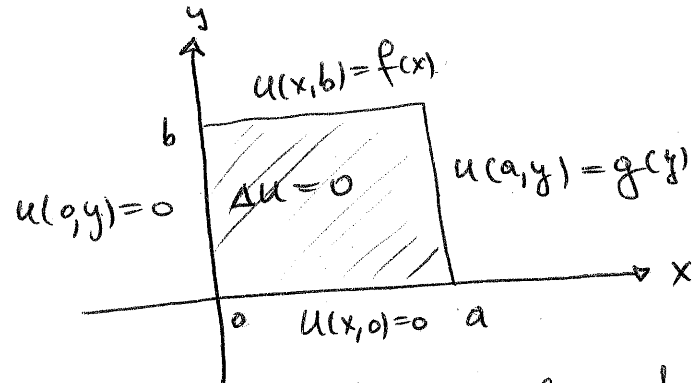
4.3. Potential in a rectangle

Example 1: $\Delta u = u_{xx} + u_{yy} = 0, \quad 0<x<a, \; 0<y<b$

$u(0,y) = 0, \; u(a,y) = g(y), \quad 0<y<b$

$u(x,0) = 0, \; u(x,b) = f(x), \quad 0<x<a$



Remark: We need to split this problem into two problems if we want to use separation of variables.

PDE1: $\Delta u_1 = 0$

$\boxed{u_1(0,y) = 0, \; u_1(a,y) = 0}$ homogeneous vertical boundary

$u_1(x,0) = 0, \; u_1(x,b) = f(x)$

PDE2: $\Delta u_2 = 0$

$u_2(0,y) = 0, \; u_2(a,y) = g(y)$

$\boxed{u_2(x,0) = 0, \; u_2(x,b) = 0}$ homogeneous horizontal boundary

We then have $u(x,y) = u_1(x,y) + u_2(x,y)$.

We solve PDE1 using separation of variables: $u_1(x,y) = X(x)\,Y(y)$ and get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

Also $X(0) = 0, \; X(a) = 0, \; Y(0) = 0$.

From $X'' + \lambda^2 X = 0$ we find $X(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$

$X(0) = 0 \Rightarrow C_2 = 0, \; X(a) = 0 \Rightarrow \sin(\lambda a) = 0$ or $\lambda = \frac{n\pi}{a}, \; n = 1, 2, \dots$

$$X_n(x) = \sin(\lambda_n x), \quad \lambda_n = \frac{n\pi}{a}.$$

$Y'' - \lambda^2 Y = 0$ gives $Y(y) = C_1 \sinh(\lambda y) + C_2 \cosh(\lambda y)$

$Y(0) = 0$ so $C_2 = 0$ and $Y(y) = \sinh(\lambda y)$

$Y_n(y) = \sinh(\lambda_n y)$ and we have found

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

From $u_1(x,b) = f(x)$ we get $\displaystyle\sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi b}{a}\right)}_{b_n} \sin\left(\frac{n\pi x}{a}\right) = f(x)$

Set $b_n = c_n \sinh\left(\frac{n\pi b}{a}\right)$ a constant.

Then $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = f(x)$ so $b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

and $c_n = \dfrac{b_n}{\sinh\left(\frac{n\pi b}{a}\right)}$. The solution of PDE1 is given by:

$$u_1(x,y) = \sum_{n=1}^{\infty} b_n \frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right), \quad b_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Using a similar strategy we solve <u>PDE2</u> to find $u_2(x,y)$:

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \sinh(\lambda_n x) \sin(\lambda_n y), \quad \lambda_n = \frac{n\pi}{b}, \; n=1,2,\ldots$$

<u>Remark</u>: We swap $x \longleftrightarrow y$ in PDE1 to get the solution for PDE2.

