Master Thesis

ENTROPY PRODUCTION OF HYPERBOLIC MAPS

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Contents

Introduction

The study of nonequilibrium statistical mechanics leads naturally to the introduction of nonequilibrium states. These are probability measures μ on the phase space of the system, suitably chosen and stationary under the nonequilibrium time evolution. In the present paper we analyze the entropy production $e(\mu)$ for such nonequilibrium states.

In short, our physical problem is to pump entropy out of the system, while keeping the energy fixed. For this, we fix our mathematical setup to a smooth dynamical system (M, f). In order to study our problem, we will first give some preliminary results.

In the first chapter, we present some basic facts concerning the ergodic properties of our system. We give the definition and some useful properties of both metric and topological entropy as given by Walters in [13]. Also, the topological pressure is introduced, along with the Variational Principle. Moreover, another useful tool from ergodic theory, the Jacobian of an invariant measure of an endomorphism, is defined following the presentation from [8].

The second chapter presents some basic facts about hyperbolic dynamics: the notion of hyperbolicity of a homeomorphism and the way this notion is extended to the more complicated, endomorphism case. Also, in order to study some important properties of SRB measures, defined in the third chapter, Lyapunov exponents are introduced. Furthermore, we mention the Livshitz Theorem, which proves to be very important in the last chapter.

Next, the notions of SRB and physical measures are introduced in order to describe the distribution of forward iterates in a neighbourhood of an attractor. Again, we study separately the classical invertible case, and the non-invertible one. Moreover, by similarity to the classical forward SRB measure, one natural question is the study the distribution of various preimages near a hyperbolic repellor, and for this the inverse SRB measures are defined as in [6].

The last chapter deals with the initially stated problem, of studying the entropy production of invariant measures for certain maps. We will see that in the classic case, the positivity of the entropy production is obtained, whereas in the non-ivertible case, there are examples of negative entropy production.

Chapter 1

Ergodic properties

1.1 Measure theoretic facts

Let $(X, \mathcal{B}, \mu)$ be a measure space. We will mainly work with Lebesgue spaces, i.e. it contains at most a countable set of points with positive measure.

Any collection ζ of non-empty, disjoint sets that cover X is said to be a partition of X. The subsets of X, which are sums of elements of a partition ζ, are called $\zeta-$sets.

Definition 1.1.1. *A countable system $\{B_\alpha, \alpha \in A\}$ of measurable $\zeta-$sets is said to be a basis for ζ if for any two elements C and C' of ζ, there exists an $\alpha \in A$ such that either $C \subset B_\alpha$ and $C' \not\subset B_\alpha$ or $C \not\subset B_\alpha$ and $C' \subset B_\alpha$.*

In the following, we will denote by $\hat{\zeta}$ the $\sigma-$algebra generated by the measurable partition ζ. We wish to use ζ to construct a new Lebesgue space $(X_\zeta, \mathcal{B}_\zeta, \mu_\zeta)$. We simply take $X_\zeta = \zeta$. Let $H_\zeta : X \to X_\zeta$ be a map which takes x to $\zeta(x)$, then put $\mathcal{B}_\zeta = H_\zeta \hat{\zeta}$. Define μ_ζ on $\mathcal{B}_\zeta$ by the formula

$$\mu_\zeta(B') = \mu(H_\zeta^{-1} B').$$

Since $(X, \mathcal{B}, \mu)$ is a Lebesgue space we also have that $(X_\zeta, \mathcal{B}_\zeta, \mu_\zeta)$ is a Lebesgue space, called the factor space of $(X, \mathcal{B}, \mu)$ with respect to ζ.

For almost all $C \in \zeta$ there is a $\sigma-$algebra $\mathcal{B}_C$ and a measure μ_C defined on $\mathcal{B}_C$ so that $(C, \mathcal{B}_C, \mu_C)$ is a Lebesgue space and

(i) if $B \in \mathcal{B}$ then $B \cap C \in \mathcal{B}_C$ for almost all C.

(ii) $\mu_C(B \cap C)$ is $\mathcal{B}_\zeta-$measurable if $B \in \mathcal{B}$.

(iii) for $B \in \mathcal{B}$,

$$\mu(B) = \int_{X_\zeta} \mu_C(B \cap C) d\mu_\zeta.$$

It can be shown that $(C, \mathcal{B}_C, \mu_C)$ are uniquely defined mod $0[\mu_\zeta]$. The system of measures so defined is called a canonical system with respect to ζ. Also, we can notice that $\{\mu_C\}$ are the conditional measures with respect to ζ.

1.2 Entropy

1.2.1 Metric entropy

Definition 1.2.1. *Let $\zeta = \{A_1, A_2, \ldots, A_k\}$ and $\xi = \{C_1, C_2, \ldots, C_p\}$ be two finite partitions. Their join is the partition $\zeta \vee \xi = \{A_i \cap C_j; 1 \leq i \leq k, 1 \leq j \leq p\}$*

If $\mathcal{A}$ and $\mathcal{C}$ are finite sub$-\sigma-$algebras of $\mathcal{B}$, then $\mathcal{A} \vee \mathcal{C}$ denotes the smallest sub$-si-$algebra of $\mathcal{B}$ containing $\mathcal{A}$ and $\mathcal{C}$.

We start by defining the conditional entropy of two partitions, following the presentation of Walters [13]. It is not required in order to define the entropy of a transformation, but it is useful in deriving its properties.

Definition 1.2.2. *The entropy of ζ, given ξ is the number*

$$\begin{aligned} H(\zeta|\xi) &= -\sum_{j=1}^{n} \mu(C_j) \sum_{i=1}^{p} \frac{\mu(A_i \cap C_j)}{\mu(C_j)} \log \frac{\mu(A_i \cap C_j)}{\mu(C_j)} \\ &= -\sum_{i,j} \mu(A_i \cap C_j) \log \frac{\mu(A_i \cap C_j)}{\mu(C_j)}, \end{aligned}$$

omitting the $j-$terms when $\mu(C_j) = 0$.

Also, the conditional information of ζ, given a sub$-\sigma-$algebra $\mathcal{C}$, is defined by the formula

$$I(\zeta|\mathcal{C}) = -\sum_{A \in \zeta} \log \mu(A|\mathcal{C}),$$

where $\mu(A|\mathcal{C}) := E[\chi_A|\zeta]$.

Definition 1.2.3. *The entropy of the partition ζ is given by*

$$H(\zeta) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

We will proceed to define the entropy of a measure-preserving transformation.

Definition 1.2.4. *Suppose $f : X \to X$ is a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$. If ζ is a finite measurable partition of X, then*

$$h(f, \zeta) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\hat{\zeta}\right)$$

is called the entropy of f with respect to ζ and

$$h(f) = \sup h(f, \zeta),$$

where the supremum is taken over all finite measurable partitions of X, is called the entropy of f.

Note that $h(f, \zeta) \geq 0$ and $h(f) \geq 0$.

We know the entropy is well-defined due to the following result:

Theorem 1.2.5. *[13] If $f : X \to X$ is measure preserving and ζ a finite measurable partition of X, then $\frac{1}{n} H(\vee_{i=0}^{n-1} f^{-i}\hat{\zeta})$ decreases to $h(f, \zeta)$.*

Remark 1.2.6. *The definitions hold if we consider countable partitions.*

1.2.2 Topological entropy

In this section we will give the definition of topological entropy using separating and spanning sets. This definition was first introduced by Bowen, even for the non-compact case.

In the following (X, d) will be a metric space, not necessarily compact. Also, $f : X \to X$ will denote a fixed uniformly continuous mapping. If $n \geq 1$ we can define a new metric d_n on X by $d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y))$. The open ball, centered in x and with radius r in the metric d_n will be denoted by $B_n(x, r)$.

Definition 1.2.7. *Let n be a natural number, $\epsilon > 0$ and K a compact subset of X. A subset F of X is said to $(n, \epsilon)-$span K with respect to f if $\forall x \in K$, $\exists y \in F$ with $d_n(x, y) \leq \epsilon$.*

Also, let $r_n(\epsilon, K)$ denote the smallest cardinality of any $(n, \epsilon)-$spanning set for K with respect to f and $r(\epsilon, K, f) = \limsup \frac{1}{n} \log r_n(\epsilon, K)$.

Definition 1.2.8. *If K is a compact subset of X let $h(f,K) = \lim_{\epsilon\to 0} r(\epsilon, K, f)$. The topological entropy of f is $h_{top}(f) = \sup_K h(f,K)$, where the supremum is taken over the collection of all compact subsets of X.*

We will give now a similar definition, using separated sets.

Definition 1.2.9. *Let n be a natural number, $\epsilon > 0$ and K a compact subset of X. A subset E of K is said to be $(n,\epsilon)-$separated with respect to f if $\forall x, y \in X$, $x \neq y$ implies $d_n(x,y) > \epsilon$.*

Also, let $s_n(\epsilon, K)$ denote the largest cardinality of any $(n,\epsilon)-$separated subsetset of K with respect to f and $s(\epsilon, K, f) = \limsup \frac{1}{n} \log s_n(\epsilon, K)$.

Remark 1.2.10. *We have $r_n(\epsilon, K) \leq s_n(\epsilon, K) \leq r_n(\epsilon/2, K)$, and using the above definitions we conclude that $h_{top}(f)$ can be defined using either spanning or separated sets.*

The following result gives the link between metric and topological entropy.

Theorem 1.2.11. *[13] [Variational Principle] Let $f : X \to X$ be a continuous map of a compact metric space X. Then*

$$h_{top}(f) = \sup_{\mu \in M(X,f)} h_\mu(f).$$

1.3 Topological pressure and the Variational Principle

Definition 1.3.1. *The topological pressure* of f is the map $P : C(X,\mathbb{R}) \to \bar{\mathbb{R}}$ defined as follows:

$$P_f(\varphi) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup \left\{ \sum_{x\in E} \exp\left(\sum_{i=0}^{n-1} \varphi(f^i(x)) \right); E\ (n,\delta) - \text{separated} \right\}.$$

Next are some well-known properties of topological pressure.

Theorem 1.3.2. *[13] [Variational Principle]*

In the above setting, $P_f(\varphi) = \sup_\mu \{h_\mu(f) + \int \varphi d\mu\}$, where the supremum is taken over all the $f-$invariant Borel probability measures μ and $h_\mu(f)$ is the measure-theoretic entropy of μ.

A measure μ for which the above supremum is attained is called an equilibrium measure for φ.

