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Modélisations déterministes et stochastiques

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An algebra of infinite matrices associated to the Weyl pseudo-differential calculus

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This talk is based on the papers:

- Horia D. Cornean, Bernard Helffer and Radu Purice:

  Matrix Representation of Magnetic Pseudo-Differential Operators via Tight Gabor Frames,smallskip

  Journal of Fourier Analysis and Applications 30, 21 (2024), 21 pages.

- Horia D. Cornean, Bernard Helffer and Radu Purice:

  A Beals criterion for magnetic pseudodifferential operators proved with magnetic Gabor frames,

  Communications in Partial Differential Equations 43 (8) (2018), pp. 1196 –1204.

Plan of the talk

1. Functional calculus with differential operators
2. The pseudo-differential calculus
3. The algebra of Hörmander type symbols
4. Frames in Hilbert spaces
5. Algebra of infinite matrices for Weyl-Hörmander calculus

Functional calculus with derivation operators.

An important tool in dealing with PDE problems is to use techniques from **Linear Operator Theory**.

We start with the partial derivation ∂_{x_j} on $\mathbb{R}^d$
defined as generator of the translation group $\big(\tau_z f\big)(x) := f(x+z)$
for f in a given linear topological space $\mathcal{V}(\mathbb{R}^d)$ of functions on $\mathbb{R}^d$:

$$f \mapsto \partial_{x_j} f := \lim_{\delta \searrow 0} \delta^{-1}\big(\tau_{\delta \mathfrak{e}_j} f - f\big), \quad \{\mathfrak{e}_j\}_{j=1}^d \text{ the canonical basis of} \mathbb{R}^d.$$

This point of view allows us to use the Fourier transform:

$$f \leadsto \big(\mathcal{F} f\big)(z) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} d\xi \, \exp\Big(-i \sum_{1 \leq j \leq d} \xi_j \, z_j\Big) f(\xi), \ \forall z \in \mathbb{R}^d$$

and define a rich functional calculus with the family of commuting operators $\{\partial_{x_j}\}_{j=1}^d$ (as convolution with the Fourier transform):

$$f \leadsto f(D) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} dx \, \big(\mathcal{F} f\big)(z) \, \tau_z$$

with integrals of operator valued functions on $\mathbb{R}^d$ as Bochner integrals.

Functional calculus with derivation operators. 2

There are several usual choices for the space $\mathcal{V}(\mathbb{R}^d)$:

- either $\mathcal{V}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d)$ with its Fréchet topology
- or $\mathcal{V}(\mathbb{R}^d) := C^\infty(\mathbb{R}^d)$ with its inductive limit topology of uniform convergence on compacts,
- or $\mathcal{V}(\mathbb{R}^d) := BC^\infty(\mathbb{R}^d)$ with its Fréchet topology,
- or $\mathcal{V}(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d)$ with its strong dual topology,
- or $\mathcal{V}(\mathbb{R}^d) := L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$ with its Banach space structure.

The basic choice used in developing the pseudo-differential calculus is $\mathcal{V}(\mathbb{R}^d) := L^2(\mathbb{R}^d)$ with its Hilbert space structure .
(with integrals of operator valued functions in the weak operatorial sense).
In agreement with the existing literature we shall denote by $U(z)$, for any $z \in \mathbb{R}^d$, the usual unitary representation of the translation τ_z on $L^2(\mathbb{R}^d)$.

All the operators $f(D)$ are translation invariant!

The inhomogeneous calculus.

For $1 \leq j \leq d$ let $\epsilon_j(\xi) := \xi_j$, so that $\epsilon_j(D) = -i\partial_{x_j}$.

A differential operator of order $N \in \mathbb{N}$ has the form $\sum_{|\alpha| \leq N} c_\alpha(x)\partial^\alpha$

that "may be associated with the function" $P(x,\xi) := \sum_{|\alpha| \leq N} c_\alpha(x)\big(i\epsilon(\xi)\big)^\alpha$.

