MAT 341 – Applied Real Analysis
Spring 2015

Midterm 2 – Solutions – April 16, 2015

NAME: ______________________________

Please turn off your cell phone and put it away. You are **NOT** allowed to use a calculator.

Please show your work! To receive full credit, you must explain your reasoning and neatly write the steps which led you to your final answer. If you need extra space, you can use the other side of each page.

Academic integrity is expected of all students of Stony Brook University at all times, whether in the presence or absence of members of the faculty.

PROBLEM	**SCORE**
1	
2	
3	
4	
5	
TOTAL	

Problem 1: Consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial t}$$

on the interval $0 < x < 2$, with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 10, \quad u(2,t) = 100, \qquad \text{for all } t > 0.$$

a) (8 points) What is the *steady-state* temperature distribution?

SOLUTION. The steady-state solution $v(x)$ satisfies $v''(x) = 0$ so $v(x) = Ax + B$. From $v'(0) = 10$ and $v(2) = 100$ we find $A = 10$ and $B = 80$. So $v(x) = 10x + 80$. □

b) (12 points) Find all the product solutions $w(x,t) = \phi_n(x)T_n(t)$ that satisfy the PDE and the boundary conditions for the *transient* solution. You are **NOT** asked to find the general solution!

SOLUTION. The transient solution $w(x,t)$ satisfies $w_{xx} = 4w_t$ and $w_x(0,t) = 0$ and $w(2,t) = 0$. We write $w(x) = \phi(x)T(t)$ and get $\phi''T = 4\phi T'$. The boundary conditions are $\phi'(0) = 0$ and $\phi(2) = 0$. Separating the variables we write $\frac{\phi''}{\phi} = \frac{4T'}{T} = -\lambda^2$, so $\phi'' + \lambda^2\phi = 0$ and $T' + \frac{1}{4}\lambda^2 T = 0$. The second equation gives $T(t) = e^{-\frac{\lambda^2}{4}t}$. The first equation gives $\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$. From $\phi'(0) = 0$ we find $c_2 = 0$, so $\phi(x) = c_1\cos(\lambda x)$. From $\phi(2) = 0$ we find $\cos(2\lambda) = 0$ so $\lambda = \frac{(2n-1)\pi}{4}$, for $n = 1, 2, \ldots$

The product solutions are

$$w(x,t) = \phi_n(x)T_n(t) = \cos\left(\frac{(2n-1)\pi}{4}x\right)e^{-\frac{(2n-1)^2\pi^2}{64}t},$$

for $n = 1, 2, \ldots$ □

Problem 2: (20 points) Find the Fourier integral representation of the function $f(x)$ given below:

$$f(x) = \begin{cases} \pi & \text{if } \ 0 < x < 1 \\ 0 & \quad \text{otherwise.} \end{cases}$$

Solution. The Fourier integral representation of the function $f(x)$ is

$$\int_0^\infty [A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]\, d\lambda,$$

where

$$A(\lambda) = \frac{1}{\pi}\int_0^\infty f(x)\cos(\lambda x)\, dx = \frac{1}{\pi}\int_0^1 \pi\cos(\lambda x)\, dx = \int_0^1 \cos(\lambda x)\, dx = \frac{\sin(\lambda)}{\lambda},$$

and

$$B(\lambda) = \frac{1}{\pi}\int_0^\infty f(x)\sin(\lambda x)\, dx = \frac{1}{\pi}\int_0^1 \pi\sin(\lambda x)\, dx = \int_0^1 \sin(\lambda x)\, dx = \frac{1-\cos(\lambda)}{\lambda}.$$

Putting everything together we find that

$$f(x) = \int_0^\infty \left[\frac{\sin(\lambda)}{\lambda}\cos(\lambda x) + \frac{1-\cos(\lambda)}{\lambda}\sin(\lambda x)\right] d\lambda.$$

□

Problem 3: (20 points) Consider the heat conduction problem in a metal rod of semi-infinite length that is insulated on the sides:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$
$$u(0,t) = 0, \qquad t > 0,$$

whose initial temperature distribution is $u(x,0) = f(x)$ for $0 < x < \infty$, where

$$f(x) = \begin{cases} 1 & \text{if } \quad 0 < x < 1 \\ 0 & \quad\text{otherwise.} \end{cases}$$

Find the temperature $u(x,t)$ if we further assume that $u(x,t)$ remains finite as $x \to \infty$.