Theorem 1.3.3 (Properties of Pressure)**.** *If $f : X \to X$ is a continuous transformation and $\varphi, \psi \in \mathcal{C}(X, \mathbb{R})$, then:*

(1) $\varphi \leq \psi \Longrightarrow P_f(\varphi) \leq P_f(\psi)$

(2) $P_f(\cdot)$ *is either finitely valued or constantly* ∞

(3) $P_f(\cdot)$ *is convex*

(4) for a strictly negative function φ, the map $t \longrightarrow P_f(t\varphi)$ is strictly decreasing

(5) If $f_i : X_i \to X_i$ is a continuous map of a compact metric space (X_i, d_i), $i = \overline{1, 2}$ and $\phi : X_1 \to X_2$ is a homeomorphism, then $P_{f_1}(\varphi) = P_{f_2}(\varphi \circ \phi^{-1})$.

1.4 The Jacobian of an invariant-measure

Let $(X, \mathcal{B}, \mu)$ a Lebesgue space. We denote by ε the point partition.

Consider now $f : X \to X$ a measure-preserving endomorphism. If we consider also the partition $\alpha = \{A_0, A_1, \ldots\}$, where $\mu(A_i) < \infty$, then $f^{-1}\varepsilon$ induces a measurable partitiom $f^{-1}\varepsilon \cap f^{-1}A_i$ on $f^{-1}A_i$ and we can obtain a canonical system of normalised measures with respect to $f^{-1}\varepsilon$ on each fibre $f^{-1}x \in f^{-1}\varepsilon$ of $f^{-1}A_i$. In fact, corresponding to any $\sigma-$finite sub$-\sigma-$algebra $\hat{\zeta}$ there will exist a measurable partition ζ whose fibres posses canonical measures. Consequently, $I(\varepsilon|f^{-1}\varepsilon)$ and $H(\varepsilon|f^{-1}\varepsilon)$ are well-defined.

If f is a countable-to-one endomorphism, we can use a theorem of Rokhlin to obtain a measurable partition $\alpha = \{A_0, A_1, \ldots\}$ such that f is one-to-one on each A_i.

We define the Jacobian J_f of a countable-to-one endomorphism to be $\frac{d\mu_f}{d\mu}$ on each set $A_i \in \alpha$. This is a well defined measurable function nowhere less than one. J_f is independent of the choice of partition α on whose elements f is one-to-one and $J_f \equiv 1$ iff f is an automorphism.

Proposition 1.4.1. *[8] If f is a countable-to-one endomorphism, then $I(\varepsilon|f^{-1}\varepsilon) = \log J_f$. $I(\varepsilon|f^{-1}\varepsilon)$ is finite almost everywhere iff f is countable-to-one.*

Chapter 2

Some facts about hyperbolic dynamics

2.1 Hyperbolic homeomorphisms

Let X be a metric space and $f : X \to X$ a homeomorphism. We define the $\epsilon-$stable and $\epsilon-$unstable sets of a point $x \in X$ as

$$W_\epsilon^s(x) := \{y; d(f^n(y), f^n(x)) \leq \epsilon, \ \forall n \geq 0\},$$

$$W_\epsilon^u(x) := \{y; d(f^{-n}(y), f^{-n}(x)) \leq \epsilon, \ \forall n \geq 0\}.$$

Definition 2.1.1. *The homeomorphism f of X is called hyperbolic if there exist $\epsilon_0 > 0$, $K > 0$ and $0 < \lambda < 1$ such that*

(a) $d(f^n(y), f^n(x)) \leq K\lambda^n$ *for all $x \in X$, $y \in W_{\epsilon_0}^s$ and $n \geq 0$.*

(b) $d(f^{-n}(y), f^{-n}(x)) \leq K\lambda^n$ *for all $x \in X$, $y \in W_{\epsilon_0}^u$ and $n \geq 0$.*

(c) *There exists $\delta > 0$ such that*

$$Card(W_{\epsilon_0}^s(x) \cap W_{\epsilon_0}^u(y)) = Card(W_{\epsilon_0}^s(y) \cap W_{\epsilon_0}^u(x)) = 1$$

for every $x, y \in X$ such that $d(x, y) \leq \delta$.

2.1.1 Examples

Example I *Hyperbolic automorphisms of the torus* $\mathbb{T}^n$,

the automorphisms $f : \mathbb{T}^n \to \mathbb{T}^n$ whose eigenvalues have absolute values different from 1.

We can prove the hyperbolicity of f considering its lift $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$. Let E^s, E^u be the subspaces of $\mathbb{R}^n$ invariant under $\tilde{f}$ corresponding to eigenvalues with absolute value less than and greater than one, respectively. Put

$$E^s_\epsilon := \{y : ||\tilde{f^n}(y)|| \leq \epsilon \; \forall n \geq 0\},$$

$$E^s_\epsilon := \{y : ||\tilde{f^n}(y)|| \leq \epsilon \; \forall n \leq 0\}.$$

Then, E^s_ϵ, respectively E^u_ϵ is a neighbourhood of 0 in E^s, respectively in E^u and it can be shown that for small ϵ,

$$W^s_\epsilon = \pi(p + E^s_\epsilon),$$

$$W^u_\epsilon = \pi(p + E^u_\epsilon),$$

where p is any point in $\pi^{-1}(x)$, and π is the projection map from $\mathbb{R}^n$ to $\mathbb{T}^n$. The first two properties from the definition of hyperbolicity are satisfied considering that the eigenvalues of $\tilde{f}|_{E^s}$, respectively of $\tilde{f}|_{E^u}$ are less than, respectively, greater that 1. The last property in the definition is a consequence of the fact that $E^s \oplus E^u = \mathbb{R}^n$, which implies that if $p, q \in \mathbb{R}^n$ are close enough,

$$Card((p + E^s) \cap (q + E^u)) = Card((p + E^s_\epsilon) \cap (q + E^u_\epsilon)) = 1,$$

so that

$$Card(W^s_\epsilon(\pi(p)) \cap W^u_\epsilon(\pi(q))) = 1.$$

Example II *Anosov diffeomorphisms*

A diffeomorphism f of a closed manifold M is called Anosov if there exists a direct sum decomposition of the tangent bundle T_xM at each point x into complementary subspaces E^s_x, E^u_x and also constants $K > 0$ and $0 < \lambda < 1$, satisfying

$$(D_xf)E^s_x = E^s_{f(x)},$$

$$(D_xf)E^u_x = E^u_{f(x)},$$

$$||(D_xf^n)E^s_x|| \leq K\lambda^n,$$

$$||(D_xf^{-n})E^u_x|| \leq K\lambda^n$$

for every $x \in M$ and $n \geq 0$.

In general, these sets can have a very complex structure. In the case of Anosov diffeomorphisms, however, for ϵ small enough, they are diffeomorphic to discs.

Theorem 2.1.2. *[4] If $f : M \to M$ is an Anosov diffeomorphism of class $\mathcal{C}^r$, there exists $\epsilon_0 > 0$ such that:*

(a) *For every $0 < \epsilon \leq \epsilon_0$ and every $x \in M$, the sets W^s_ϵ and W^u_ϵ are $\mathcal{C}^r$ diffeomorphic to discs, and $T_xW^s_\epsilon = E^s_x$, $T_xW^u_\epsilon = E^u_x$.*

(b) *For every $0 < \epsilon \leq \epsilon_0$ there exists $\delta = \delta(\epsilon)$ such that $Card(W^s_\epsilon(x) \cap W^u_\epsilon(y)) = 1$, for all $x, y \in M$ such that $d(x, y) \leq \delta$.*

Hyperbolicity of Anosov diffeomorphisms is implied by the previous result.

Example III *Hyperbolic sets*

Give a diffeomorphism of a manifold M, we say that Λ is a hyperbolic set if Λ is compact, $f(\Lambda) = \Lambda$ and there exist $K > 0$ and $0 < \lambda < 1$ such that at every point $x \in \Lambda$ there is a decomposition

$$T_xM = E^s_x \oplus E^u_x$$

such that

$$(D_xf)E^s_x = E^s_{f(x)},$$

$$(D_xf)E^u_x = E^u_{f(x)},$$

$$||(D_xf^n)E^s_x|| \leq K\lambda^n,$$

$$||(D_xf^{-n})E^u_x|| \leq K\lambda^n$$

for every $x \in M$ and $n \geq 0$. Observe that f is an Anosov diffeomorphism if the whole set M is hyperbolic. Put

$$\tilde{W}^s_\epsilon(x) := \{y \in M; d(f^n(x), f^n(y)) \leq \epsilon, \ \forall n \geq 0\},$$
$$\tilde{W}^u_\epsilon(x) := \{y \in M; d(f^{-n}(x), f^{-n}(y)) \leq \epsilon, \ \forall n \geq 0\}.$$

We observe that the $\epsilon-$stable and $\epsilon-$unstable sets W^s_ϵ, W^u_ϵ of the diffeomorphism $h|_\Lambda$ are exactly

$$W^s_\epsilon(x) = \tilde{W}^s_\epsilon(x) \cap \Lambda,$$
$$W^u_\epsilon(x) = \tilde{W}^u_\epsilon(x) \cap \Lambda.$$

For $\delta > 0$ small enough, two points $x, y \in \Lambda$ which are less than δ apart, have manifolds $\tilde{W}^u_\epsilon(x)$ and $\tilde{W}^s_\epsilon(x)$ which intersectin a single poitn, but this point need not belong to Λ. This fact motivates the following definition:

Definition 2.1.3. *A hyperbolic set Λ has a local product structure if there exists $\epsilon > 0$ such that $\tilde{W}^u_\epsilon(x) \cap \tilde{W}^s_\epsilon(x) \in \Lambda$ for all $x, y \in \Lambda$.*

2.1.2 Properties

Before giving our result, we need the following definition:

Definition 2.1.4. *Let X be a metric space and $f : X \to X$ a continuous map. Then f is sais to be topologically transitive on X if for every pair of non-empty open sets U and V in X, there is a non-negative integer n such that $f^n U \cap V \neq \emptyset$.*

Almost equivalently, we can say that f is topologically transitive on X, if there exists a point $x_0 \in X$ such that its orbit $orb(x_0) = \{f^n(x_0); n \in \mathbb{Z}\}$ is dense in X.

