## MAT 341 – Applied Real Analysis Spring 2015

**Final** – May 18, 2015

Please turn off your cell phone and put it away. You are  ${f NOT}$  allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	SCORE
1	
2	
3	
4	
5	
6	
7	
TOTAL	

**Problem 1:** (12 points) Consider the function

$$f(x) = \begin{cases} 0 & \text{if } -3 \le x < -1, \\ 1 & \text{if } -1 \le x < 1, \\ 0 & \text{if } 1 \le x < 3; \end{cases} \qquad f(x+6) = f(x).$$

a) Sketch the graph of f on the interval [-7, 7].

b) Find the Fourier series for f. Explain why the series converges to 0.5 when x = 7.

**Problem 2:** (14 points) Suppose  $u(x,t) = e^{-\lambda t}X(x)$  is a nontrivial solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad t > 0;$$
$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad u(a, t) = 0, \quad t > 0.$$

Find an ordinary differential equation that is satisfied by X(x). What initial conditions must X(x) satisfy? Determine X(x) and the possible values of  $\lambda$ .

**Problem 3:** (14 points) Consider the following eigenvalue problem

$$(e^{-x}\phi')' + e^{x}\gamma^{2}\phi = 0, \quad 0 < x < a$$
  
 $\phi(0) + \beta^{2}\phi'(0) = 0, \quad \phi(a) + \beta^{2}\phi'(a) = 0$ 

Check true or false, no other explanation is necessary.

- a) This is a regular Sturm-Liouville problem for all values of the parameter  $\beta$ . True False
- b) If  $\beta = 0$  and  $\phi_1, \phi_2, \phi_3, \dots$  are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x) \, dx = 0.$$

TRUE FALSE

c) If  $\beta = 0$  and  $\phi_1, \phi_2, \phi_3, \dots$  are eigenfunctions of this problem then

$$\int_0^a \phi_2(x)\phi_4(x)e^x \, dx = 0.$$

TRUE FALSE

d) If  $\beta = 0$  and  $\phi_1, \phi_2, \phi_3, \dots$  are eigenfunctions of this problem then

$$\int_0^a \phi_m(x)\phi_n(x)e^x dx = 0.$$

TRUE FALSE

e) If  $\beta = 4$  and  $\phi_1, \phi_2, \phi_3, \dots$  are eigenfunctions of this problem then

$$\int_0^a \phi_3(x)\phi_5(x)e^x \, dx = 0.$$

TRUE FALSE

f)  $\gamma = 0$  is not an eigenvalue, regardless of the parameter  $\beta$ .

TRUE FALSE

g) If  $\beta = 0$  and  $\phi_1, \phi_2, \phi_3, \ldots$  are eigenfunctions of this problem then  $\sum_{n=1}^{\infty} c_n \phi_n(x) = e^{-x}$ , for 0 < x < a, where

$$c_n = \frac{\int_0^a \phi_n(x) \, dx}{\int_0^a \phi_n^2(x) e^x \, dx}.$$

TRUE FALSE

Problem 4: (20 points) Consider the dispersive wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \gamma^2 u, \qquad 0 < x < a, \quad t > 0$$

subject to the following boundary conditions and initial conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(a,t) = 0, \qquad t > 0; \qquad \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < a.$$

a) Set  $u(x,t) = \phi(x)T(t)$  to separate the variables and find the associated equations for  $\phi$  and T. Solve these equations and show that the solution u(x,t) can be written as

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(t\sqrt{\frac{n^2\pi^2}{a^2} + \gamma^2}\right) \cos\left(\frac{n\pi}{a}x\right).$$

## (Problem 4 continued)

b) Find the formula for the coefficients  $c_n$  using f(x).

c) By using trigonometric identities, rewrite the solution as

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} c_n \left[ \cos \left( \frac{n\pi}{a} (x - b_n t) \right) + \cos \left( \frac{n\pi}{a} (x + b_n t) \right) \right]$$

Determine  $b_n$ , the speed of wave propagation. When is  $b_n$  independent of n?

**Problem 5:** (10 points) Consider the potential equation in a vertical strip 0 < x < a, 2 < y:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \qquad 2 < y;$$
  
$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 2 < y;$$

We know that the bounded solutions to this equation are given by

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{n\pi y}{a}\right).$$

Find the coefficients  $c_n$  if in addition  $u(x,2) = 1, \ 0 < x < a$ . Is the solution periodic in y? What happens to u(x,y) and  $\frac{\partial u}{\partial x}(x,y)$  as  $y \to \infty$ ?

**Problem 6:** (12 points) We know that the following functions  $v(r, \theta)$ :

1, 
$$r^n \cos(n\theta)$$
,  $r^{-n} \cos(n\theta)$ ,  $r^n \sin(n\theta)$ ,  $r^{-n} \sin(n\theta)$ , (where  $n = 1, 2, ...$ )

are all solutions to the Laplace equation in polar coordinates:  $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$ . Let  $v_1$  and  $v_2$  be any two functions from the given list.

- a) Any combination  $c_1v_1 + c_2v_2$  is a solution of the Laplace equation. Check true or false, no other explanation is necessary: True False
- b) Consider the Laplace equation on the disk  $0 \le r < 2$ . Which of the listed functions would you try for a bounded solution  $v = c_1v_1 + c_2v_2$ ? You are asked to list possible values for  $v_1$  and  $v_2$ , not to find  $c_1$  and  $c_2$ .

c) Find coefficients  $c_1$  and  $c_2$  such that  $v = c_1v_1 + c_2v_2$  is a solution of the Laplace equation on the disk  $0 \le r < 2$ , subject to the boundary condition  $v(2, \theta) = \cos(3\theta)$ ,  $-\pi \le \theta < \pi$ .

**Problem 7:** (18 points) Find the solution u(x,y) of Laplace's equation  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  in the rectangle  $0 < x < \pi$ , 0 < y < 1, that satisfies the boundary conditions

$$\begin{split} \frac{\partial u}{\partial x}(0,y) &= 0, \qquad \frac{\partial u}{\partial x}(\pi,y) = 0, & 0 < y < 1, \\ u(x,0) &= 0, \qquad u(x,1) = 1 + \cos(5x), & 0 < x < \pi. \end{split}$$

Some useful formulas & trigonometric identities:

$$\int x \cos(ax) \, dx = \frac{\cos(ax)}{a^2} + \frac{x \sin(ax)}{a} + C$$

$$\int x \sin(ax) \, dx = \frac{\sin(ax)}{a^2} - \frac{x \cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin(ax) \sin(bx) = \frac{\cos((a - b)x) - \cos((a + b)x)}{2}$$

$$\sin(ax) \cos(bx) = \frac{\sin((a - b)x) + \sin((a + b)x)}{2}$$

$$\cos(ax) \cos(bx) = \frac{\cos((a - b)x) + \cos((a + b)x)}{2}$$