \displaystyle\sum_{0\le s<t} \Delta A_s, & \text{if} \quad t > 0, \end{cases}$$

where $\Delta A_s = A_{s_+} - A_s$, and ${}^c\!A_t := A_t - {}^d\!A_t$. It is easy to check (see e.g. page 182 in [Sh 88]) that ${}^d\!A$ (resp. ${}^c\!A$) is a PLAF of X (resp. a PCAF of X), having the same defining and exceptional sets as A.

d) We denote by $\widetilde{A} = (\widetilde{A}_t)_{t\ge 0}$ the family of maps $\widetilde{A}_t : \Omega_A \to [0,\infty]$ defined for all $t \ge 0$ and $\omega \in \Omega_A$ by

$$\widetilde{A}_t(\omega) := A_{t+}(\omega) = \int_{[0,t]} dA_s(\omega).$$

Clearly $t \longmapsto \widetilde{A}_t(\omega)$ is increasing, right continuous, and for all $s > 0$ and $t \ge 0$ we have

$$\widetilde{A}_{s+t}(\omega) = A_s(\omega) + \widetilde{A}_t(\theta_s(\omega)).$$

e) For all $n \ge 1$ we shall define now inductively two types of "compensated nth powers" of the PLAF $A = (A_t)_{t\ge 0}$: the $\mathcal{F}_t$-measurable functionals $A^{[n]} = (A^{[n]}_t)_{t\ge 0}$ and $\widetilde{A}^{[n]} = (\widetilde{A}^{[n]}_t)_{t\ge 0}$, $A^{[n]}_t$, $\widetilde{A}^{[n]}_t : \Omega_A \to [0,\infty]$ given by, $A^{[0]}_t = \widetilde{A}^{[0]}_t = 1$ and

$$A^{[n+1]}_t = (n+1)\int_{[0,t)} A^{[n]}_s dA_s, \quad \widetilde{A}^{[n+1]}_t = (n+1)\int_{[0,t]} \widetilde{A}^{[n]}_s dA_s.$$

It is easy to see that for all $\omega \in \Omega_A$ and $n \geq 1$ the map $t \longmapsto A_t^{[n]}(\omega)$ (resp. $t \longmapsto \widetilde{A}_t^{[n]}(\omega)$) is increasing and left continuous (resp. right continuous). Notice that if $t > 0$ then $\widetilde{A}_{t-} = \sup_{s<t} \int_{[0,s]} dA_u = \int_{[0,t)} dA_u = A_t$. We have also $\quad {}^d\!\widetilde{A}_t = \sum_{0 \leq s \leq t} \Delta A_s$.

The proofs of the following three propositions will be presented in (A2) in Appendix.

Proposition 2.1. *For every $n \in \mathbb{N}^*$ and $t \geq 0$ we have*

$$A_{s+t}^{[n]} = \sum_{k=0}^{n} C_n^k A_s^{[k]} A_t^{[n-k]}(\theta_s) \;\; \text{if} \;\; s \geq 0 \quad , \quad \widetilde{A}_{s+t}^{[n]} = \sum_{k=0}^{n} C_n^k \widetilde{A}_{s-}^{[k]} \widetilde{A}_t^{[n-k]}(\theta_s) \;\; \text{if} \;\; s > 0.$$

Proposition 2.2. *For every $n \in \mathbb{N}^*$ and $t > 0$ we have*

$${}^c\!A_t^{[n]} = {}^c\!\widetilde{A}_t^{[n]} = ({}^c\!A_t)^n ,$$

$${}^d\!A_t^{[n]} = n! \sum_{0 \leq s_1 < s_2 < \ldots < s_n < t} \Delta A_{s_1} \Delta A_{s_2} \cdot \ldots \cdot \Delta A_{s_n} \quad , \quad A_t^{[n]} = \sum_{k=0}^{n} C_n^k \, {}^c\!A_t^{[k]} \, {}^d\!A_t^{[n-k]},$$

$$\widetilde{A}_t^{[n]} = \sum_{k=0}^{n} C_n^k \, {}^c\!A_t^{[k]} \, {}^d\!\widetilde{A}_t^{[n-k]} \quad , \quad {}^d\!\widetilde{A}_t^{[n]} = n! \sum_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq t} \Delta A_{s_1} \Delta A_{s_2} \cdot \ldots \cdot \Delta A_{s_n}.$$