$a \longleftrightarrow b$

$f \longleftrightarrow g$

So one needs to pay attention on constants.

$$u_2(a,y) = g(y) = \sum_{n=1}^{\infty} \underbrace{c_n \sinh(\lambda_n a)}_{a_n} \sin(\lambda_n y)$$

$a_n = c_n \sinh(\lambda_n a)$ so $\displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{b}\right) = g(y)$ gives $a_n = \frac{2}{b}\int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy$

by Fourier series

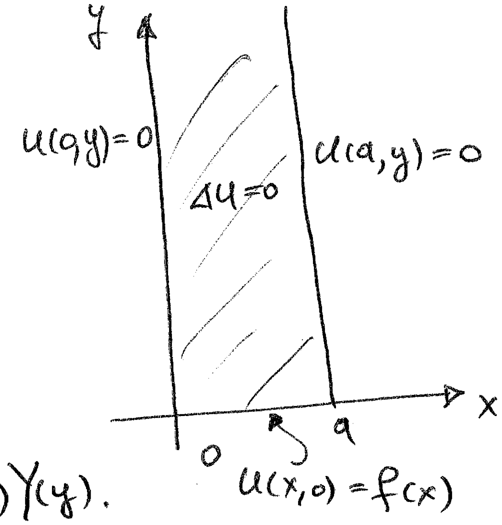
and $c_n = \dfrac{a_n}{\sinh\left(\frac{n\pi a}{b}\right)}$

The general solution of PDE2 is:

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi y}{b}\right), \text{ where}$$

$$a_n = \frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right) dy.$$

4.4. Potential in unbounded regions



Example 2: $\Delta u = 0, \quad 0<x<a, \; y>0$

$u(x,0) = f(x), \quad 0<x<a$

$u(0,y) = 0, \; u(a,y) = 0, \quad 0<y$

$u(x,y)$ bounded as $y \to \infty$

we set up separation of variables $u(x,y) = X(x)Y(y)$.

Then $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda^2$ (we need to set it to $-\lambda^2$, not λ^2, because otherwise $X(x)$ will become ~~unbounded~~ zero.)

$X(0) = X(a) = 0$
$X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

We find $X_n(x) = \sin(\lambda_n x)$, $\lambda_n = \frac{n\pi}{a}$, $n = 1, 2, \ldots$

and $Y(y) = C_1 e^{-\lambda y} + C_2 e^{\lambda y}$ (It is more convenient to use this notation instead of $C_1 \sinh(\lambda y) + C_2\cosh(\lambda y)$.)

$Y(y)$ bounded as $y \to \infty$ means that $C_2 = 0$ so $Y(y) = C_1 e^{-\lambda y}$. We found $Y_n(y) = e^{-\lambda_n y}$, $\lambda_n = \frac{n\pi}{a}$.

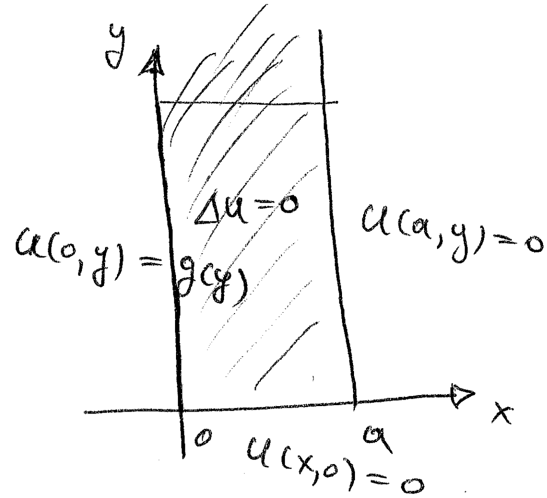
Putting all together we find:

$$u(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{n\pi}{a}y}$$

From $u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)$ we get $c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

<u>Example 3:</u>

$$\Delta u = 0, \quad 0 < x < a, \; 0 < y$$
$$u(x,0) = 0, \quad 0 < x < a$$
$$u(0,y) = g(y), \quad 0 < y$$
$$u(a,y) = 0, \quad 0 < y$$
$u(x,y)$ bounded as $y \to \infty$



$u(x,y) = X(x)\,Y(y)$ so $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda^2$

and:
$X(a) = 0$
$Y(0) = 0$

(here we set it equal to λ^2 because if $-\dfrac{Y''}{Y} = -\lambda^2$ then $Y'' - \lambda^2 Y = 0$ and $Y(0) = 0$ will give
$Y(y) = c_1 e^{\lambda y} + c_2 e^{-\lambda y}$, $c_2 = -c_1$
$Y(y) = c_1(e^{\lambda y} - e^{-\lambda y})$ but $Y(y)$ is bounded as $y \to \infty$ so $c_1 = 0$ and $Y \equiv 0$)

we find $\begin{cases} Y'' + \lambda^2 Y = 0 \\ X'' - \lambda^2 X = 0. \end{cases}$

So $Y(y) = c_1 \cos(\lambda y) + c_2 \sin(\lambda y)$, $Y(0) = 0$ gives $c_1 = 0$

$Y(y) = \sin(\lambda y)$ (can take $c_1 = 1$ at this step)

$X(x) = c_1 e^{-\lambda x} + c_2 e^{\lambda x}$ or

$X(x) = A \sinh(\lambda x) + B \cosh(\lambda x)$ ← this is a more convenient notation in this problem.