Given a function $\phi \in L^1(\mathbb{R}^d)$ and its inverse Fourier transform:

$$\phi \rightsquigarrow (\mathcal{F}^-\phi)(\zeta) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} dx \, \exp\Big(i \sum_{1 \leq j \leq d} \zeta_j x_j\Big)\, \phi(x), \ \forall \zeta \in \mathbb{R}^d,$$

we notice that: $\phi(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} d\zeta \, \exp\Big(- i \sum_{1 \leq j \leq d} \zeta_j x_j\Big)\, (\mathcal{F}^-\phi)(\zeta)$,

if we denote by $\phi(Q)$ the operator of multiplication with $\phi(x)$ in $L^2(\mathbb{R}^d_x)$ and by $V(\zeta)$ the multiplication with $\exp\Big(- i \sum_{1 \leq j \leq d} \zeta_j x_j\Big)$ in $L^2(\mathbb{R}^d_x)$,

we can write: $\phi(Q) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} d\zeta \, V(\zeta)\, (\mathcal{F}^-\phi)(\zeta)$.

The phase space.

The above constructions put into evidence the following frame:

- two copies of $\mathbb{R}^d$ in duality: $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{X}^* \cong \mathbb{R}^d$
  with duality map: $\mathcal{X}^* \times \mathcal{X} \ni (\xi, x) \mapsto < \xi, x > := \sum_{1 \leq j \leq d} \xi_j \, x_j \in \mathbb{R}$;
- the role of the *'phase space'*: $\Xi := \mathcal{X} \times \mathcal{X}^*$
  with points denoted by: $X := (x, \xi), Y := (y, \eta), Z := (z, \zeta)$.
- a *'phase space'* Fourier transform:
  $$\big(\mathcal{F}_\Xi F\big)(z, \zeta) := (2\pi)^{-d} \int_\Xi dx \, d\xi \, \exp\big(i < \zeta, x > - i < \xi, z > \big) \, F(x, \xi)$$
- two unitary representations on $L^2(\mathcal{X})$:
  - $\mathcal{X} \ni z \mapsto U(z) \in \mathbb{U}\big(L^2(\mathcal{X})\big)$, $U(z) \equiv \tau_z$ - translation on $\mathcal{X}$,
  - $\mathcal{X}^* \ni \zeta \mapsto V(\zeta) \in \mathbb{U}\big(L^2(\mathcal{X})\big)$, $V(\zeta) := e^{-i < \zeta, Q >}$.

The Weyl commutation relation.

A simple calculus with some $f \in L^2(\mathcal{X})$ shows that:

$$\big(U(z)V(\zeta)f\big)(x) = \exp\big(-i < \zeta, x+z >\big) f(x+z) \tag{1}$$
$$= \exp\big(-i < \zeta, z >\big)\big(V(\zeta)U(z)f\big)(x) \tag{2}$$

The weyl system:

$W : \Xi \to \mathbb{U}\big(L^2(\mathcal{X})\big)$:

$$W(z,\zeta) := \exp\big(-(i/2) < \zeta, z >\big)\, V(\zeta)U(z).$$

Then: for any $Z = (z, \zeta)$ and $Y = (y, \eta)$ in Ξ

$$W(Z+Y) = \exp\big(-(i/2)[< \zeta, y > - < \eta, z >]\big) W(Z)W(Y);$$

$$W(Z)W(Y) = \exp\big(i[< \zeta, y > - < \eta, z >]\big) W(Y)W(Z).$$

a projective representation of $\Xi = \mathcal{X} \times \mathcal{X}^*$ on $L^2(\mathcal{X})$

The Weyl functional calculus.