Stieltjes exponentials

In the sequel we shall consider the Stieltjes exponentials of the positive left continuous additive functional $A = (A_t)_{t \geq 0}$, corresponding to the two compensated nth powers $A^{[n]}$ and $\widetilde{A}^{[n]}$ respectively (Stieltjes exponentials for *right continuous* additive functionals have been considered in [Yi 97] and [StumSt 00]). The functionals $\operatorname{Exp}(A) = (\operatorname{Exp}(A)_t)_{t \geq 0}$ and $\widetilde{\operatorname{Exp}}(A) = (\widetilde{\operatorname{Exp}}(A)_t)_{t \geq 0}$, $\operatorname{Exp}(A)_t$, $\widetilde{\operatorname{Exp}}(A)_t : \Omega_A \to [0, \infty]$ are defined by

$$\operatorname{Exp}(A)_t(\omega) := \sum_{n=0}^{\infty} \frac{1}{n!} A_t^{[n]}(\omega) \quad , \quad \widetilde{\operatorname{Exp}}(A)_t(\omega) := \sum_{n=0}^{\infty} \frac{1}{n!} \widetilde{A}_t^{[n]}(\omega).$$

Clearly, for $\omega \in \Omega_A$ the functionals $t \longmapsto \operatorname{Exp}(A)_t(\omega)$ and $t \longmapsto \widetilde{\operatorname{Exp}}(A)_t(\omega)$ are increasing, and $\operatorname{Exp}(A)_0(\omega) = 1$, $\widetilde{\operatorname{Exp}}(A)_0(\omega) = \sum_{n=0}^{\infty} a(X_0(\omega))^n \geq 1$. By Proposition 2.1 we obtain

$$\operatorname{Exp}(A)_{s+t} = \operatorname{Exp}(A)_s \cdot \operatorname{Exp}(A)_t \circ \theta_s, \quad \widetilde{\operatorname{Exp}}(A)_{s+t} = \widetilde{\operatorname{Exp}}(A)_{s-} \cdot \widetilde{\operatorname{Exp}}(A)_t \circ \theta_s$$

with the convention $\widetilde{\operatorname{Exp}}(A)_{0-} = 1$. From Proposition 2.2 we get

$$\operatorname{Exp}({}^c\!A)_t = e^{{}^c\!A_t}$$

and since

$$\operatorname{Exp}(A)_t = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} C_n^k \, {}^c\!A_t^{[k]} \, {}^d\!A_t^{[l]} = \Big(\sum_{k=0}^{\infty} \frac{1}{k!} {}^c\!A_t^{[k]}\Big) \sum_{l=0}^{\infty} \frac{1}{l!} {}^d\!A_t^{[l]} = \operatorname{Exp}({}^c\!A)_t \cdot \operatorname{Exp}({}^d\!A)_t$$

we get

$$\operatorname{Exp}(A)_t = e^{{}^c\!A_t} \operatorname{Exp}({}^d\!A)_t$$

and analogously

$$\widetilde{\operatorname{Exp}}(A)_t = e^{{}^c\!A_t} \widetilde{\operatorname{Exp}}({}^d\!A)_t.$$

Proposition 2.3. *For $t > 0$ we have*

$$\operatorname{Exp}(A)_t = e^{{}^c\!A_t} \prod_{0 \leq s < t} (1 + \Delta A_s) \quad , \quad \frac{1}{\operatorname{Exp}(A)_{t+}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \widetilde{A}_t^{[n]},$$

and particularly $d\left(\frac{1}{\mathrm{Exp}(A)_{t+}}\right) = -\frac{1}{\mathrm{Exp}(A)_{t+}} dA_t$. *If* $\omega \in \Omega_A$ *is such that* $\widetilde{\mathrm{Exp}}(A)_t(\omega) < \infty$ *then*

$$\widetilde{\mathrm{Exp}}(A)_t(\omega) = e^{{}^c\!A_t(\omega)} \frac{1}{\prod\limits_{0 \le s \le t} (1 - \Delta A_s(\omega))}.$$

Corollary 2.4. *For every real number* $p > 1$ *and* $t > 0$ *we have* $(\widetilde{\mathrm{Exp}}(A)_t)^p \le \widetilde{\mathrm{Exp}}(pA)_t$.