Theorem 2.1.5. *If Λ is a hyperbolic set with local product structure of a diffeomorphism f and the periodic points of $f|_\Lambda$ are dense in Λ, there exists a decomposition $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_k$ into disjoint invariant compact sets such that $f|_{\Lambda_i}$ is transitive for all $1 \leq i \leq k$.*

Let $\epsilon_0 > 0$, $\delta > 0$ be the constants given in the definition of a hyperbloic homeomorphism. Take $0 < \delta_1 < \min(\delta, \epsilon_0)$ such that the image under f or f^{-1} of stets with diameter smaller or equal to δ_1 has diameter smaller or equal to $\min(\frac{1}{2}\delta, \epsilon_0)$. We say that a set $R \subset X$ is a rectangle if $diam(R) \leq \delta_1$ and

$$W^s_{\epsilon_0}(x) \cap W^u_{\epsilon_0}(y) \in R$$

for all $x, y \in R$. Since $diam(R) \leq \delta_1 < \delta$, this intersection contains exactly one point. For $x \in R$, we put

$$W^s(x, R) = W^s_{\epsilon_0}(x) \cap R$$

and

$$W^u(x, R) = W^u_{\epsilon_0}(x) \cap R$$

A rectangle is proper if it is the closure of its interior.

Definition 2.1.6. *A family $\mathcal{R} = \{R_1, \ldots, R_m\}$ of closed proper rectangles is a Markov partition if it satisfies*

(a) $R_i \cap R_j \subset \partial R_i \cap \partial R_j$, $1 \leq i, j \leq m$;

(b) *If $x \in R_i$ and $f(x) \in Int(R_j)$ then*

$$fW^s(x, R_i) \subset W^s(fx, R_j)$$

(c) *If $x \in R_i$ and $f^{-1}(x) \in Int(R_j)$ then*

$$f^{-1}(W^u(x, R_i)) \subset W^u(fx, R_j)$$

The following theorem due to Bowen is very important in proving many results in the theory of hyperbolic homeomorphisms:

Theorem 2.1.7. *[1] Every transitive hyperbolic homeomorphism has a Markov partition.*

2.2 Hyperbolic endomorphisms

In this section we will deal with endomorphisms. This non-invertible case is very different from the diffeomorphic one. We begin by presenting some definitions.

Definition 2.2.1. *[7] Let (X, d) be a compact metric space and $f : X \to X$ a continuous map on X. Then the natural extension of X with respect to f is the space $\hat{X} := \{\hat{x}; \hat{x} = (x, x_{-1}, x_{-2}, \ldots),$ where $f(x_{-i}) = x_{-i+1}\}$. The shift map on $\hat{X}$ is $\hat{f} : \hat{X} \to \hat{X}$, defined by $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \ldots)$. The canonical projection map*

$\pi : \hat{X} \to X$ is given by $\pi(\hat{x}) = x$, $\hat{x} \in \hat{X}$. There exists also a natural metric on $\hat{X}$, for every $K > 1$, given by $d_K(\hat{x}, \hat{y}) = d(x, y) + \frac{d(x_{-1}, y_{-1})}{K} + \frac{d(x_{-2}, y_{-2})}{K^2} + \cdots$, if $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$ and $\hat{y} = (y, y_{-1}, y_{-2}, \ldots)$ belong to $\hat{X}$.

In the following, assume that Λ is a compact basic set for the smooth endomorphism f, i.e. $f|_\Lambda$ is topologically transitive and there exists a neighbourhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$. Let us notice that in the definition of hyperbolicity for non-invertible maps, the unstable tangent space $E^u_{\hat{x}}$ depends a priori on the whole prehistory $\hat{x}$ of x. One can also define local stable and local unstable manifolds, $W^s_\epsilon(x) := \{y \in X, d(f^i x, f^i y) < \epsilon, i \geq 0\}$ and $W^u_\epsilon(\hat{x}) := \{y \in X; y \text{ has a prehystory } \hat{y} = (y, y_{-1}, \ldots), \text{ with } d(y_{-i}, x_{-i}) < \epsilon, i \geq 0\}$, where $\hat{x} = (x, x_{-1}, \ldots) \in \hat{\Lambda}$ and $\epsilon > 0$ is some small positive number.

For a map f as above, define also the global stable set of a point $x \in \Lambda$ as the union $\bigcup_{n \geq 0} f^{-n} W^s_\epsilon(x)$ and denote it by $W^s(x)$. We also define the global unstable set of a prehistory $\hat{x} \in \hat{\Lambda}$ as $W^u(x) := \bigcup_{n \geq 0} f^n W^u_\epsilon(\hat{x})$. The global unstable set of Λ, $W^u(\hat{\Lambda})$ is defined as the union of all global unstable sets $W^u(\hat{x})$.

We also mention that if μ is an $f-$invariant probability measure on Λ, then there exists an unique $\hat{f}-$invariant probability measure $\hat{\mu}$ on $\hat{\Lambda}$ such that $\pi_*(\hat{\mu}) = \mu$. It can be seen that μ is ergodic if and only if $\hat{\mu}$ is ergodic on $\hat{\Lambda}$. Also, the topological pressure of ϕ is equal to the topological pressure of $\phi \circ \pi$; and μ is an equilibrium measure for $\phi : \Lambda \to \mathbb{R}$ if and only if $\hat{\mu}$ is an equilibrium measure for $\phi \circ \pi$.

2.3 Lyapunov exponents

We follow the presentation from [4] in order to give the definition and some properties concerning Lyapunov exponents.

Let f be a diffeomorphism of a compact manifold M. A point $x \in M$ is called regular for f if there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_m(x)$ and a decomposition

$$T_x M = E_1(x) \oplus \cdots \oplus E_m(x)$$

such that

$$\lim_{n \to \infty} \frac{1}{n} \log ||(D_x f^n) u|| = \lambda_j(x)$$

for every $0 \neq u \in E_j(x)$ and every $1 \leq j \leq m$. The numbers $\lambda_i(x)$ and the above decomposition are unique. $\lambda_i(x)$ are called the Lyapunov exponents and $E_i(x)$ the

Lyapunov eigenspaces of f at the regular point x.

Remark 2.3.1. *In general, regular points are few from the topological point of view - they form a set of the first category. But, the next result due to Oseledec, states that the situation from the ergodic point of view is exactly the opposite.*

Theorem 2.3.2. *(Oseledec) If M is compact and μ is $f-$invariant, then the set of regular points of a diffeomorphism $f : M \to M$ is a Borel set with total $\mu-$measure.*

Remark 2.3.3. *Since $Df(E_i(x)) = E_i(f(x))$, the functions $x \mapsto \lambda_i(x)$ and $\dim E_i(x)$ are constant $\mu-$a.e. if (f, μ) is ergodic, i.e. $f^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$.*

Now, we concentrate on results which relate entropy to Lyapunov exponents. If f is a diffeomorphism of a closed manifold M and Ω is the set of its regular points, we define a function $\chi : \Omega \to \mathbb{R}$ as

$$\chi(x) := \sum_{\lambda_j(x) \geq 0} \lambda_j(x) \dim E_j(x).$$

If all Lyapunov exponents at x are negative, we put $\chi(x) = 0$.

Theorem 2.3.4. **(a)** *(Ruelle) For $\mu \in \mathcal{M}_f(M)$ an $f-$invariant measure, we have*

$$h_\mu(f) \leq \int_M \chi d\mu,$$

where $h_\mu(f)$ is the metric entropy of f.

(b) *(Pesin) If f is Holder $\mathcal{C}^1$ and $\mu \in \mathcal{M}_f(M)$ is absolutely continuous with respect to the Lebesgue measure of M, then*

$$h_\mu(f) = \int_M \chi d\mu.$$

Remark 2.3.5. *The integrals above are well defined due to Oseledec's Theorem.*

2.4 Some facts about cohomology and Livshitz theory

The following results will be usefull in the last chapter of the paper.

Definition 2.4.1. *Let $f : X \to X$ be an endomorphism. We say that a continuous observable $\varphi : X \to \mathbb{R}$ is cohomologous to a constant if there exists $h : X \to \mathbb{R}$ continuous and $c \in \mathbb{R}$ such that $\varphi = h \circ f - h + c$.*

The main tool for obtaining global information from periodic data is the following result due to Livshitz:

Theorem 2.4.2. *[2] Let $f : M \to M$ be an endomorphims of a smooth manifold M, Λ a compact topologically transitive hyperbolic set and $\varphi : \Lambda \to \mathbb{R}$ Holder continuous. If $\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0$ whenever $f^n(x) = x$, then there is a function $\phi : \Lambda \to \mathbb{R}$, unique up to an additive constant, such that $\varphi = \phi \circ f - \phi$. Moreover, ϕ has the same holder exponent as φ.*

Chapter 3

SRB measures

3.1 SRB measures for diffeomorphisms

3.1.1 Existence of SRB measures: General results

SRB (Sinai-Ruelle-Bowen) measures have first been introduced for Axiom A attractors.

Definition 3.1.1. *Let $f : M \to M$ be a diffeomorphism of a compact Riemannian manifold onto itself. A compact $f-$invariant set $\Lambda \subset M$ is called an attractor if there is a neighbourhood U of Λ called its basin such that $f^n x \to \Lambda$ for every $x \in U$. Λ is called an Axiom A attractor if the tangent bundle over Λ is split into $E^u \oplus E^s$, where E^u and E^s are $Df-$invariant subspaces, $Df|_{E^u}$ is uniformly expanding and $Df|_{E^s}$ is uniformly contracting.*

Theorem 3.1.2. *[14] Let f be a $\mathcal{C}^2-$diffeomorphism having an Axiom A attractor Λ. Then there exists an unique $f-$invariant Borel probability measure μ on Λ that is characterized by each of the following equivalent conditions:*

(i) *μ has absolutely continuous conditional measures on unstable manifolds;*

(ii)

$$h_\mu(f) = \int |\det(Df|_{E^u})| d\mu$$

where $h_\mu(f)$ is the metric entropy of f;

(iii) there is a set $V \subset U$ having full Lebesgue measure such that for every continuous observable $\varphi : u \to \mathbb{R}$ we have, for every $x \in V$

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi d\mu.$$

The invariant measure μ from the previous theorem is called the SRB measure of f.

We will present now a more general point of view. Let $f : M \to M$ be a $\mathcal{C}^2$ diffeomorphism of a compact Riemannian manifold to itself.

We say that a measurable partition ζ of M is said to be subordinate to W^u if for μ-a.e. x, $\zeta(x)$, the element of ζ containing x, is contained in $W^u(x)$. We will be interested in the following two families of measures on the elements of ζ: $\{\mu_x^\zeta\}$, the conditional measures of μ, and m_x^ζ, The restriction of the Riemannian measure on W^u to $\zeta(x)$.