Now we can extend the functional calculi $f(D)$ and $\phi(Q)$ above,
to a functional calculus $F(Q,D)$ for functions on the phase space Ξ,
using the 'phase space' Fourier transform
and the unitary 'projective' representation $W:\Xi\to\mathbb{U}\big(L^2(\mathcal{X})\big)$
We define:

$$\mathfrak{Op}:\mathscr{S}(\Xi)\to\mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}(\mathcal{X})\big),$$

$$\mathfrak{Op}(F):=(2\pi)^{-d}\int_{\Xi}dz\,d\zeta\,\big(\mathscr{F}_{\Xi}F\big)(z,\zeta)\,W(z,\zeta)$$

Explicitely we have for any $F\in\mathscr{S}(\Xi)$ and $\phi\in\mathscr{S}(\mathcal{X})$:

$$\big(\mathfrak{Op}(F)\phi\big)(x)\;=\;(2\pi)^{-d/2}\int_{\mathcal{X}}dy\,\int_{\mathcal{X}^*}d\eta\,e^{i<\eta,x-y>}\,F\big((x+y)/2,\eta\big)\,\phi(y)$$

and $\mathfrak{Op}(F)\phi\in\mathscr{S}(\mathcal{X})$.

We use the bijective linear map $\Upsilon:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ defined by:

$$\Upsilon(x,y)\;:=\;\big((x+y)/2\,,\,x-y\big).$$

The Weyl functional calculus 2

Proposition

For $F \in \mathscr{S}(\Xi)$, the operator $\mathfrak{Op}(F)$ extends by continuity to a linear operator in $\mathbb{B}\big(L^2(\mathcal{X})\big)$ with the estimation:

$$\big\|\mathfrak{Op}(F)\big\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq \sup_{z\in\mathcal{X}}\big\|\big(\mathbf{1}\otimes\mathcal{F}^{-}\big)F(z,\cdot)\big\|_{L^1(\mathcal{X})}.$$

It is an integral operator with integral kernel $\mathfrak{K}[F] \in C^\infty(\mathcal{X}\times\mathcal{X})$ having rapid decay in the variables orthogonal to the diagonal $\{(x,x)\in\mathcal{X}\times\mathcal{X}\}$.

Proposition

The map $\mathfrak{Op}:\mathscr{S}(\Xi)\to\mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}(\mathcal{X})\big)$ extends (by continuity in the weak distribution topology) to a map $\mathfrak{Op}:\mathscr{S}(\Xi)\to\mathcal{L}\big(\mathscr{S}'(\mathcal{X});\mathscr{S}(\mathcal{X})\big)$ that is a linear and topological isomorhism, for the topology of uniform convergence on bounded sets on $\mathcal{L}\big(\mathscr{S}'(\mathcal{X});\mathscr{S}(\mathcal{X})\big)$.

The Moyal product.

The last Proposition implies:

There exists a bilinear continuous map:

$$\mathscr{S}(\Xi)\times\mathscr{S}(\Xi)\ni(\Phi,\Psi)\mapsto\Phi\sharp\Psi\in\mathscr{S}(\mathcal{X})$$

such that: $\mathfrak{Op}(\Phi\sharp\Psi)=\mathfrak{Op}(\Phi)\,\mathfrak{Op}(\Psi).$

Explicitely:

$$(\Phi\sharp\Psi)(X)=\pi^{-2d}\int_{\Xi}dY\int_{\Xi}dZ\,e^{-2i[<\eta,z>-<\zeta,y>]}\,\Phi(X-Y)\,\Psi(X-Z).$$

Lemma

For any $(F,G)\in\big[\mathscr{S}(\Xi)\big]^2$ the following equality holds true:

$$\int_{\Xi}dX\,F(X)\,G(X)=\int_{\Xi}dX\,\big(F\sharp\,G\big)(X).$$

The Moyal algebra.

The previous Lemma allows to extend by duality the map $\mathfrak{Op}$ to a linear and topological isomorphism:

$$\mathfrak{Op}:\mathscr{S}'(\Xi)\to\mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$$

and define

The Moyal algebra:

$$\mathfrak{M}(\Xi):=\big\{F\in\mathscr{S}'(\Xi)\,|\ \mathfrak{Op}(F)\big[\mathscr{S}(\Xi)\big]\subset\mathscr{S}(\Xi)\big\}$$

with the Moyal product extended by the formula:

$$\big\langle (F\sharp\, G)\,,\,\phi\big\rangle:=\big\langle\, F\,,\,(G\sharp\Phi)\big\rangle$$

and involution given by the complex conjugation.