$X(a) = 0$ gives $A \sinh(\lambda a) + B \cosh(\lambda a) = 0$ and $A = -B \dfrac{\cosh(\lambda a)}{\sinh(\lambda a)}$

so $X(x) = B\left(-\dfrac{\cosh(\lambda a)\sinh(\lambda x)}{\sinh(\lambda a)} + \cosh(\lambda x)\right)$

$= B\left(\dfrac{\cosh(\lambda x)\sinh(\lambda a) - \sinh(\lambda x)\cosh(\lambda a)}{\sinh(\lambda a)}\right)$

$= B\left(\dfrac{\sinh((a-x)\lambda)}{\sinh(\lambda a)}\right)$ | we have used the identity: $\sinh(a \pm b) = \sinh a \cosh b \pm \sinh b \cosh a$

The constant B can depend on λ.

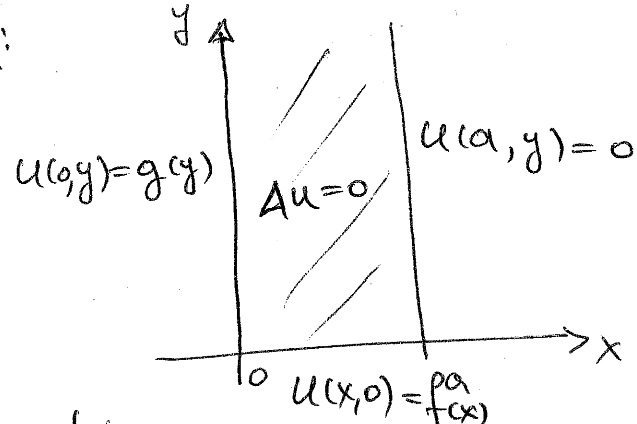
So

$$u(x,y) = \int_0^\infty B(\lambda)\,\frac{\sinh(a-x)\lambda}{\sinh(\lambda a)}\,\sin(\lambda y)\,d\lambda$$

From $u(0,y) = g(y)$ we find $u(0,y) = \int_0^\infty B(\lambda)\sin(\lambda y)\,d\lambda$

so $B(\lambda) = \frac{2}{\pi}\int_0^\infty g(y)\sin(\lambda y)\,dy$. (Review ch. 2.10 and 1.9).

Example 4:



the solution to this PDE is Example 2 + Example 3.