Proof. Assume that $\omega \in \Omega_A$ is such that $\widetilde{\mathrm{Exp}}(pA)_t(\omega) < \infty$. In this case we get $p\Delta A_t(\omega) < 1$ and the assertion follows from Proposition 2.3 and since

$$0 < x \quad , \quad px < 1 \Longrightarrow \frac{1}{(1-x)^p} \le \frac{1}{1-px}.$$

□

3 Feynman-Kac formula for PLAFs

In this section we assume, as in Section 2, that the process X is transient.

Let $A = (A_t)_{t \ge 0}$ be a PLAF of X. If $q \ge 0$ we consider the kernel U^q_A on $\left(E \backslash N_A, \mathcal{B}^n \mid_{E \backslash N_A}\right)$ defined by

$$U^q_A f(x) = E^x \int_{[0,\zeta)} e^{-qt} f \circ X_t dA_t, \quad x \in E \backslash N_A.$$

The kernel $U_A = U^\circ_A$ is called the *potential kernel of* A.

The *Revuz measure* of A with respect to m is the σ-finite measure on $(E, \mathcal{B})$ defined by

$$\nu_A(M) := \sup\{\mu(U_A(1_M))/\mu \circ U \le m\},\ M \in \mathcal{B}.$$

One can show that (cf. [BeBo 04]):
- The Revuz measure ν_A of A is a smooth measure.
- Every smooth measure is the Revuz measure of a PLAF.

Let $B = (B_t)_{t \ge 0}$ be a second PLAF of X. In the sequel, considering the restriction of X to $E \backslash (N_A \cup N_B)$, we may assume that $N_A = N_B = \emptyset$. It is known (see Theorem 6.5.8 in [BeBo 04]) that for each $q \ge 0$ the kernel U^q_A is regular strongly supermedian (with respect to $\mathcal{U}_q$) and there exists a sub-Markovian resolvent of kernels $\mathcal{V}^q = (V^q_\alpha)_{\alpha > 0}$ on $(E, \mathcal{B}^n)$ having U^q_A as initial kernel (i.e., $U^q_A = \sup\limits_{\alpha > 0} V^q_\alpha$).

For each $\alpha > 0$ we consider the following kernels on $(E, \mathcal{B}^n)$:

$$W^{q,\alpha}_B = U^q_B - \alpha V^q_\alpha U^q_B \quad , \quad W^{q,\alpha} = U_q - \alpha V^q_\alpha U_q,$$

$$W'^{q,\alpha}_B = \sum_{n=0}^{\infty} (W^{q,\alpha}_B)^n W^{q,\alpha}, \quad \widetilde{W'^q_B} = \sum_{n=0}^{\infty} (U^q_B)^n U_q.$$

Proposition 3.1. *If* $q \ge 0$ *and* $\alpha > 0$ *then the following assertions hold.*

i) $U_A = U^q_A + qU_qU_A$.

ii)

$$V^q_\alpha f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f \circ X_t\, dA_t, \quad W^{q,\alpha}_B f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f \circ X_t\, dB_t.$$

iii) Assume further that the jump moments of A *and* B *are disjoint a.s., i.e.,* P^x*-a.s. we have*

$$\inf(\Delta A_t, \Delta B_t) = 0, \quad t \ge 0.$$

Then for all $n \ge 1$ *we have*

$$(W^{q,\alpha}_B)^n f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}\, \widetilde{B}^{[n-1]}_t}{(n-1)!\, \mathrm{Exp}(\alpha A)_{t+}} f \circ X_t\, dB_t = E^x \int_{[0,\zeta)} \frac{e^{-qt}\, \widetilde{B}^{[n-1]}_t}{(n-1)!\, \mathrm{Exp}(\alpha A)_t} f \circ X_t\, dB_t,$$

$$(U_A^q)^n f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}\,\widetilde{A}_t^{[n-1]}}{(n-1)!} f\circ X_t\, dA_t.$$

iv) $W_B'^{q,\alpha} f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}\,\widetilde{\mathrm{Exp}}(B)_t}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t dt = E^x \int_{[0,\zeta)} \frac{e^{-qt}\,\widetilde{\mathrm{Exp}}(B)_t}{\mathrm{Exp}(\alpha A)_t} f\circ X_t dt.$
In particular we have

$$\widetilde{W_B'^q} f(x) = E^x \int_{[0,\zeta)} e^{-qt}\widetilde{\mathrm{Exp}}(B)_t\, f\circ X_t dt.$$

