"Ilia Vekua" Institute of Applied Mathematics

Random multiple–fragmentation and flow of particles on a surface

Oana Lupașcu-Stamate
"Gheorghe Mihoc - Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of Romanian Academy Bucharest, Romania
IRN *ECO-Math*
- collaboration with Lucian Beznea (Bucharest) and Ioan R. Ionescu (Paris)

Motivation: A probabilistic approach for the fragmentation phase of an avalanche. The particles are characterised by the mass, velocity and the kinetic energy.

Why modeling the avalanche by a random fragmentation process ?

- we give a model closer to the real life taking into account the spatial position and the movement of the fragments

The model has:

- deterministic part: by a flow of particles on a surface driven by a given gravitational force (Newton's laws) with a random fragmentation.

`simulation.avi`

- stochastic part: initiated by [Beznea, Deaconu, **O.L**, 2015, 2016, 2019], based on branching-fragmentation processes.

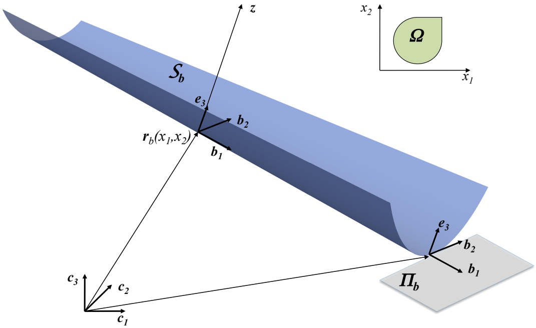
the multiple-fragmentation process, solution of a SDE, describes the time evolution of a typical particle in the fragmentation process.

[Beznea, Ionescu, **O.L-S**,J. of Ev. Eqs.,2021].

Geometrical description of the surface.

Bottom surface $\mathcal{S}_b$: parametric representation by $\mathbf{r}_b(x_1, x_2)$, parametric coordinates $x = (x_1, x_2) \in \Omega$, covariant basis

$$\mathbf{b}_1 = \frac{\partial \mathbf{r}_b}{\partial x_1}(x), \quad \mathbf{b}_2 = \frac{\partial \mathbf{r}_b}{\partial x_2}(x), \quad \beta_3 = \frac{\mathbf{b}_1 \wedge \mathbf{b}_2}{g}.$$



Motion equations between two fragmentation moments

- we consider a **motion of N particles on the surface $\mathcal{S}_b$** during the time interval $[t_0, t_1]$: the number of particles N will be constant.
- we denote by $\mathbf{r}^p(t) = \mathbf{r}_b(x^p(t))$, $x^p(t) = (x_1^p(t), x_2^p(t))$ the position of each particle p at $t \in [t_0, t_1]$.
- we compute the **velocity** and the **acceleration of each particle**:

$$\mathbf{v}^p = \frac{d}{dt}\mathbf{r}^p = \dot{x}_1^p \mathbf{b}_1(x^p) + \dot{x}_2^p \mathbf{b}_2(x^p),$$

$$\mathbf{a}^p = \frac{d}{dt}\mathbf{v}^p = \ddot{x}_1^p \mathbf{b}_1(x^p) + \dot{x}_1 \frac{\partial \mathbf{b}_1(x^p)}{\partial x_1}\dot{x}_1 + \dot{x}_1 \frac{\partial \mathbf{b}_1(x^p)}{\partial x_2}\dot{x}_2 + \ddot{x}_2 \mathbf{b}_2(x^p) +$$

$$\dot{x}_2 \frac{\partial \mathbf{b}_2(x^p)}{\partial x_1}\dot{x}_1 + \dot{x}_2 \frac{\partial \mathbf{b}_2(x^p)}{\partial x_2}\dot{x}_2.$$

Let $\mathbf{F}^p = \mathbf{F}^p(\mathbf{r}^1, ..., \mathbf{r}^N, \mathbf{v}^1, ..., \mathbf{v}^N)$ be the force acting on the particle p (the gravity forces acting on the vertical direction $-c_3$, $\mathbf{F}^p = -m^p \mathcal{G} c_3$, m^p is the mass of the particle P and $\mathcal{G}$ is the gravitational acceleration).

The **movement of a particle** is described by the Newton evolution equation

$$m^p \mathbf{a}^p = \mathbf{F}^p(\mathbf{r}^1, ..., \mathbf{r}^N, \mathbf{v}^1, ..., \mathbf{v}^N) + M^p(t)\beta_3, \qquad \text{for all} \qquad P = 1, ..., N, \tag{0.1}$$

where M^p is the reaction force of the surface $\mathcal{S}_b$.