Definition 3.1.3. *An f invaiant Borel probability measure μ is said to have absolutely continuous conditional measures on unstable manifolds if (f, μ) has positive Lyapunov exponents a.e. and for every measurable partition ζ subordinate to W^u, we have $\mu_x^\zeta << m_x^\zeta$ for $\mu-$a.e. x.*

Theorem 3.1.4. *Let f be an arbitrary diffeomorphism and μ an $f-$invariant Borel probability measure with a positive Lyapunov exponent a.e. Then μ has absolutely continuous conditional measures on W^u if and only if*

$$h_\mu(f) = \int \sum_{\lambda_i > 0} \lambda_i \dim E_i d\mu.$$

Without the absolute continuity assumption on μ, "$=$" above is replaced by "$\leq$".

Definition 3.1.5. *Let f be a $\mathcal{C}^2$ diffeomorphism of a compact Riemannian manifold. An $f-$invariant Borel probability measure μ is called a SRB measure if (f, μ) has a positive Lyapunov exponent a.e. and μ has absolutely continuous conditional measures on unstable manifolds.*

Definition 3.1.6. *Let $f : M \to M$ be an arbitrary map and μ an invariant probability measure. We call μ a physical measure if there is a positive Lebesgue measure set $V \subset M$ such that for every continuous observable $\varphi : M \to \mathbb{R}$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi d\mu \tag{3.1}$$

for every $x \in V$.

We say a point $x \in M$ is $\mu-$generic if the time averages of continuous observables along the trajectory of x converge to their space averages with respect to μ, i.e. if (3.1) holds. Therefore a measure is physical if the set of its generic points has positive Lebesgue measure.

Definition 3.1.7. *[14] Let (f, μ) be ergodic and assume it has a negative Lyapunov exponent a.e. We say its W^s-foliation is absolutely continuous if the following holds: Let Σ and Σ' be embedded disks having complementary dimension to W^s and let $\{D^s_\alpha\}$ be a positive $\mu-$measure set of local stable manifolds such that each D^s_α meets both Σ and Σ' transversally. Let $\Phi : (\cup D^s_\alpha) \cap \Sigma \to \Sigma'$ be the holonomy map, and let m_Σ and $m_{\Sigma'}$ denote the Lebesgue measure on Σ and Σ' respectivelly. Then for $E \subset (\cup D^s_\alpha) \cap \Sigma$, $m_\Sigma(E) > 0$ if and only if $m_{\Sigma'}(\Phi(E)) > 0$.*

Theorem 3.1.8. *Assume (f, μ) has a negative Lyapunov exponent a.e. Then its W^s-foliation is absolutely continuous. It follows that every ergodic SRB measure with no zero Lyapunov exponents is a physical measure.*

Theorem 3.1.9. *Let $f : M \to M$ be a diffeomorphism, and suppose there is a positive Lebesgue measure set $R \subset M$ such that the following hold for every $x \in R$:*

(i) $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}$ *converges weakly to an ergodic measure which we will denote by μ_x;*

(ii) the Lyapunov exponents at x as $n \to \infty$ coincide with those of μ_x;

(iii) μ_x has no zero and at least one positive Lyapunov exponent.

Then μ_x is a SRB measure for Lebesgue-a.e. $x \in R$.

3.2 SRB measures for endomorphisms

In the case of endomorphisms, Qian and Zhang [10], gave a construction of the Pesin entropy formula for Axiom A endomorphisms, as follows:

Theorem 3.2.1. *[10] Let $f : M \to M$ be a $\mathcal{C}^2$ smooth endomorphism of a compact Riemannian manifold M and $\Lambda \in M$ be an Axiom A basic set. Then Λ is an*

Axiom A attractor if and only if there is an $f-$invariant probability measure μ on Λ satisfying Pesin's entropy formula:

$$h_\mu(f) = \int_\Lambda \sum_{\lambda_i(x_0)>0} \lambda_i(x_0)\mu(dx_0) \tag{3.2}$$

where $\lambda_1(x_0) \leq \ldots \leq \lambda_m(x_0)$, $(m = \dim(M))$, are the Lyapunov exponents of f at point x_0. And, in this case, we will have

(i) μ is ergodic;

(ii) if $\epsilon > 0$ is small enough and the set of critical points of f, C_f, has $0-$Lebesgue measure, then for Lebesgue-almost all $x_0 \in B(\Lambda, \epsilon)$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi \cdot f^i(x_0) = \int_\Lambda \varphi d\mu$$

for any continuous observable φ.

We now give a formula of the SRB property for invariant measures of $\mathcal{C}^2$ endomorphisms of a compact manifold via their inverse limit spaces and show that this property is necessary and sufficient for Pesin's entropy formula.

Definition 3.2.2. *[9] We say a measure $\mu \in \mathcal{M}(f)$ is called a SRB measure if for any measurable partition γ of $\hat{M}$, subordinate to the W^u-manifolds of (f, μ), we have that for $\hat{\mu}-$a.e. $\hat{x} \in \hat{\Lambda}$*

$$\pi(\hat{\mu}^\gamma_{\hat{x}}) << \lambda^u_{\hat{x}},$$

where $\{\hat{\mu}^\gamma_{\hat{x}}\}$ is a canonical system of conditional measures of $\hat{\mu}$ associated to γ, $\pi(\hat{\mu}^\gamma_{\hat{x}})$ is the projection of $\hat{\mu}^\gamma_{\hat{x}}$ under $\pi|_{\gamma(\hat{x})}$, iar $\lambda^u_{\hat{x}}$ is The Lebesgue measure of $W^u(\hat{x})$ induced by the Riemannian structure of M.

Theorem 3.2.3. *[9] Let f be a $\mathcal{C}^2$ endomorphism on M with an $f-$invariant Borel probability measure μ satisfying the integrability condition*

$$\log |D_x f| \in L^1(M, \mu).$$

Then the entropy formula (3.2) holds if and only if μ has the SRB property.

Corollary 3.2.4. *Let f be a $\mathcal{C}^2$ endomorphism on M with an invariant Borel probability measure μ. If μ is absolutely continuous with respect to the Lebesgue measure on M, then μ has the SRB property.*

The following result is a direct consequence of the previous two theorems:

Corollary 3.2.5. *Let Λ be an Axiom A attractor of f, and assume that $D_x f$ is non-degenerate for every $x \in \Lambda$. Then, there exists a unique $f-$invariant Borel probability measure μ on Λ which is characterized by each of the following properties:*

(a) μ has the SRB property;

(b) (f, μ) satisfies the entropy formula (3.2);

(c) when $\epsilon > 0$ is small enough, $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}$ converges to μ as $n \to \infty$ for Lebesgue almost every $x \in B(\Lambda, \epsilon)$.

3.3 Inverse SRB measures

3.3.1 Existence of inverse SRB measures

Definition 3.3.1. *Let $f : M \to M$ be a smooth map on a Riemannian manifold and let Λ be a compact set which is $f-$ invariant and such that $F|_\Lambda$ is topologically transitive. Also, assume there exists a neighbourhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$. Λ will be called basic set for f. We say that Λ is a repellor for f if Λ is a basic set for f, $C_f \cap \Lambda = \emptyset$ and if there exists a neighbourhood U of Λ such that $\bar{U} \subset f(U)$.*

Proposition 3.3.2. *In the setting of Definition 3.3.1, if Λ is a repellor for f, then $f^{-1}\Lambda \cap U = \Lambda$. If moreover Λ is assumed to be a connected, the number of $f-$preimages that a point has in Λ is constant.*

We call stable potential, the function

$$\phi^s(x) := \log|det(Df_s(x))|, \quad \forall x \in \Lambda.$$

In the sequel, given $y \in \Lambda$, $n \geq 1$ and $\epsilon > 0$, we denote by $B_n(y, \epsilon) := \{z \in M; d(f^i z, f^i y) < \epsilon, i = 0, \dots, n-1\}$ a Bowen ball. For a continuous real function ϕ defined on Λ and for a positive integer n , we define the consecutive sum by

$$S_n\phi(y) := \phi(y) + \dots + \phi(f^{n-1}y), y \in \Lambda.$$

Proposition 3.3.3. *[6] Let Λ be a hyperbolic basic set for a smooth endomorphism $f : M \to M$, and let ϕ be a Holder continuous function on Λ. Then there exists a* *unique equilibrium measure μ_ϕ for ϕ on Λ such that for any $\epsilon > 0$, there exist positiv constants A_ϵ, B_ϵ so that for any $y \in \Lambda$, $n \geq 1$,*

$$A_\epsilon e^{S_n\phi(y)-nP(\phi)} \leq \mu_\phi(B_n(y,\epsilon)) \leq B_\epsilon e^{S_n\phi(y)-nP(\phi)}.$$

Proof.

The shift $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ is an expansive homeomorphism. The existence of a unique equilibrium measure for the Holder potential $\phi \circ \pi$ with respect to the homeomorphism $\hat{f}$ follows fromthe classic theory of expansive homeomorphisms, and we will denote this measure by $\hat{\mu}_\phi$. We know that there exists a unique probability measure μ_ϕ with $\mu_\phi := \pi_* \hat{\mu}_\phi$, and μ_ϕ is the unique equilibrium measure for ϕ on Λ. The uniqueness is implied by the fact that the is a bijection between $\mathcal{M}(f)$ and $\mathcal{M}(\hat{f})$ and from tha fact that $\hat{\phi} = \phi \circ \pi : \hat{\Lambda} \to \mathbb{R}$ is Holder continuous, since π is Lipschitz and ϕ is Holder. We will the following inclusions

$$\exists k = k(\epsilon) \geq 1 \text{ s.t. } \hat{f}^k(\pi^{-1}B_n(y,\epsilon)) \subset B_{n-k}(\hat{f}^k\hat{y}, 2\epsilon) \subset \hat{\Lambda},\ \forall y \in \Lambda$$

$$\pi(B_n(\hat{y},\epsilon)) \subset B_n(y,\epsilon),\ \forall \hat{y} \in \hat{\Lambda}$$

the $\hat{f}-$invariance of $\hat{\mu}_\phi$ and the estimates for the $\hat{\mu}_\phi-$measure of the Bowen balls in $\hat{\Lambda}$ to get the existence of the positive constants A_ϵ, B_ϵ such that the estimates in our statement hold.

□

Lemma 3.3.4. *Let $f : M \to M$ a differentiable endomorphism and Λ a basic set on which f is hyperbolic. Then, for a small, fixed $\epsilon > 0$ there exist positive constants $A, B > 0$ such that for any $n \geq 1$ we have:*

$$Ae^{S_n\phi^s(y)} \leq m(f^n B_n(y,\epsilon)) \leq Be^{S_n\phi^s(y)},$$

where by m we have denoted the Lebesgue measure.