v) With the notation $W_B'^q := W_B'^{q,1}$ *we have*

$$W_B'^q + U_A^q W_B'^q = U_q + U_B^q W_B'^q \le \widetilde{W_B'^q}.$$

Proof. i) and *ii)*. Let R_α^q be the right hand side of the first equality of *ii)*. If $f \in p\mathcal{B}$ is such that $U_B^q f < \infty$, then using also Proposition 2.3 we have

$$R_\alpha^q U_B^q f(x) = E^x \left(\int_{[0,\zeta)} e^{-qt} \frac{1}{\mathrm{Exp}(\alpha A)_{t+}} E^{X_t} \left(\int_{[0,\zeta)} e^{-qs}\, f\circ X_s\, dB_s \right) dA_t \right) =$$

$$E^x \int_{[0,\zeta)} \frac{1}{\mathrm{Exp}(\alpha A)_{t+}} \left(\int_{[t,\zeta)} e^{-qs}\, f\circ X_s\, dB_s \right) dA_t = E^x \int_{[0,\zeta)} e^{-qs}\, f\circ X_s \left(\int_{[0,s]} \frac{1}{\mathrm{Exp}(\alpha A)_{t+}} dA_t \right) dB_s =$$

$$\frac{1}{\alpha} E^x \int_{[0,\zeta)} e^{-qs}\, f\circ X_s \left(1 - \frac{1}{\mathrm{Exp}(\alpha A)_{s+}} \right) dB_s = \frac{1}{\alpha} \left(U_B^q f(x) - E^x \int_{[0,\zeta)} e^{-qs} \frac{1}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_s\, dB_s \right).$$

Taking $B = A$ we get $U_A^q = R_\alpha^q + \alpha R_\alpha^q U_A^q$ and therefore $(R_\alpha^q)_{\alpha>0}$ is a sub-Markovian resolvent of kernels having U_A^q as initial kernel, i.e., $R_\alpha^q = V_\alpha^q$ for all $\alpha > 0$. Consequently assertion *ii)* holds. Assertion *i)* is a consequence of *ii)*, letting $\alpha \to 0$ and using the equality $E^x \int_{[0,\zeta)} e^{-qt} f\circ X_t dA_t = U_A f(x) - qU_q U_A f(x)$.

iii) The second equality is a particular case of the first one. To prove this one we shall proceed by induction. By *ii)* the assertion holds for $n = 1$. If we assume that it holds for n then we have

$$(W_B^{q,\alpha})^{n+1} f(x) = E^x (\int_{[0,\zeta)} \frac{e^{-qt}\widetilde{B}_t^{[n-1]}}{(n-1)!\,\mathrm{Exp}(\alpha A)_{t+}} E^{X_t} (\int_{[0,\zeta)} \frac{e^{-qs}}{\mathrm{Exp}(\alpha A)_{s+}} f\circ X_s\, dB_s)\, dB_t) =$$

$$E^x (\int_{[0,\zeta)} \frac{\widetilde{B}_t^{[n-1]}\mathrm{Exp}(\alpha A)_t}{(n-1)!\,\mathrm{Exp}(\alpha A)_{t+}} (\int_{[t,\zeta)} \frac{e^{-qs}}{\mathrm{Exp}(\alpha A)_{s+}} f\circ X_s\, dB_s)\, dB_t) =$$

$$E^x (\int_{[0,\zeta)} \frac{e^{-qs}}{\mathrm{Exp}(\alpha A)_{s+}} f\circ X_s (\int_{[0,s]} \frac{\mathrm{Exp}(\alpha A)_t}{(n-1)!\,\mathrm{Exp}(\alpha A)_{t+}} \widetilde{B}_t^{[n-1]} dB_t)\, dB_s) =$$

$$E^x \int_{[0,\zeta)} \frac{e^{-qs}}{n!\,\mathrm{Exp}(\alpha A)_{s+}} f\circ X_s\, \widetilde{B}_s^{[n]} dB_s.$$

iv) By *iii)* we get $\sum_{n=1}^{\infty} (W_B^{q,\alpha})^n f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(\alpha A)_{s+}} f\circ X_s\, dB_s$ and so

$$\sum_{n=1}^{\infty} (W_B^{q,\alpha})^n W^{q,\alpha} f(x) = E^x (\int_{[0,\zeta)} \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(\alpha A)_{s+}} E^{X_s} (\int_{[0,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t dt)\, dB_s) =$$

$$E^x (\int_{[0,\zeta)} \widetilde{\mathrm{Exp}}(B)_s (\int_{[s,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t dt)\, dB_s) =$$

$$E^x (\int_{[0,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t (\int_{[0,t]} \widetilde{\mathrm{Exp}}(B)_s\, dB_s) dt) = E^x \int_{[0,\zeta)} \frac{e^{-qt}}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t\, (\widetilde{\mathrm{Exp}}(B)_t - 1) dt.$$