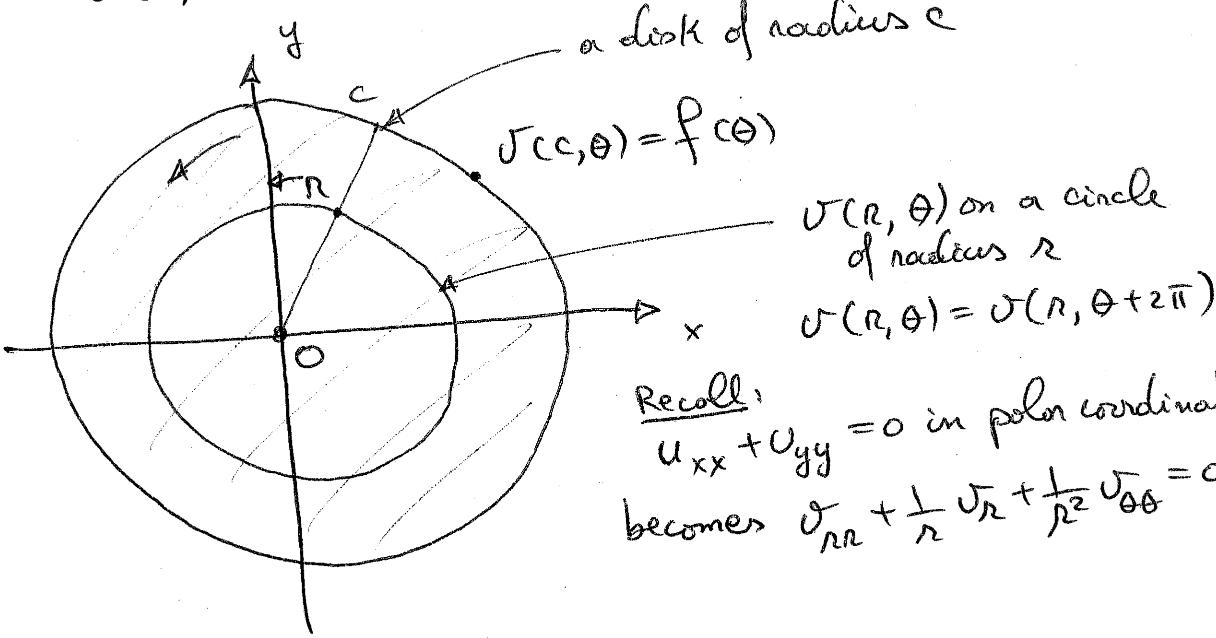
4.5. Potential in a disk

Example 5: $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0, \quad 0 \le r < c$

$v(c,\theta) = f(\theta) \quad , -\pi < \theta < \pi$

$v(r, \theta + 2\pi) = v(r,\theta) \quad , 0 < r < c$

$v(r,\theta)$ is bounded as $r \to 0$.



Recall:

$u_{xx} + u_{yy} = 0$ in polar coordinates

becomes $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0$.

Set $v(r,\theta) = R(r)\Theta(\theta)$ so

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \text{ gives}$$

$$\left(R'' + \frac{1}{r}R'\right)\Theta = -\frac{1}{r^2}R\Theta'' \quad \text{or} \quad \frac{R'' + \frac{1}{r}R'}{\frac{1}{r^2}R} = -\frac{\Theta''}{\Theta}$$

or $\dfrac{r^2R'' + rR'}{R} = -\dfrac{\Theta''}{\Theta} = \lambda^2$ (if we set $= -\lambda^2$ then Θ would be exponential so not periodic; we need Θ to be periodic)

$\Theta'' + \lambda^2\Theta = 0$, $\Theta(\theta + 2\pi) = \Theta(\theta)$

so $\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$ if this is periodic of period 2π then λ is an integer, so $\underline{\lambda = n}$, $n = 0, 1, \ldots$

if $\lambda = 0$ we get $\Theta_0(\theta) = $ constant so we pick $\Theta_0(\theta) = 1$.

if $\lambda = n$, for $n = 1, 2\ldots$ $\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$, $n = 1, 2\ldots$

The equation for R is $r^2R'' + rR' - n^2R = 0$, $n = 0, 1, 2\ldots$

There are 2 cases:

① if $n = 0$ then $r^2R'' + rR' = 0$ so $\dfrac{R''}{R'} = -\dfrac{1}{r}$ and so $R' = \dfrac{1}{r}$ or $R = \ln(r)$
if $R' \neq 0$

If $R' = 0$ then $R = $ constant so we take $R = 1$. Note that $R = \ln(r)$ does not work since $\lim_{r \to 0} R(r) = -\infty$.

$R_0(r) = 1$.

② $n \neq 0$. This is a Cauchy-Euler equation which cannot be solved by a characteristic equation. We know (and not prove) that the general solution is $R(r) = C_1 r^n + C_2 r^{-n}$. Now, since $R(r)$ is bounded as $r \to 0$ we must have $C_2 = 0$.

$R_n(r) = r^n$.

The fundamental solutions for this problem are

$$1, \quad r^n\cos(n\theta), \quad r^n\sin(n\theta)$$

or 1 and $A_n r^n\cos(n\theta) + B_n r^n(\sin(n\theta))$ but they are the same.

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} A_n r^n\cos(n\theta) + B_n r^n\sin(n\theta)$$

from the initial condition $v(c,\theta)=f(\theta)$ we find:

$$v(c,\theta)=a_0+\sum_{n=1}^{\infty} A_n c^n \cos(n\theta)+B_n c^n \sin(n\theta)=f(\theta)$$

$$\text{so}\quad a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$A_n c^n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \quad \text{so} \quad A_n=\frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$B_n c^n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \quad \text{so} \quad B_n=\frac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

using Fourier series.