Multiplying now by $\mathbf{b}^1$, respectively $\mathbf{b}^2$, we obtain the second order nonlinear system of equations:

$$\begin{cases} \ddot{x}_1^p + \Gamma_{11}^1(\dot{x}_1^p)^2 + \Gamma_{22}^1(\dot{x}_2^p)^2 + \dot{x}_1^p\dot{x}_2^p(\Gamma_{21}^1 + \Gamma_{12}^1) = \\ \qquad\qquad \mathbf{F}^p(\mathbf{r}^1, ..., \mathbf{r}^N, \mathbf{v}^1, ..., \mathbf{v}^N) \cdot \mathbf{b}^1(x^p) \\ \ddot{x}_2^p + \Gamma_{11}^2(\dot{x}_1^p)^2 + \Gamma_{22}^2(\dot{x}_2^p)^2 + \dot{x}_1^p\dot{x}_2^p(\Gamma_{21}^1 + \Gamma_{12}^1) = \\ \qquad\qquad \mathbf{F}^p(\mathbf{r}^1, ..., \mathbf{r}^N, \mathbf{v}^1, ..., \mathbf{v}^N) \cdot \mathbf{b}^2(x^p) \end{cases} \tag{0.2}$$

for all $p = 1, ..., N$,
with the initial conditions

$$\mathbf{r}^p(t_{0+}) = \mathbf{r}_0^p \quad \text{and} \quad \mathbf{v}^p(t_{0+}) = \mathbf{v}_0^p, \quad \text{for all} \quad p = 1, ..., N. \tag{0.3}$$

Random binary fragmentation

Our aim: to introduce a random binary fragmentation process for N particles characterised by their **mass** $m^1, ..., m^N$, their **positions** $\mathbf{r}^1, ..., \mathbf{r}^N$, and their **velocities** $\mathbf{v}^1, ..., \mathbf{v}^N$.

- Let t_1 be the first random fragmentation time greater then t_0, having a Poisson distribution.

The position $\mathbf{r}_1(t_1-)$ and the velocity $\mathbf{v}_1(t_1-)$ are computed solving the above nonlinear system (0.2).

- A fragmentation process $\mathcal{F}^p$ of a particle p is a set of equations allow at each given $(m^p, \mathbf{r}^p, \mathbf{v}^p)$ the couple $[(m_1^p, \mathbf{r}_1^p, \mathbf{v}_1^p),\ (m_2^p, \mathbf{r}_2^p, \mathbf{v}_2^p)]$,

$$\mathcal{F}^p(m^p, \mathbf{r}^p, \mathbf{v}^p) = [(m_1^p, \mathbf{r}_1^p, \mathbf{v}_1^p),\ (m_2^p, \mathbf{r}_2^p, \mathbf{v}_2^p)],$$

which represents **the masses, the positions, and the velocities of the resulting two particles at** $t = t_1+$**.**

Description of our choice of the fragmentation process.

- We suppose that the position at $t = t_1+$ of the resulting two particles coincide with the position of the mother particle, i.e.

$$\mathbf{r}_1^p(t_1+) = \mathbf{r}_2^p(t_1+) = \mathbf{r}^p(t_1-). \tag{0.4}$$

- **For the mass and the velocity fragmentation we choose a random procedure.** We take the mass of the fragments to be

$$m_1^p = \xi m^p, \qquad m_2^p = (1-\xi)m^p, \tag{0.5}$$

where ξ is a fixed uniform random variable. We have the mass conservation property, i.e.

$$m_1^p + m_2^p = m^p.$$

The **velocity fragmentation** needs more physical restrictions.

1. We assume that a part $(1-\theta)\mathcal{E}^p$ of the particle kinetic energy $\mathcal{E}^p = \frac{1}{2} m^p |\mathbf{v}^p(t_1-)|^2$ is lost in the fragmentation process, where $\theta \in (0,1)$ is a fixed rupture parameter.
2. The resulting two fragments will have the kinetic energy $\mathcal{E}_1^p = \theta\gamma\mathcal{E}^p$ and $\mathcal{E}_2^p = \theta(1-\gamma)\mathcal{E}^p$ with $\mathcal{E}_1^p + \mathcal{E}_2^p = \theta\mathcal{E}^p$, where γ is a fixed $\mathcal{U}([0,1])$:
$$\begin{cases} \mathcal{E}_1^p = \frac{1}{2} m_1^p |\mathbf{v}_1^p(t_1+)|^2 = \theta\gamma\mathcal{E}^p \\ \\ \mathcal{E}_2^p = \frac{1}{2} m_2^p |\mathbf{v}_2^p(t_1+)|^2 = \theta(1-\gamma)\mathcal{E}^p, \end{cases} \tag{0.6}$$
3. Finally, we suppose that we have an additive law:
$$\mathbf{v}_1^p(t_1+) + \mathbf{v}_2^p(t_1+) = \mathbf{v}^p(t_1-). \tag{0.7}$$

- We compute the resulting velocity $\mathbf{v}_1^p(t_1+)$ and $\mathbf{v}_2^p(t_1+)$ from the fragmentation's laws (0.5),(0.6), and (0.7). For the first fragment:

$$\begin{cases} \dot{x}_1^p = \alpha_1 \dot{x_1}^p + \gamma_1 \left(g^{12}\ \dot{x_1}^p - g^{11}\ \dot{x_2}^p \right) \\ \\ \dot{x}_2^p = \alpha_1 \dot{x_2}^p + \gamma_1 \left(g^{22}\ \dot{x_1}^p - g^{12}\ \dot{x_2}^p \right) \end{cases} \tag{0.8}$$

where g^{11}, g^{12}, g^{22} are the contra fundamental magnitudes of the first order and $\mathbf{v}_1^p = \alpha_1 \mathbf{v}^p + \gamma_1 (\mathbf{v}^p)^\perp$.