It follows that

$$W_B'^{q,\alpha} f(x) = W^{q,\alpha} f(x) + \sum_{n=1}^{\infty} (W_B^{q,\alpha})^n W^{q,\alpha} f(x) = E^x \int_{[0,\zeta)} \frac{e^{-qt}\,\widetilde{\mathrm{Exp}}(B)_t}{\mathrm{Exp}(\alpha A)_{t+}} f\circ X_t dt.$$

$v)$ We have

$$U^q_A W'^q_B f(x) = E^x(\int_{[0,\zeta)} e^{-qt} E^{X_t}(\int_{[0,\zeta)} \frac{e^{-qs}\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}} f\circ X_s\, ds)\, dA_t) =$$

$$E^x(\int_{[0,\zeta)} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}(\int_{[t,\zeta)} \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}} f\circ X_s\, ds)\, dA_t) =$$

$$E^x(\int_{[0,\zeta)} \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}} f\circ X_s(\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dA_t) ds),$$

$$W'^q_B f(x) + U^A_q W'^q_B f(x) = E^x(\int_{[0,\zeta)} \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}} f\circ X_s(1+\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dA_t) ds) \le$$

$$E^x \int_{[0,\zeta)} e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s\, f\circ X_s\, ds = \widetilde{W'^q_B} f(x).$$

Further we get

$$U^q_B W'^q_B f(x) = E^x(\int_{[0,\zeta)} f\circ X_s \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}}(\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dB_t)\, ds)$$

and therefore

$$U_q f(x) + U^q_B W'^q_B f(x) = E^x(\int_{[0,\zeta)} e^{-qs} f\circ X_s(1+\frac{\widetilde{\mathrm{Exp}}(B)_s}{\mathrm{Exp}(A)_{s+}}\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dB_t)\, ds).$$

From $d\,\mathrm{Exp}(A)_t = \mathrm{Exp}(A)_t\, dA_t,\ d(\frac{1}{\widetilde{\mathrm{Exp}}(B)_t}) = \frac{-1}{\widetilde{\mathrm{Exp}}(B)_t}\, dB_t$ it follows that

$$-\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dB_t + \int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dA_t = \int_{[0,s]} d(\frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}) = \frac{\mathrm{Exp}(A)_s}{\widetilde{\mathrm{Exp}}(B)_{s-}} - 1.$$

We conclude that

$$U_q f(x) + U^q_B W'^q_B f(x) = E^x(\int_{[0,\zeta)} f\circ X_s \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_{s-}}{\mathrm{Exp}(A)_s}(\frac{\mathrm{Exp}(A)_s}{\widetilde{\mathrm{Exp}}(B)s-} + \int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dB_t) ds) =$$

$$E^x(\int_{[0,\zeta)} f\circ X_s \frac{e^{-qs}\,\widetilde{\mathrm{Exp}}(B)_{s-}}{\mathrm{Exp}(A)_s}(1+\int_{[0,s]} \frac{\mathrm{Exp}(A)_t}{\widetilde{\mathrm{Exp}}(B)_{t-}}\, dA_t) ds) = W'^q_B f(x) + U^q_A W'^q_B f(x).$$

□

Remark. For the first equality of assertion $ii)$ in Proposition 3.1 see also Theorem 7.3 in [FiGe 03].

The next result is a "Khas'minskii Lemma" for Stieltjes exponentials of positive left additive functionals.