Theorem 3.3.5. *[6] Consider Λ to be a connected hyperbolic repellor for the smooth endomorphism $f : M \to M$ and we assume that the constant number of $f-$ preimages belonging to Λ of any point from Λ is equal to d. Then $P(\phi^s - \log d) = 0$.*

Theorem 3.3.6. *[6] Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$. There exists a neighbourhood V of Λ, $V \subset U$ such that if we denote by*

$$\mu_n^z := \frac{1}{n} \sum_{y \in f^{-n}z \cap U} \frac{1}{d(f(y)) \cdots d(f^n(y))} \sum_{i=1}^{n} \delta_{f^i y}, \quad z \in V, \tag{3.3}$$

where $d(y)$ is the number of $f-$preimages belonging to U of a point from V, then for any continuous function $g \in \mathcal{C}(U, \mathbb{R})$ we have

$$\int_V |\mu_n^z(g) - \mu_s(g)| dm(z) \xrightarrow{n\to\infty} 0,$$

where μ_s is the equilibrium measure of the stable potential $\phi^s(x)$, $x \in \Lambda$.

Sketch of the proof.

We will assume that U is the neighbourhood of Λ from Definition 3.3.1. Using Proposition 3.3.2 we know that any point in Λ has exactly d preimages belonging to Λ, where d is a positive integer. Moreover, it can be shown that there exists a neighbourhood V of Λ s.t. any point in V has exactly d^n preimages in U, for $n \geq 1$.

Since Λ is a hyperbolic repellor we can deduce that all the local stable manifolds have to be contained in Λ. We will denote by $\mathcal{C}(U)$ the space of the real continuous functions on U. Fix now $g \in \mathcal{C}(U)$ a Holder continuous function. We will apply Birkhoff's L^1 Ergodic Theorem on $\hat{\Lambda}$ for the homeomorphism $\hat{f}^{-1}$ to abtain an estimation for the measure of the set of ill-behaved points. The Holder continuity implies the existence of a unique equilibrium measure for the stable potenatial on $\hat{\Lambda}$, hence there exists a unique equilibrium measure for ϕ^s on Λ, denoted by μ_s. This measure is ergodic so we can apply again the Birkhoff's L^1 Ergodic Theorem for the function $g \circ \pi$ on $\hat{\Lambda}$:

$$\left|\left|\frac{1}{n}(g(x) + g\circ\pi(\hat{f}^{-1}(\hat{x})) + \cdots + g\circ\pi(\hat{f}^{-n+1}(\hat{x}))) - \int_\Lambda g\circ\pi d\hat{\mu}_s\right|\right|_{L^1(\hat{\Lambda},\hat{\mu}_s)} \xrightarrow{n\to\infty} 0. \tag{3.4}$$

A general observation is that if $f : \Lambda \to \Lambda$ is a continuous map on a compact metric space Λ, μ an $f-$invariant Borel probability measure on Λ and $\hat{\mu}$ is the only probability measure $\hat{f}$-invariant on $\hat{\Lambda}$ with $\pi_*(\hat{\mu}) = \mu$, then for an arbitrary closed set $\hat{F} \subset \hat{\Lambda}$, we have that

$$\hat{\mu}(\hat{F}) = \lim_n \mu(\{x_{-n}; \exists \hat{x} = (x, \ldots, x_{-n}, \ldots) \in \hat{F}\}). \tag{3.5}$$

For a positive integer n, a real continuous function g defined on a neighbourhood U of Λ and a point y such that $y, f(y), \ldots, f^{n-1}(y)$ are all in U, we will make the following notation:

$$\Sigma_n(g, y) := \frac{g(y) + \cdots + g(f^{n-1}y)}{n} - \int g d\mu_s, \quad n \geq 1, y \in \Lambda.$$

The norm convergence given by (3.4) implies the convergence in the $\hat{\mu}_s-$measure: if we consider for $\eta > 0$ small and an integer $n > 1$ the closed set

$$\hat{F}_n(\eta) = \{\hat{x} = (x, x_{-1}, x_{-2}, \ldots) \in \hat{\Lambda}, |\Sigma_n(g, x_{-n}) \geq \eta\},$$

then we have the convergence

$$\hat{\mu}_s(\hat{F}_n(\eta)) \xrightarrow{n\to\infty} 0, \quad \forall \eta > 0. \tag{3.6}$$

So, from (3.6), (3.5) and the f-invariance of μ_s we get that for any $\eta, \chi > 0$ small, there exists an integer $N(\eta, \chi) \geq 1$ such that:

$$\mu_s(x_{-n'} \in \Lambda \cap f^{-n'+n}(x_{-n}), |\Sigma_n(g, x_{-n})| \geq \eta) = \mu_s(x_{-n} \in \Lambda, |\Sigma_n(g, x_{-n})| \geq \eta) < \chi, \tag{3.7}$$

for $n' > n > N(\eta, \chi)$.

Let $\epsilon > 0$. It can be proven that if $y \in \Lambda$ and $z \in B_n(y, \epsilon)$ for n large enough, then the behavior of $\Sigma_n(g, z)$ is simillar to that of $\Sigma_n(g, y)$. More precisely, let $\eta > 0$ and $y \in \Lambda$ which satisfies $|\Sigma_n(g, y)| \geq \eta$. Then we will show that there exists $N(\eta) \geq 1$ such that

$$|\Sigma_n(g, z)| \geq \frac{\eta}{2}, \quad \forall z \in B_n(y, \epsilon), n > N(\eta). \tag{3.8}$$

Denote by

$$I_n(g, x) := \frac{1}{d^n} \sum_{y \in f^{-n}x \cap U} |\Sigma_n(g, y)|,$$

where $g : U \to \mathbb{R}$ is a real continuous function and $x \in V$. Consider a $(n, \epsilon)-$separated set of maximum cardinality in Λ, denoted by $F_n(\epsilon)$. We are interested to estimate the quantity $\int_V I_n(g, x) dm(x)$ and for this we proceed in the following way: replace $|\Sigma_n(g, y)|$ by $|\Sigma_n(g, z)|$ for all $y \in f^{-n}x \cap U$, unde $y \in B_n(z, 3\epsilon)$, $z \in F_n(\epsilon)$, apoi am integrat sumele respective ale lui $|\Sigma_n(g, z)|$, $z \in F_n(\epsilon)$ on small pieces of tubular overlap $V_n(z_1, \ldots, z_{d^n} := \bigcap_{1 \leq i \leq d^n} f^n B_n(z_i, 3\epsilon)$, where $z_i \in F_n(\epsilon)$ for $1 \leq i \leq d^n$; and last keep $|\Sigma_n(g, z)|$ fixed for $z \in F_n(\epsilon)$ arbitrary and add the measures of all the intersections of $f^n B_n(z, 3\epsilon)$ with other tubular sets of type $f^n B_n(w, 3\epsilon)$, with $w \in F_n(\epsilon)$. Adding the measures of this overlaps, $m(f^n B_n(z, 3\epsilon))$ is recovered. In conclusion, we get:

$$\int_V I_n(g, x) dm(x) \leq C \cdot \sum_{y \in F_n(\epsilon)} |\Sigma_n(g, y)| \cdot \frac{m(f^n(B_n(y, 3\epsilon)))}{d^n} + \frac{\eta}{2} \cdot m(V).$$

We know from Lemma 3.3.4 that $m(f^n(B_n(y, 3\epsilon)))$ is comparable with $e^{S_n \phi^s(y)}$, independently of n and $y \in \Lambda$. From Theorem 3.3.5 we know that $P(\phi^s) = \log d$.So, we can deduce from Proposition 3.3.3 that, if μ_s is the uniques equilibrium measure of ϕ^s, then $\mu_s(B_n(y, \epsilon/2))$ is comparable with $\frac{e^{S_n \phi^s(y)}}{d^n}$, independently of n, y. Therefore we get that there exists a constant $c_1 > 0$ s.t.:

$$\int_V I_n(g, x) dm(x) \leq C_1 \left(\sum_{y \in F_n(\epsilon)} |\Sigma_n(g, y)| \mu_s(B_n(y, \epsilon/2)) + \eta \right),$$

for $n > N(\eta)$.

Manipulating in a convenient way the points of the set $F_n(\epsilon)$, for a constant $C_\epsilon > 0$ and $n > \sup\{N(\eta), N(\eta, \chi)\}$, we can obtain

$$\int_V I_n(g, x) dm(x) \leq 2C_1(\eta + \chi \cdot C_\epsilon ||g||).$$

We know that $\eta, \chi > 0$ have been chosen arbitrary, and recalling the formula of $I_n(g, x)$ and the definition of μ_n^z, we get that

$$\int_V |\mu_n^z(g) - \mu_s(g)| dm(z) \xrightarrow{n \to \infty} 0.$$

Since Holder continuous functions are dense in the uniform norm on $\mathcal{C}(U)$, we obtain the conclusion of the theorem for any $g \in \mathcal{C}(U)$. □

Corollary 3.3.7. *In the same setting as the previous theorem, it follows that there exists a Borelian set $A \subset V$ with $m(V \setminus A) = 0$ and a subsequence $(n_k)_k$ such that, for any point $z \in A$ we have*

$$\mu_{n_k}^z \xrightarrow{k \to \infty} \mu_s. \tag{3.9}$$

Proof.

Fix $g \in \mathcal{C}(U)$. From the convergence in the Lebesgue measure of the sequence $(\mu_n^z(g))_{z \in V}$, $n \geq 1$, obtained in Theorem 3.3.6, it follows that there exists a Borelian set $A(g)$ with $m(A \setminus A(g)) = 0$ and a subsequence $(n_p)_p$ so that $\mu_{n_p}^z(g) \xrightarrow{p} \mu_s(g)$, $z \in A(g)$. We will consider now a sequence of functions $(g_m)_{m \geq 1}$ dense in $\mathcal{C}(U)$. Applying a diagonal sequence procedure, we get a subsequence $(n_k)_k$ such that $\mu_{n_k}^z(g_m) \xrightarrow{k} \mu_s(g_m)$, $\forall z \in \cap_m A(g_m)$, $\forall m \geq 1$. $m(V \setminus A(g_m)) = 0$, $\forall m \geq 1$ implies that $m(V \setminus \cap_m A(g_m)) = 0$. But, since $(g_m)_m$ is dense in $\mathcal{C}(U)$, we know that every real continuous function g can be approximated in the uniform norm by g_m

functions, hence it follows that $\mu^z_{n_k} \xrightarrow{k} \mu_s(g)$, $\forall z \in A := \cap_m A(g_m)$. We showed above that $m(V \setminus A) = 0$, so our conclusion holds for all points a set of full Lebesgue measure in V.