- Following the same procedure for $\mathbf{v}_2^p$ we get the corresponding expressions of the components of second fragment.

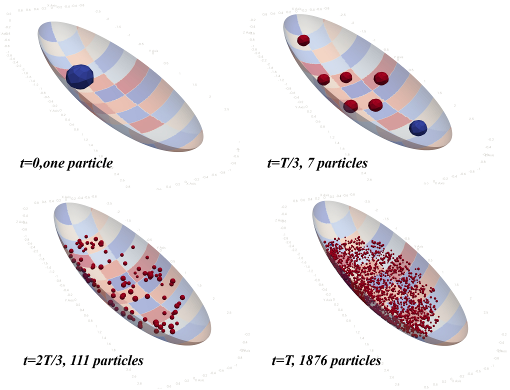
Remarks:

1. an algorithm to simulate the flow and the fragmentation of a set of particles on a half of an ellipsiod.
2. the link between the simulations of a system of particles (deterministic part) and the Markov process involving a set of sizes is as follows (stochastic part) :

   we start with one particle and simulate the time evolution of each fragment resulting after the fragmentation procedure presented above (deterministic part) while the multiple-fragmentation process, solution of the further coming SDEnDF (0.9), describes the time evolution of a typical particle in the fragmentation process (stochastic part).
3. numerical illustrations.

Binary fragmentation and flow of particles on a half-ellipsoid

A partition into i regions $\mathcal{D}_i = \mathbb{R}_+ \times \Omega_i \times \mathbb{R}^2$; $n_{max} = 2000$;
A particle P of mass m^P located at $\mathbf{r}^P$: a sphere of center $\mathbf{r}^P$, radius $R^P = C(m^P)^{1/3}$.



t=0,one particle *t=T/3, 7 particles*

t=2T/3, 111 particles *t=T, 1876 particles*

Since at each fragmentation process the kinetic and total energies are decreasing the particles are slower and slower and they are accumulated on the bottom of the ellipsoid (lowest potential energy).

Stochastic differential equation of n-dimensional fragmentation driven by an Euclidian gradient flow

$$
\begin{cases}
Y_t = \int_0^t \mathbf{d}(X_s)\mathbf{B}(Y_s)\mathrm{d}s \\
X_t^k = X_0^k - \int_0^t \int_F \int_0^1 y_k 1_{[0<y_k<X_s^k]} 1_{\left[u \leqslant \mathbf{c}(Y_s) \frac{X_{s-}^k - y_k}{X_{s-}^k} \mathbf{F}^k(y_k, X_{s-}^k - y_k)\right]} \\
\qquad\qquad p_c(\mathrm{d}s, \Pi_{i=1}^n \mathrm{d}y_i, \mathrm{d}u), \text{ if } 1 \leqslant k \leqslant n_c, \\
X_t^k = X_0^k - \int_0^t \int_F X_{s-}^k \left((1-\beta) 1_{\left[\frac{y_k}{\mathbf{c}(Y_s)\beta\lambda_o} < X_{s-}^k \leqslant 1\right]} + \beta 1_{\left[\frac{y_k}{\mathbf{c}(Y_s)\lambda_o} < X_{s-}^k \leqslant \frac{y_k}{\mathbf{c}(Y_s)\beta\lambda_o}\right]}\right) \\
\qquad\qquad p^k(\mathrm{d}s, \Pi_{i=1}^n \mathrm{d}y_i), \text{ if } n_c < k \leqslant n.
\end{cases}
\tag{0.9}
$$

where $p(\sum_{i=1}^{n_c} dsdy_i du)$ is a Poisson measure with intensity $q = \sum_{i=1}^{n_c} dsdy_i du$.

Example: $\mathbf{B}(x)$: the flow induced by the ODE Netwon's eqs.(**Part I** of the model)

Stochastic part of the model

- We construct a process $(X_t, Y_t)_{t \geqslant 0}$ (solution of a SDE) can be seen as the evolution of the couple (position, size) of a sort of typical particle moving according to an Euclidean gradient continuous flow.
- The spatial Markov process Y_t has a coefficient **d** depending on the size of the particles while the fragmentation process X_t has a coefficient **c** depending on their spatial position.

Theorem

Let $z \in F$ and $x \in E$. Then the following SDEnDF with flow (0.9) has a weak solution with the initial distribution $\delta_{(z,x)}$ which is equal in distribution with the process $(Z_t, \mathbb{P}^{(z,x)})_{t \geqslant 0}$, induced by L.

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