Proposition 3.2. *The following assertion hold for $q > 0$.*
$i)$ We have $\widehat{c_q}(\nu_A) = \inf\{\alpha > 0 / U^q_A 1 \le \alpha$ m-q.e.$\}$.
$ii)$ If $\widehat{c}_q(\nu_A) \le \gamma < 1$ then the following inequalities hold m-q.e. (in x) on E for each $t > 0$:

$$E^x(\widetilde{A}^{[n]}_t) \le n!\,\gamma^n e^{qt} \quad \textit{for all}\, n\in\mathbb{N}^*, \quad E^x(\widetilde{\mathrm{Exp}}(A)_t) \le \frac{e^{qt}}{1-\gamma}.$$

Proof. $i)$ By Proposition 2.2 in [BeBo 05] and assertion $i)$ of Proposition 3.1 we have $V^q_\mu = U^q_A$.
$ii)$ Since $U^q_A 1 \le \gamma$ m-q.e. we deduce inductively that we have $(U^q_A)^n 1 \le \gamma^n$ m-q.e. and therefore by Proposition 3.1 $iii)$ we get m-q.e. (in x)

$$E^x(\widetilde{A}^{[n+1]}_t) = (n+1)E^x\int_{[0,t]} \widetilde{A}^{[n]}_s dA_s \le (n+1)e^{qt}E^x\int_{[0,t]} e^{-qs}\widetilde{A}^{[n]}_s dA_s \le$$

$$(n+1)!\, e^{qt}(U^q_A)^{n+1}1 \le (n+1)!\,\gamma^{n+1}\, e^{qt}.$$

Consequently the second inequality of assertion $ii)$ also holds. □

We can present now the perturbed semigroup defined by a *Feynman-Kac formula*. For each $t \geq 0$ we define the kernel Q_t on $(E, \mathcal{B}^u)$ by

$$Q_t f(x) := E^x\Big(\frac{\widetilde{\mathrm{Exp}}(B)_{t-}}{\mathrm{Exp}(A)_t} f \circ X_t\Big), \quad f \in p\mathcal{B}^u, \quad x \in E,$$

where recall that $\widetilde{\mathrm{Exp}}(B)_{0-} = 1$. Notice that by Proposition 3.1 *iv)* we have for every $f \in \mathcal{B}^u$ and $q > 0$:

$$W'^q_B f = \int_0^\infty e^{-qt} Q_t f \, dt.$$

Proposition 3.3. *The following assertions hold.*

i) The family $(Q_t)_{t\geq 0}$ is a semigroup of kernels on $(E, \mathcal{B}^u)$.

ii) Assume that q_0 is such that $\widehat{c}_{q_0}(\nu_B) < 1$ and let $p_0 > 1$ be such that $\gamma_0 := \dfrac{p_0}{p_0 - 1}\widehat{c}_{q_0}(\nu_B) < 1$. Then for each $p \in [p_0, \infty]$, $t > 0$, $q > q_0$, the kernels Q_t, W'^q_B and $\widetilde{W}'^q_B$ are bounded linear operators on $L^p(E, m)$ and

$$\|Q_t\|_{L^p \to L^p} \leq \frac{e^{q_0 t}}{1 - \gamma_0}, \quad \|W'^q_B\|_{L^p \to L^p} \leq \|\widetilde{W}'^q_B\|_{L^p \to L^p} \leq \frac{1}{(1 - \gamma_0)(q - q_0)}.$$

If $p \in [p_0, \infty)$ and $f \in L^p$ then $\lim_{t \to 0} \|Q_t f - f\|_p = 0$, $\lim_{q \to \infty} \|q W'^q_B f - f\|_p = 0$.

Proof. *i)* The semigroup property follows since the functional $t \longmapsto N_t := \frac{\widetilde{\mathrm{Exp}}(B)_{t-}}{\mathrm{Exp}(A)_t}$ is a multiplicative functional:

$$Q_t(Q_s f)(x) = E^x(N_t \cdot E^{X_t}(N_s \cdot f \circ X_s)) = E^x(N_t \cdot N_s \circ \theta_t \cdot f \circ X_{t+s} = E^x(N_{t+s} \cdot f \circ X_{t+s}) = Q_{t+s} f(x).$$

ii) If $p < \infty$ and $p' > 1$ is such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ then by Corollary 2.4 we get $(\widetilde{\mathrm{Exp}}(B)_{t-})^{p'} \leq \widetilde{\mathrm{Exp}}(p'B)_{t-}$ and we have also $\widehat{c}_{q_0}(\nu_{p'B}) = p'\widehat{c}_{q_0}(\nu_B) < 1$. Therefore by Proposition 3.2 *ii)* we obtain

$$E^x((\widetilde{\mathrm{Exp}}(B)_{t-})^{p'}) \leq \frac{e^{q_0 t}}{1 - \gamma_0} \quad m\text{-q.e. (in } x).$$