$\square$

3.3.2 Inverse Pesin entropy formula

First we present the notion of the Jacobian of an endomorphism, relative to an invariant probability measure. Let $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ a measure preserving endomorphism on a Lebesgue probability space. Assume that the fibers of f are countable. It can be proved that in this case f is positively non-singular, i.e. $\mu(A) = 0$ implies $\mu(f(A)) = 0$ for an arbitrary measurable set $A \subset X$. Also, there exists a measurable partition $\alpha = (A_0, A_1, \cdots)$ of X such that $f|_{A_i}$ is injective. Then, using the absolute continuity of $\mu \circ f$ with respect to μ, we define the Jacobian $J_{f,\mu}$ on each set A_i, to be equal to the Radon-Nikodym derivative $\frac{d\mu \circ f}{d\mu}$. $J_{f,\mu}$ is a well-defined measurable function, which is larger or equal to 1 everywhere.

Other properties include the fact that the Jacobian $J_{f,\mu}(\cdot)$ is independent on the choice of the partition α and that it satisfies a Chain Rule, namely $J_{f\circ g,\mu} = J_{f,\mu} \cdot J_{g,\mu}$ if $f, g : X \to X$ and both preserve μ. We also know that

$$\log J_{f,\mu}(x) = I(\varepsilon / f^{-1}\varepsilon)(x),$$

for $\mu-$a.e $x \in X$, where ε is the partition of X into single points, and $I(\varepsilon / f^{-1}\varepsilon)(\cdot)$ is the conditional information function of ε given the partition $f^{-1}\varepsilon$. Also, by the definition of the Jacobian we see that

$$\mu(fA) = \int_A J_{f,\mu}(x) d\mu(x),$$

for all sets A such that $f|_A : A \to f(A)$ is injective.

Theorem 3.3.8. *[6] Let Λ be a connected hyperbolic repellor for a smooth endomorphism $f : M \to M$ of a Riemann manifold M and assume that f is $d - to - 1$ on Λ. Then there exists an unique $f-$invariant probability measure μ^- on Λ which satisfies an inverse Pesin entropy formula:*

$$h_{\mu^-}(f) = \log d - \int_\Lambda \sum_{i, \lambda_i(x) < 0} \lambda_i(x) m_i(x) d\mu^-(x). \tag{3.10}$$

Moreover, the measure μ^- has absolutely continuous conditional measures on local stable manifolds.

Proof of existence and uniqueness:

From the Chain Rule and Birkhoff's Theorem, for an $f-$inavriant measure μ suported on Λ, we have:

$$\begin{aligned}\int_\Lambda \phi^s d\mu &= \int_\Lambda \lim_n \frac{1}{n}\sum_{i=0}^{n-1}\phi^s(f^i x)d\mu(x)\\ &= \int_\Lambda \lim_n \frac{1}{n}\log|\det(Df^n_{s,x})|d\mu(x) = \int_\Lambda \sum_{i,\lambda_i(x)<0}\lambda_i(x)m_i(x)d\mu(x).\end{aligned}$$

Therefore the inverse Pesin entropy formula is satisfied for $\mu = \mu_s$, because μ_s is the equilibrium measure of ϕ^s, hence it satisfies the Variational Principle 1.3.2

$$P_f(\varphi) = \sup_\mu\{h_\mu(f) + \int \varphi d\mu\}$$

and from Theorem 3.3.5 we know that $P(\phi^s - \log d) = 0$.

If the formula would be satisfied also by another invariant measure μ then we would have $h_\mu(f) = \log d - \int_\Lambda \sum_{i,\lambda_i(x)<0}\lambda_i(x)m_i(x)d\mu(x)$, hence:

$$P(\phi^s - \log d) \geq h_\mu - \log d + \int_\Lambda \phi^s d\mu = 0.$$

Applying again the result of Theorem 3.3.5 we know that $P(\phi^s - \log d) = 0$, so μ is an equilibrium measure for ϕ^s. But ϕ^s is Holder continuous and has, thus, a unique equilibrium measure. Therefore:

$$\mu = \mu_s = \mu^-.$$

□

Chapter 4

Entropy production

4.1 Entropy production for diffeomorphisms

Let M be a compact manifold and $f : M \to M$ a $\mathcal{C}^1$ diffeomorphism. We define the entropy production for an arbitrary $f-$invariant probability measure μ as in [12]

$$e_f(\mu) = -\int \mu(dx) \log J(x),$$

where $J(x)$ is the absolute value of the Jacobian of f at x.

In the following, we take μ to be ergodic, so that the Lyapunov exponents are constant $\mu-$a.e.

Lemma 4.1.1. *The entropy production $e_f(\mu)$ is independent of the choice of Riemannian metric and equal to minus the sum of the Lyapunov exponents of μ with respect to f.*

The result follows from the Oseledec Multiplicative Ergodic Theorem.

□

We have the following result:

Theorem 4.1.2. *[12] Let f be a $\mathcal{C}^1$ diffeomorphism and μ an $f-$invariant probability measure on the compact manifold M.*

(a) *If μ is a SRB measure then $e_f(\mu) \geq 0$.*

(b) *Let f be a $\mathcal{C}^{1+\alpha}$ with $\alpha > 0$ and μ be a SRB measure. If μ is singular with respect to dx and has no vanishing Lyapunov exponent, then $e_f(\mu) > 0$.*

(c) *For every a*

$$vol\{x : \frac{1}{n}\sum_{k=0}^{n-1} \log J(f^k x) \geq a\} \leq e^{-ma} vol M.$$

In particular, if $\nu(\mu) = \{x : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{k=n-1} \delta_{f^k x} = \mu\}$ *and* $e_f(\mu) < 0$, *then* $vol\nu(\mu) = 0$.

Proof.

(a) If μ is a SRB measure, denoting by λ_i^f the Lyapunov exponents of f, and by $\lambda_i^{f^{-1}}$ those of f^{-1}, we have

$$\begin{aligned} e_f(\mu) &= -\sum_i \lambda_i^f \\ &= [h_\mu(f) - \sum_{\lambda_i^f > 0} \lambda_i^f] - [h_\mu(f) - \sum_{\lambda_i^f < 0} \lambda_i^f] \\ &= [h_\mu(f) - \sum_{\lambda_i^f > 0} \lambda_i^f] - [h_\mu(f) - \sum_{\lambda_i^{f^{-1}} > 0} \lambda_i^{f^{-1}}] \\ &\geq 0 \end{aligned}$$

(b) If $e_f(\mu) = 0$, from (a) we have that the entropy $h_\mu(f)$ is minus the sum of the negative Lyapunov exponents. This implies that μ is absolutely continuous with respect to the Lebesgue measure if f is of class $\mathcal{C}^{1+\alpha}$ and μ has no vanishing Lyapunov exponent. This is a contradiction with our assumption, hence $e_f(\mu) > 0$.

(c) Next, we write

$$\nu(\mu) = \{x : \frac{1}{n}\sum_{k=0}^{n-1} \log J(f^k x) \geq a\}$$

Thus, we have

$$\begin{aligned} vol M \geq vol f^n \nu(\mu) &= \int_{\nu(\mu)} \Pi_{k=0}^{n-1} J(f^k x) dx \\ &\geq e^{na} vol \nu(\mu), \ \forall n, \end{aligned}$$

as announced.

□

Corollary 4.1.3. *If* μ *is a SRB measure with respect to both* f *and* f^{-1}, *then* $e_f(\mu) = 0$.

4.2 Entropy production for non-invertible maps

Definition 4.2.1. *Let $f : M \to M$ be a smooth endomorphism and μ an $f-$invariant probability on M, then the folding entropy $F_f(\mu)$ of μ is the conditional entropy:*

$$F_f(\mu) := H_\mu(\varepsilon|f^{-1}\varepsilon),$$

where ε is the partition into single points. Also, define the entropy production of μ by:

$$e_f(\mu) := F_f(\mu) - \int \log|\det Df(x)| d\mu(x).$$

4.2.1 A relation between entropy, folding entropy and negative Lyapunov exponents

Next, the following result due to Liu [3] will be presented: if f has no degenerate points, the formula

$$h_\mu(f) = F_\mu(f) - \int \sum_i \lambda_i^-(x) m_i(x) d\mu \tag{4.1}$$

holds, under a a somewhat restrictive condition on the Jacobian of (f, μ), only if μ has absolutely continuous conditional measures on the stable manifolds of (f, μ).

Theorem 4.2.2. *[3] (Sufficiency)*

Let $f : M \to M$ be a $\mathcal{C}^2$ non-invertible map such that $D_x f$ is non-degenerate at every point $x \in M$, and let μ be a invariant measure of f. If μ has absolutely continuous conditional measures on the stable manifolds, then the equality (4.1) holds true.

In order to give the converse of the statement above, we need to make an assumption on the Jacobian of f which seems rather restrictive:

(H) There is a Holder continuous funtion $J_f : M \to [1, \infty)$ such that

$$\mu(fB) = \int_B J_f(y) d\mu(y)$$

for any Borel set $B \subset M$ for which $f|_B : B \to fB$ is injective.

Actually, some weaker conditions are needed:

(H') For $\mu-$a.e. x, $J_f(y)$ is well defined on $V^s(x)$, which is the arc connected component of $W^s(x)$ which contains x. Also, $\Pi_{k=0}^{\infty} \frac{J_f(f^k x)}{J_f(f^k(y))}$ converges and is bounded away from 0 and $+\infty$ on any given neighbourhood of x in $V^s(x)$ whose d^s-diameter is finite, where d^s is the distance along $V^s(x)$. Moreover,denoting λ_x^s the Lebesgue measure on $W^s(x)$, we have for λ_x^s-a.e. $y \in V^s(x)$

$$\sum_{z \in f^{-1}(y)} \frac{1}{J_f(z)} = 1.$$

Theorem 4.2.3. *[3] (Necessity)*

Let (f, μ) be as in the previous theorem and assume one of the conditions (H) or (H') holds. If (4.1) is true then μ has absolutely continuous conditional measures on the stable manifolds.