The integrals in the formulas are to be interpreted as oscillatory integrals.

The algebra of Hörmander symbols.

$$C^\infty_{\mathtt{pol,u}}(\mathcal{X}) := \big\{ f \in C^\infty(\mathcal{X}) \big| \, \exists N \in \mathbb{N}, \ \forall \alpha \in \mathbb{N}^d, \ \exists C_\alpha, \ \big(\partial^\alpha f\big)(x) \leq C_\alpha < x >^N \big\}.$$

- $C^\infty_{\mathtt{pol,u}}(\mathcal{X})$ and the polynomials in $\xi \in \mathcal{X}^*$ are subalgebras of $\mathfrak{M}(\Xi)$.
- the Moyal product restricted tu functions constant on $\mathcal{X}$ or on $\mathcal{X}^*$ is pointwise multiplication.

We consider the norms on $C^\infty_{\mathtt{pol,u}}(\mathcal{X})$ indexed by $(p,n,m) \in \mathbb{N}^3$:

$$\nu^p_{n,m}(F) := \sup_{(x,\xi)\in\Xi} \sum_{|\alpha|\leq n} \sum_{|\beta|\leq m} < \xi >^{-p} \, \big| \big(\partial^\alpha_x \partial^\beta_\xi F\big)(x,\xi) \big|$$

Let: $S^p_0(\Xi) := \big\{ F \in C^\infty_{\mathtt{pol,u}}(\Xi) \big| \, \nu^p_{n,m}(F) < \infty, \ \forall (n,m) \in \mathbb{N}^2 \big\}$

$$S^\infty_0(\Xi) := \bigcup_{p\in\mathbb{R}} S^p_0(\Xi) \text{ and } S^{-\infty}(\Xi) := \bigcap_{p\in\mathbb{R}} S^p_0(\Xi).$$

The Weyl-Hörmander pseudo-differential calculus.

The composition property:

The Moyal product extends to a continuous map:

$$\sharp : S^p_0(\Xi) \times S^q_0(\Xi) \to S^{p+q}_0(\Xi), \quad \forall (p,q) \in \mathbb{R} \times \mathbb{R}.$$

The Calderón-Vaillancourt Theorem:

$\mathfrak{Op}\big[S^0_0(\Xi)\big] \subset \mathbb{B}\big(L^2(\mathcal{X})\big)$ and:

$$\exists n(d) > 0, \exists c(d) > 0 : \qquad \big\|\mathfrak{Op}(F)\big\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq c(d)\, \nu^0_{n(d),n(d)}(F).$$

The Beals criterion:

If $Q_j := \mathfrak{Op}(e_j)$ with $e_j(x,\xi) := x_j$ and $D_k := \mathfrak{Op}(\epsilon_k)$ with $\epsilon_k(x,\xi) := \xi_k$

$$F \in S^0_0(\Xi) \iff \big[Q_{j_1}, \ldots \big[Q_{j_n}, \big[D_{k_1}, \ldots \big[D_{k_m}, \mathfrak{Op}(F)\big] \ldots\big] \in \mathbb{B}\big(L^2(\mathcal{X})\big)$$

$$\forall (n,m) \in \mathbb{N}^2 \text{ and for all families } \{j_1, \ldots, j_n\} \text{ and } \{k_1, \ldots, k_m\}.\}\}$$

Parseval frames

Definition

Given a Hilbert space $\mathcal{H}$ and an infinite set of indices $\mathbb{A}$, a family $\{w_a\}_{a\in\mathbb{A}} \subset \mathcal{H}$ is called a Parseval frame, when the following equalities hold true:

$$\forall\, v \in \mathcal{H} : \qquad \|v\|_{\mathcal{H}}^2 = \sum_{a\in\mathbb{A}} (w_a\,,\,v)_{\mathcal{H}}^2.$$

Given a Parseval frame $\{w_a\}_{a\in\mathbb{A}}$ in a Hilbert space $\mathcal{H}$,

- let us denote by $\ell^2(\mathbb{A})$ the Hilbert space of square-summable complex sequences indexed by $\mathbb{A}$
- and let us define the map: $\mathcal{H} \ni v \overset{\mathfrak{F}_{\mathbb{A}}}{\mapsto} \{(w_a\,,\,v)_{\mathcal{H}}\}_{a\in\mathbb{A}} \in \ell^2(\mathbb{A})$.