Hence if $f \in p\mathcal{B}^m \cap L^p(E, m)$ and $t > 0$ then

$$|Q_t f(x)|^p \leq |E^x(\widetilde{\mathrm{Exp}}(B)_{t-} \cdot f \circ X_t)|^p \leq E^x(|f|^p \circ X_t) \cdot E^x((\widetilde{\mathrm{Exp}}(B)_{t-})^{p'})^{p-1} \leq P_t(|f|^p)(x) \cdot (\frac{e^{q_0 t}}{1 - \gamma_0})^{p-1}.$$

It follows that if $f = 0$ m-a.e. then $Q_t f = 0$ m-a.e. We conclude that if $f \in L^p(E, m)$ then $Q_t f \in L^p(E, m)$ and

$$\|Q_t f\|_p \leq \frac{e^{q_0 t}}{1 - \gamma_0} \|f\|_p.$$

The case $p = \infty$ follows by Proposition 3.2 *ii)*.

By assertion *iv)* of Proposition 3.1 it follows that the family $(\widetilde{W'^q}_B)_{q>1}$ is dominated by the resolvent of kernels associated with the semigroup $(Q^\circ_t)_{t\geq 0}$ (where $(Q^\circ_t)_{t\geq 0}$ is $(Q_t)_{t\geq 0}$ in the case $A = 0$) and by *v)* we have $W'^q_B \leq \widetilde{W}'^q_B$. Consequently we get

$$\|W'^q_B\|_{L^p \to L^p} \leq \|\widetilde{W}'^q_B\|_{L^p \to L^p} \leq \int_0^\infty e^{-qt} \|Q^\circ_t\|_{L^p \to L^p} dt \leq \frac{1}{1 - \gamma_0} \int_0^\infty e^{-qt + q_0 t} dt = \frac{1}{(1 - \gamma_0)(q - q_0)}.$$

If $p \in [p_0, \infty)$ then

$$|Q_t f(x) - P_t f(x)| \leq E^x\left(|\frac{\widetilde{\mathrm{Exp}}(B)_{t-}}{\mathrm{Exp}(A)_t} - 1| \cdot |f| \circ X_t \right) \leq$$

$$E^x\left((\widetilde{\mathrm{Exp}}(B)_{t-} - \frac{\widetilde{\mathrm{Exp}}(B)_{t-}}{\mathrm{Exp}(A)_t}) |f| \circ X_t \right) + E^x\left((\widetilde{\mathrm{Exp}}(B)_{t-} - 1) |f| \circ X_t \right),$$

$$E^x\left(\widetilde{\mathrm{Exp}}(B)_{t-}(1-\frac{1}{\mathrm{Exp}(A)_t})|f|\circ X_t\right)\le (E^x((\widetilde{\mathrm{Exp}}(B)_{t-})^{p'}))^{1/p'}(P_t(|f|^p)(x))^{1/p}\le (\frac{e^{q_0t}}{1-\gamma_0})^{1/p'}(P_t(|f|^p)(x))^{1/p},$$

$$E^x((\widetilde{\mathrm{Exp}}(B)_{t-}-1)|f|\circ X_t)\le (E^x((\widetilde{\mathrm{Exp}}(B)_{t-}-1)^{p'}))^{1/p'}\cdot(P_t(|f|^p)(x))^{1/p}\le (\frac{e^{q_0t}}{1-\gamma_0})^{1/p'}\cdot(P_t(|f|^p)(x))^{1/p}.$$

Since we have m-a.e. (in x)

$$\lim_{t\to 0}E^x(\widetilde{\mathrm{Exp}}(p'B)_{t-}(1-\frac{1}{\mathrm{Exp}(A)_t})^{p'})=0,\quad \lim_{t\to 0}E^x((\widetilde{\mathrm{Exp}}(B)_{t-}-1)^{p'})=0$$

it follows that $\lim_{t\to 0}\int |Q_tf-P_tf|^p dm=0$ and because $\|Q_tf-f\|_p\le \|Q_tf-P_tf\|_p+\|P_tf-f\|_p$, $\lim_{t\to 0}\|P_tf-f\|_p=0$, we deduce that $\lim_{t\to 0}\|Q_tf-f\|_p=0$. From $W'^q_Bf=\int_0^\infty e^{-qt}Q_tfdt$ we conclude that $\lim_{q\to\infty}\|qW'^q_Bf-f\|_p=0$. □

The strongly continuous semigroup of bounded operators on $L^p(E,m)$ given by Proposition 3.3 is called *Feynman-Kac semigroup*.

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