Solution. In this problem the constant k is 1. The general solution of this PDE is given by

$$u(x,t) = \int_0^\infty B(\lambda)\sin(\lambda x)e^{-\lambda^2 t}\,d\lambda,$$

where

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)\,dx = \frac{2}{\pi}\int_0^1 \sin(\lambda x)\,dx = \frac{2}{\pi}\frac{(-\cos(\lambda x))}{\lambda}\bigg|_0^1 = \frac{2(1-\cos(\lambda))}{\pi\lambda}.$$

Therefore the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\lambda)}{\lambda}\sin(\lambda x)e^{-\lambda^2 t}\,d\lambda.$$

□

Problem 4:

a) (10 points) Find the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the problem:

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) = 0$$

SOLUTION. If $\lambda = 0$ then $\phi(x) = Ax + B$, but $\phi'(1) = A = 0$ and $\phi(0) = B = 0$. It follows that $\lambda = 0$ is not an eigenvalue. We get that $\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$ is the general solution of this ODE. From $\phi(0) = 0$ we find that $c_1 = 0$. From $\phi'(1) = c_2\lambda\cos(\lambda) = 0$ we find that $\cos(\lambda) = 0$ so $\lambda = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, \ldots$ The eigenvalues are $\lambda_n = \frac{(2n-1)\pi}{2}$, while the eigenfunctions are $\phi_n(x) = \sin(\lambda_n x)$, for $n = 1, 2, \ldots$ □

b) (10 points) Find the expression of the function $f(x) = x$, $0 < x < 1$ in terms of these eigenfunctions. Does this series converge at $x = 1$?

SOLUTION. We write $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$, where

$$c_n = \frac{\int_0^1 \phi_n(x)f(x)\,dx}{\int_0^1 \phi_n^2(x)\,dx} = \frac{\int_0^1 x\sin\left(\frac{(2n-1)\pi}{2}x\right)\,dx}{\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right)\,dx}.$$

Using the formulas at the end of the exam we compute

$$\int_0^1 x\sin\left(\frac{(2n-1)\pi}{2}x\right)\,dx = \frac{\sin\left(\frac{(2n-1)\pi}{2}\right)}{\frac{\pi^2}{4}(2n-1)^2} = \frac{4}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}$$

and

$$\int_0^1 \sin^2\left(\frac{(2n-1)\pi}{2}x\right)\,dx = \left.\frac{1-\cos((2n-1)\pi x)}{2}\right|_0^1 = \frac{1}{2}$$

It follows that for $0 < x < 1$ we have

$$x = \sum_{n=1}^{\infty}\frac{8}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi}{2}x\right).$$

When $x = 1$ the sum becomes $\sum_{n=1}^{\infty}\frac{8}{\pi^2}\frac{1}{(2n-1)^2} < \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges. □

Problem 5: (20 points) Solve the vibrating string problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 1, \qquad t > 0;$$
$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0;$$
$$u(x,0) = \sin(3\pi x), \qquad 0 < x < 1;$$
$$\frac{\partial u}{\partial t}(x,0) = \sin(5\pi x), \qquad 0 < x < 1.$$

Explain why $u(x,t+1) = u(x,t)$, which means that the solution to this problem is a function that is periodic in time of period 1.

SOLUTION. In this problem $a = 1$ and $c = 2$. The general solution to this PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(2n\pi t) + b_n \sin(2n\pi t)] \sin(n\pi x).$$

We check the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sin(3\pi x),$$

so $a_3 = 1$ and $a_n = 0$ otherwise. From

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n 2n\pi \sin(n\pi x) = \sin(5\pi x),$$

we find $10\pi b_5 = 1$ and so $b_5 = \frac{1}{10\pi}$. The remaining b_n are all zeros. The solution to this problem is

$$u(x,t) = \cos(6\pi t)\sin(3\pi x) + \frac{1}{10\pi}\sin(10\pi t)\sin(5\pi x).$$

Clearly $u(x,t+1) = u(x,t)$ since $\cos(6\pi t)$ and $\sin(10\pi t)$ are both periodic of period 1. □

Some useful formulas & trigonometric identities:

$$\int x\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(ax)\sin(bx) = \frac{\cos((a-b)x) - \cos((a+b)x)}{2}$$

$$\sin(ax)\cos(bx) = \frac{\sin((a-b)x) + \sin((a+b)x)}{2}$$

$$\cos(ax)\cos(bx) = \frac{\cos((a-b)x) + \cos((a+b)x)}{2}$$