By the above Definition, the map $\mathfrak{F}_{\mathbb{A}} : \mathcal{H} \to \ell^2(\mathbb{A})$ is an isometry but in general it is not surjective!

We are interested only in the situation when:

- $\mathscr{H}$ is a separable Hilbert space.
- $\mathbb{A}$ is countable.

The uncountable situation is rather easily handled but is out of our interest.

The adjoint of the Parseval frame isometry.

Let us consider the adjoint of the isometry $\mathfrak{F}_{\mathbb{A}}:\mathcal{H}\rightarrow\ell^2(\mathbb{A})$, taking into account the Riesz antilinear isomorphism:

$$\big(\mathfrak{F}_{\mathbb{A}}^*\{(w_a\,,\,v)_{\mathcal{H}}\}_{a\in\mathbb{A}}\,,\,u\big)_{\mathcal{H}}=\big(\{(w_a\,,\,v)_{\mathcal{H}}\}_{a\in\mathbb{A}}\,,\,\mathfrak{F}_{\mathbb{A}}u\big)_{\ell^2(\mathbb{A})}$$
$$=\sum_{a\in\mathbb{A}}\overline{(w_a\,,\,v)_{\mathcal{H}}}\,(w_a\,,\,u)_{\mathcal{H}}$$

Fixing any bijection $\mathbb{N}\xrightarrow{\sim}\mathbb{A}$, approximating with finite sums and using the Cauchy-Schwartz inequality we prove that the above series is absolutely convergent and has the limit:

$$(v\,,\,u)_{\mathcal{H}}\;=\;\big(\mathfrak{F}_{\mathbb{A}}^*\{(w_a\,,\,v)_{\mathcal{H}}\}_{a\in\mathbb{A}}\,,\,u\big)_{\mathcal{H}},\quad\forall u\in\mathcal{H},$$

i.e. $\mathfrak{F}_{\mathbb{A}}^*\{(w_a\,,\,v)_{\mathcal{H}}\}_{a\in\mathbb{A}}=v$ for any $v\in\mathcal{H}$.

Notice that: $\mathfrak{F}_{\mathbb{A}}^*\,\mathfrak{F}_{\mathbb{A}}=\mathbf{1}_{\mathcal{H}}$
and $\mathfrak{F}_{\mathbb{A}}\,\mathfrak{F}_{\mathbb{A}}^*=\mathfrak{P}_{\mathcal{H}}$ orthogonal projection on $\mathbb{R}\mathrm{ge}\,\mathfrak{F}_{\mathbb{A}}\subset\ell^2(\mathbb{A})$.

Parseval frames: matrices of operators.

Suppose given a Parseval frame $\{w_a\}_{a\in\mathbb{A}}$ in a Hilbert space $\mathcal{H}$.

- We may now define the map
  $\widetilde{\mathfrak{F}}_{\mathbb{A}} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\ell^2(\mathbb{A})), \quad \widetilde{\mathfrak{F}}_{\mathbb{A}} T := \mathfrak{F}_{\mathbb{A}} T \mathfrak{F}_{\mathbb{A}}^*$
  and notice that it is a C^*-algebra homomorphism.
- If we consider some linear operator $T: \mathcal{D}(T) \to \mathcal{H}$ and suppose that $w_a \in \mathcal{D}(T)$ for any $a \in \mathbb{A}$. We define the infinite matrix :
  $$\mathbf{M}[T]_{a,a'} := \big(w_a \, , \, Tw_{a'} \big)_{\mathcal{H}}$$
- If also $w_a \in \mathcal{D}(T^*)$ for any $a \in \mathbb{A}$, then:
  $$T\varphi = \sum_{(a,a')\in\mathbb{A}^2} \mathbf{M}[T]_{a,a'} \big(w_a \, , \, \varphi\big)_{\mathcal{H}} w_{a'}$$
- We notice that for any $T \in \mathbb{B}(\mathcal{H})$, the matrix $\mathbf{M}[T]_{a,a'}$ coincides with the matrix of the operator $\widetilde{\mathfrak{F}}_{\mathbb{A}}$ in the canonical Hilbertian bases of $\ell^2(\mathbb{A})$.