4.2.2 An estimate for the Jacobian of an invariant measure of a smooth endomorphism

Theorem 4.2.4. *[5] Let f be a smooth hyperbolic endomorphism on a folded basic set Λ, which has no critical points in Λ; let also ϕ be a Holder continuous potential on Λ and denote by μ_ϕ the unique equilibrium measure of ϕ on Λ. Then for all $m \geq 1$, the Jacobian of μ_ϕ w.r.t. f^m is comparable to the ratio $\frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}}$, i.e. there exists a comparability constant $C > 0$ (independent of m, x) s.t. for $\mu_\phi-$a.e. $x \in \Lambda$:*

$$C^{-1} \cdot \frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}} \leq J_{f^m}(\mu_\phi)(x) \leq c \cdot \frac{\sum_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m \phi(\zeta)}}{e^{S_m \phi(x)}}, \quad (4.2)$$

Let $\{\mu_x\}$ be a canonical family of conditional measures supported on the finite fibers $\{f^{-1}(x)\}$ for $\mu-$a.e. x, obtained by desintegrating the invariant measure μ, using the measurable single point partition ε. Then the entropy of the conditional measure of μ restricted to $f^{-1}(x)$ is $H(\mu_x) = -\sum_{y \in f^{-1}(x)} \mu_x(y) \log \mu_x(y)$. We also know that

$$J_f(\mu)(x) = \frac{1}{\mu_{f(x)}(x)}, \quad \mu - \text{a.e. } x,$$

from where we get that

$$F_f(\mu) = \int \log J_f(\mu)(x) d\mu(x) \quad (4.3)$$

Let Λ be a hyperbolic basic set for the smooth endomorphism f and consider a Holder potential ϕ on Λ with its unique equilibrium measure μ_ϕ. In the following, a formula for the folding entropy of the equilibrium measure μ_ϕ will be given. This will take into account the $n-$preimages of the generic points with respect to μ_ϕ. We will define for an $f-$invariant probability measure μ on Λ, for any small $\tau > 0$, $n > 0$ integer and $x \in \Lambda$, the set

$$G_n(x, \mu, \tau) := \left\{ y \in f^{-n}(f^n x) \cap \Lambda, s.t. \left| \frac{S_n\phi(y)}{n} - \int \phi d\mu \right| < \tau \right\}, \qquad (4.4)$$

where $S_n\phi(y) := \phi(y) + \ldots + \phi(f^{n-1}y)$, $y \in \Lambda$ is the consecutive sum of ϕ on y.

Next, we will denote by $d_n(x, \mu, \tau) := CardG_n(x, \mu, \tau)$, $x \in \Lambda$, $n > 0$, $\tau > 0$. The function $d_n(\cdot, \mu, \tau)$ is measurable, nonnegative and finite on Λ.

4.2.3 A formula for the folding entropy of the equilibrium measure of a potential

Theorem 4.2.5. *[5] Let $f : M \to M$ be a smooth endomorphism and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ be a Holder continuous potential on Λ and μ_ϕ the equilibrium measure associated to ϕ. Then we have the following formula for the folding entropy of μ_ϕ:*

$$F_f(\mu_\phi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) d\mu_\phi(x).$$

Proof.

The Chain Rule for Jacobians gives us $J_{f^n}(\mu)(x) = J_f(\mu)(x) \ldots J_f(\mu)(f^{n-1}(x))$ $\mu-$a.e., for any $n \geq 1$. But, since μ is $f-$invariant, we have that

$$\int \log J_f(\mu)(x) d\mu(x) = \int \log J_f(\mu)(f(x)) d\mu(x) = \int \log J_f(\mu)(f^k x) d\mu(x),$$

for all $k \geq 1$.

From formula (4.3) we have for any $n \geq 1$,

$$F_f(\mu) = \frac{1}{n} \int \log J_{f^n}(\mu)(x) d\mu(x).$$

Using Theorem 4.2.4 and the fact that the constant C is independent of n, we get

$$F_f(\mu_\phi) = \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log \frac{\sum_{y \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(y)}}{e^{S_m\phi(x)}} d\mu_\phi(x). \qquad (4.5)$$

Λ is compact, so each point $x \in \Lambda$ has only finitely many $f-$preimages in Λ, i.e. there exists a positive integer d s.t. $Card(f^{-1}x) \leq d$, $x \in \Lambda$.

We know that μ_ϕ is ergodic so applying Birkhoff's Ergodic Theorem we obtain that $\mu_\phi(x \in \Lambda, \left|\frac{S_n\phi(x)}{n} - \int \phi d\mu\right| > \frac{\tau}{2}) \xrightarrow{n\to\infty} 0$, for any small $\tau > 0$. Thus for any $\eta > 0$ there exists a large integer $n(\eta)$ s.t. $n \geq n(\eta)$,

$$\mu_\phi\left(x \in \Lambda, \left|\frac{S_n\phi(x)}{n} - \int \phi d\mu\right| > \frac{\tau}{2}\right) < \eta \tag{4.6}$$

Now, take a point $x \in \Lambda$ with $|\frac{S_n\phi(x)}{n} - \int \phi d\mu| < \tau$. From the definition of $d_n(x, \mu, \tau)$ we have

$$\frac{e^{n(\int \phi d\mu_\phi - \tau)} d_n(x,\mu_\phi,\tau) + r_n(x,\mu_\phi,\tau)}{e^{n(\int \phi d\mu_\phi + \tau)}} \leq \frac{\sum_{y\in f^{-n}(f^n x)\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq \frac{e^{n(\int \phi d\mu_\phi + \tau)} d_n(x,\mu_\phi,\tau) + r_n(x,\mu_\phi,\tau)}{e^{n(\int \phi d\mu_\phi - \tau)}}$$

where $r_n(x, \mu_\phi, \tau) = \sum_{y\in f^{-n}(f^n(x))\setminus G_n(x,\mu_\phi,\tau)} e^{S_n\phi(y)}$.

Given n large, we will consider the following partition $(A_i^n)_{1\leq i\leq K}$ of Λ (modulo μ_ϕ) so that for each $0 \leq i \leq K$ there exists a point $z_i \in A_i^n$ so that for any $n-$preimage $\xi_{ij} \in f^{-n}(z_i)\Lambda$, $1 \leq j \leq d_{n,i}$ we have $A_i^n \subset f^n(B_n(\xi_{ij}, \epsilon))$, $1 \leq j \leq d_{n,i}$, $1 \leq i \leq K$. We will denote by $A_{ij}^n := f^{-n}(A_i^n) \cap B_n(\xi_{ij}, \epsilon)$, for $1 \leq j \leq d_{n,i}$, $1 \leq i \leq K$. Since the sets A_i^n were taken to be disjoint, so will the sets A_{ij}^n , i, j.

Since ϕ is Holder continuous and $A_{ij}^n \subset B_n(\xi_{ij}, \epsilon)$, for $y, z \in A_{ij}^n$, we get

$$|S_n\phi(y) - S_n\phi(z)| \leq C(\epsilon), \tag{4.7}$$

where $C(\epsilon)$ is a positive function with $C(\epsilon) \xrightarrow{\epsilon\to 0} 0$. Decomposing the integral in (4.5) over the sets A_{ij}^n, we obtain:

$$\int_\Lambda \log \frac{\sum_{y\in f^{-n}(f^n x)\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x) = \sum_{0\leq j\leq d_i, 0\leq i\leq K} \int_{A_{ij}^n} \log \frac{\sum_{y\in f^{-n}(f^n x)\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} d\mu_\phi(x). \tag{4.8}$$

Consider now k_0 fixed. We will denote by $R_n(i, \mu_\phi, \tau)$ the set of preimages ξ_{ij} with $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$ and denote by $R_{n,i}$ the set of indices j with $1 \leq j \leq d_{n,i}$ with $\xi_{ij} \in R_n(i, \mu_\phi, \tau)$ for every $1 \leq i \leq K$. Since $Card(f^{-1}x \cap \Lambda) \leq d$ for $x \in \Lambda$ and $-M \leq \phi(x) \leq M$, $x \in \Lambda$ we have

$$1 \leq \frac{\sum_{y\in f^{-n}f^n x\cap\Lambda} e^{S_n\phi(y)}}{e^{S_n\phi(x)}} \leq d^n e^{2nM},$$

so we notice that in the decomposition from (4.8) the integral from those sets A^n_{ij} with $j \in R_{n,i}$ will not matter significantly.

We know that each $A^n_{ij} \subset B_n(\xi_{ij}, \epsilon)$ and the sets A^n_{ij}, i, j are mutually disjoint, so using inequalities (4.6) and (4.7) and the fact that $\xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)$ whenever $j \in R_{n,i}$, we obtain

$$\sum_{0 \le i \le K, j \in R_{n,i}} \frac{1}{n} \int_{A^n_{ij}} \log \frac{\sum_{y \in f^{-n}(f^n x) \cap \Lambda} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) \le \frac{1}{n} \log(d^n e^{2nM}) \cdot \eta = \eta(\log d + 2M) \tag{4.9}$$

Using Theorem 4.2.4, we deduce that the last term of formula (4.8) is comparable to

$$\sum_{i,j} \mu_\phi(A^n_{ij}) \log \frac{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij}) + \tilde{r_n}(z_i, \mu_\phi, \tau)}{\mu_\phi(A^n_{ij})} \tag{4.10}$$

where $\tilde{r_n}(z_i, \mu_\phi, \tau) := \sum_{\xi_{ij} \in f^{-n}(z_i) \cap \Lambda, \xi_{ij} \notin G_n(\xi_{ik_0}, \mu_\phi, \tau)} \mu_\phi(A^n_{ij})$.

From relations (4.9) and (4.10), using the general-known inequality $\log(1+x) \le x$ for $x = \frac{\tilde{r_n}(z_i, \mu_\phi, \tau)}{d_n(z_i, \mu_\phi, \tau)\mu_\phi(A^n_{ij})}$ we have for $n \ge n(\eta)$ that

$$\left| \frac{1}{n} \int_\Lambda \log \frac{\sum_{y \in f^{-n}(f^n x) \cap \Lambda} e^{S_n \phi(y)}}{e^{S_n \phi(x)}} d\mu_\phi(x) - \frac{1}{n} \int_\Lambda \log d_n(z, \mu_\phi, \tau) d\mu_\phi(z) \right| \le \delta(\tau) + \eta,$$

where $\delta(\tau) \to 0$. Then, by taking $n \to \infty$ and $\tau \to 0$, we will obtain the desired conclusion, namely

$$F_f(\mu_\phi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau) d\mu_\phi(x).$$

□

Corollary 4.2.6. **(a)** *Let $f : \mathbb{T}^m \to \mathbb{T}^m$, $m \ge 2$ be a hyperbolic toral endomorphism, and ϕ be an arbitrary Holder continuous potential on $\mathbb{T}^m$, with its associated equilibrium measure μ_ϕ. Then the entropy production of μ_ϕ is non-positive.*

In the same setting, the entropy production of the Haar (Lebesgue) measure is equal to 0*.*

(b) *The same conclusions as above hold also for any Anosov endomorphism $f : \mathbb{T}^m \to \mathbb{T}^m$ with constant Jacobian with respect to the Riemannian metric, i.e. for which* $\det Df$ *is constant on $\mathbb{T}^m$.*

Proof.

(a) If f is given by the integer-valued matrix A, then $\det Df$ is constant and equal to $\det A$. Hence

$$\int_{\mathbb{T}^m} \log|\det Df| d\mu_\phi = \log d,$$

where we have denoted $d := |\det A|$. Also, the number of $f-$preimages of any point from $\mathbb{T}^m$ is exactly d. Therefore, for $\Lambda = \mathbb{T}^m$, we get that

$$d_n(x, \mu_\phi, \tau) \leq d^n, \forall x \in \mathbb{T}^m, n > 0, \tau > 0.$$

So, from Theorem 4.2.5, it follows that $e_f(\mu_\phi) \leq 0$.

Now, in order to prove the last statement of a), we know that f invariates the Lebesgue measure m, that $|\det Df|$ is constant and equal to d and that $d_n(x, m, \tau)$ is constant in x and equal to d since the Lebesgue measure is the unique measure of maximal entropy. Therefore the entropy production of the Lebesgue measure m with respect to f is equal to 0.

(b) We have the same argument as before. □

4.2.4 Examples

Next, it will be proven that the entropy production of the respective inverse SRB measure of a perturbation g of a hyperbolic toral endomorphism, is less than or equal to 0, and the cases when it is 0 will be identified.

Theorem 4.2.7. *[5] Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m$, $m \geq 2$, given by the integer-valued matrix A without zero eigenvalues, and let g be a $\mathcal{C}^1$ perturbation of f. Consider μ_g^- the inverse SRB measure of g and μ_g^+ the forward SRB measure. Then:*

(a) *$e_g(\mu_g^-) \leq 0$ and $F_g(\mu_g^-) = \log d$. Moreover $e_g(\mu_g^+) \geq 0$.*

(b) *$e_g(\mu_g^-) = 0$ if and only if $|\det Dg|$ is cohomologous to a constant on $\mathbb{T}^m$. The same condition on $|\det Dg|$ holds if and only if $e_g(\mu_g^+) = 0$. In either case, we get $\mu_g^- = \mu_g^+$, and the common value is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.*

Proof:

(a) Assume f is given by the integer-valued matrix A. Then f is $d - to - 1$ on $\mathbb{T}^m$, where $d := |\det A|$. If g is a $\mathcal{C}^1$ perturbation of f, then it is clar that g is also hyperbolic on $\mathbb{T}^m$. We know that we can construct the SRB measure of g, denoted by μ_g^+, which is the projection by π_* of the equilibrium measure of $\Phi_g^u(\hat{x}) = -\log|\det|Dg_u(\hat{x})|$, $\hat{x} \in \hat{\mathbb{T}}^m$. In particular, μ_g^+ is ergodic, hence its Lyapunov exponents are constant μ_g^+-a.e.

Also, since f has no critical points, we can construct the inverse SRB measure μ_g^- which is the equilibrium measure of the stable potential $\Phi_g^s(\hat{x}) = \log|\det|Dg_s(\hat{x})|$, $\hat{x} \in \hat{\mathbb{T}}^m$; hence μ_g^- is also ergodic and its Lyapunov exponents are constant μ_g^-−a.e.

Since g is a perturbation of f, it follows that every point in $\mathbb{T}^m$ has exactly d $g-$preimages. So, from Theorem 3.3.6 it follows that μ_g^- is the weak limit of a sequence of measures of type (3.3), where the degree function $d(\cdot)$ is constant and equal to d everywhere on $\mathbb{T}^m$. This implies that the Jacobian of μ_g^- is constant and equal to d, since for any small borelian set B, we have that a point $x \in g(B)$ if and only if there is exactly one $g-$preimage x_{-1} of x in B, and we use this fact in the convergence (3.9) of measures toward μ^-. Therefore

$$F_g(\mu_g^-) = \int \log J_g(\mu_g^-)(x) d\mu_g^-(x) = \log d.$$

From the inverse Pesin entropy formula (3.10), we have that

$$h_{\mu_g}^-(g) = \log d - \sum_{\lambda_i(\mu_g^-)<0} \lambda_i(\mu_g^-).$$

So, if $e_g(\mu_g^-) > 0$, it would follow that $F_g(\mu_g^-) > \int \log|\det Dg| d\mu_g^- =$
$= \frac{1}{n}\int \log|\det Dg^n| d\mu_g^-, n \geq 1$. Thus, from the last displayed formula and from Birkhoff's Ergodic Theorem, we obtain $h_{\mu_g^-}(g) > \sum_{\lambda_i(\mu_g^-)>0} \lambda_i(\mu_g^-)$, which gives a contradiction with Ruelle's inequality. Hence, we have for any perturbation g,

$$e_g(\mu_g^-) \leq 0.$$

Coming back to the SRB measure μ_g^+, if the entropy production $e_g(\mu_g^+)$ were strictly negative, then $F_g(\mu_g^+) < \int \log|\det Dg| d\mu_g^+$. Since we know that, $h_{\mu_g^+}(g) \leq F_g(\mu_g^+) - \sum_{\lambda_i(\mu_g^+)<0} \lambda_i(\mu_g^+)$, it would follow that $h_{\mu_g^+(g)} < \sum_{\lambda_i(\mu_g^+)<0} \lambda_i(\mu_g^+)$, which is a contradiction to the fact that the SRB measure satisfies Pesin entropy formula. Consequently,

$$e_g(\mu_g^+) \geq 0.$$

(b) If $e_g(\mu_g^-) = 0$, then $F_g(\mu_g^-) = \int \log|\det Dg| d\mu_g^-$, so from Birkhoff's Ergodic Theorem we obtain

$$h_{\mu_g^-}(g) = \int \log|\det Dg| d\mu_g^- - \sum_{\lambda_i(\mu_g^-)<0} \lambda_i(\mu_g^-) = \sum_{\lambda_i(\mu_g^-)>0} \lambda_i(\mu_g^-).$$

From the uniqueness of the $g-$invariant measure satisfying Pesin's entropy formula, we obtain that $\mu_g^- = \mu_g^+$. Recall that μ_g^- is the equilibrium measure of the stable potential Φ^s and μ_g^+ is the equilibrium measure for the unstable potential Φ^u, so from Livshitz's Theorem 2.4.2, we see that $\mu_g^- = \mu_g^+$ if and only if $\det Dg$ is cohomologous to a constant.

Assume now that $\mu_g^+ = \mu_g^-$. Since μ_g^+ has absolutely continuous conditional measures associated to a partition subordinated to local unstable manifolds and μ_g^- has absolutely continuous conditional measures associated to a partition subordinated to local stable manifolds, we obtain that μ_g^+ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^m$.

Corollary 4.2.8. **(a)** *Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m$, $m \geq 2$. Then there exists a neighbourhood V of f in $\mathcal{C}^1(\mathbb{T}^m, \mathbb{T}^m)$ and a set $W \subset V$ such that W is open and dense in the $\mathcal{C}^1$ topology in V and s.t. for any $g \in W$ we have $e_g(\mu_g^-) < 0$.*

(b) *Consider the hyperbolic toral endomorphism on $\mathbb{T}^2$ given by $f(x, y) = (2x + 2y, 2x + 3y)(mod 1)$ and its smooth perturbation*

$$g(x, y) = (2x + 2y + \epsilon \sin 2\pi y, 2x + 3y + 2\epsilon \sin 2\pi y)(mod 1).$$

Then the inverse SRB measure of g has negative entropy production, while the SRB measure of g has positive entropy production, i.e.

$$e_g(\mu_g^-) < 0 \text{ and } e_g(\mu_g^+) > 0.$$

(c) *The same conclusion as in (b) holds if we take* $f(x, y) = (5x + 2y, 3x + 2y)$ *and the perturbation to be*

$$g(x, y) = (5x + 2y + \epsilon \sin 2\pi y, 3x + 2y + \epsilon \sin 2\pi y).$$

Proof.

(a) If f is a hyperbolic toral endomorphism on $\mathbb{T}^m$ then there exists a neighbourhood V of f in the $\mathcal{C}^1$ topology, such that any $g \in V$ is hyperbolic and $d - to - 1$, where $d = |\det Df|$. In Theorem 4.2.7 it was shown that $e_g(\mu_g^-) < 0$ unless $|\det Dg|$ is cohomologous to a constant. From the Livshitz Theorem 2.4.2 it follows that this is equivalent to the existance of a constant c such that for any $n \geq 1$,

$$S_n(|\det Dg|)(x) = nc, \forall x \in Fix(g^n).$$

Since the set of g's not satisfying the above equalities is open and dense in V, we obtain the conclusion of part (a).

(b) First of all we notice that f is indeed hyperbolic, since it is given by an integer valued matrix A with eigenvalues $0 < \frac{1}{2}(5 - \sqrt{17}) < 1$ and $\frac{1}{2}(5 + \sqrt{17}) > 1$. Thus, for $\epsilon > 0$ small enough, we have that g (which is well defined as an endomorphism on $\mathbb{T}^m$) is hyperbolic as well. We compute now the determinant of the derivative of g as

$$\det Dg(x, y) = 2 + 4\pi\epsilon \cos 2\pi y.$$

Now, from Theorem (4.2.7) we see that $e_g(\mu_g^-) < 0$ if and only if $|\det Dg|$ is cohomologous to a constant. But, as we have seen earlier, this is equivalent to the fact that there exists a constant c such that

$$S_n(|\det Dg|)(x) = nc,\ x \in Fix(g^n), n \geq 1.$$

Now, notice that both $(0, 0)$ and $(0, \frac{1}{2})$ are fixed points for g. But, $|\det Dg(0, 0)| = 2 + 4\pi\epsilon$, whereas $|\det Dg(0, \frac{1}{2})| = 2 - 4\pi\epsilon$. So the Livshitz condition above is not satisfied. Hence $|\det Dg|$ is not cohomologous to a constant. According to Theorem 4.2.7 we obtain

$$e_g(\mu_g^-) < 0 \text{ and } e_g(\mu_g^+) > 0.$$

(c) The proof in this case is analoguous to the one above. f is a hyperbolic toral endomorphism, since it is given by an integer-valued matrix with eigenvalues $0 < \frac{1}{2}(7-\sqrt{33}) < 1$ and $\frac{1}{2}(7+\sqrt{33}) > 1$.

The determinant of the derivative of g is

$$\det Dg(x, y) = 4 + 4\pi\epsilon \cos 2\pi y.$$

Now, observe that $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ are both fixed points of g. But $|\det Dg(0, 0)| = 4+4\pi\epsilon$, whereas $|\det Dg(\frac{1}{2}, \frac{1}{2})| = 4-4\pi\epsilon$. Hence we obtain the same conclusion as above.

□

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