Construction of a Parseval frame in $L^2(\mathcal{X})$.

- $\{\mathfrak{e}_j\}_{j=1}^d$ is the canonical basis of $\mathcal{X} = \mathbb{R}^d$ and $\Gamma := \bigoplus_{1\leq j\leq d} \mathbb{Z}\mathfrak{e}_j$;
- let $\{\mathfrak{e}_j^*\}_{j=1}^d \subset \mathcal{X}^*$ such that $< \mathfrak{e}_j^*, \mathfrak{e}_k >:= 2\pi\delta_{j,k}$ and $\Gamma_* := \bigoplus_{1\leq j\leq d} \mathbb{Z}\mathfrak{e}_j^*$;
- let $\{\tau_\gamma g\}_{\gamma\in\Gamma}$ a quadratic Γ-partition of unity, i.e.: $g \in C_0^\infty\big(\mathcal{X}; [0,1]\big)$ such that
  - $\sum_{\gamma\in\Gamma} \big[\big(\tau_\gamma g\big)(x)\big]^2 = 1$ for any $x \in \mathcal{X}$;
  - $\operatorname{supp} g \subset (-1,1)^d$.
- for any $\gamma^* \in \Gamma_*$ let $\vartheta_{\gamma^*}(x) := (2\pi)^{-d/2} \exp\Big(i\frac{<\gamma^*,x>}{(2\pi)}\Big)$.
- for any pair $(\gamma, \gamma^*) \in \Gamma \times \Gamma_*$ let us define the function:

$$\mathcal{G}_{\gamma,\gamma^*} := \vartheta_{\gamma^*}\,\big(\tau_\gamma g\big) \in C_0^\infty(\mathcal{X}).$$

- let $\widetilde{\Gamma} := \Gamma \times \Gamma_*$ and denote its points by $\widetilde{\gamma} := (\gamma, \gamma^*)$.

Construction of a Parseval frame 2.

Remark

Given any $v \in L^2(\mathcal{X})$ the product $[v\,\mathcal{G}_{\widetilde{\alpha}}](x) = \vartheta_{\alpha^*}(x)\,[v\,g_\alpha](x)$ has $\mathrm{supp}[v\,\mathcal{G}_\alpha] \subset (-1,1)^d$.

As $(-1,1) \subset (-\pi,\pi)$, if we denote by $[v\,g_\alpha]_\circ$ the 2π-periodic extension of $[v\,g_\alpha]$, that is a function of class $BC^\infty(\mathcal{X})$, then:

$$\int_{\mathcal{X}} dx\,\vartheta_{\alpha^*}(x)\,[v\,g_\alpha]_\circ(x) = \int_{(-1,1)^d} dx\,\vartheta_{\alpha^*}(x)\,[v\,g_\alpha]_\circ(x) = \int_{(-\pi,\pi)^d} dx\,\vartheta_{\alpha^*}(x)\,[v\,g_\alpha]_\circ(x)$$

is the Fourier coefficient $\widehat{[v\,g_\alpha]}_{-\alpha^*}$ of the periodic function $[v\,g_\alpha]_\circ$.

Proposition A

The family $\{\mathcal{G}_{\widetilde{\gamma}}\}_{\widetilde{\gamma}\in\widetilde{\Gamma}}$ defines a Parseval frame in $L^2(\mathcal{X})$.

Proof: We prove the following apparently stronger result:

$$\forall v, f \in L^2(\mathcal{X}), \quad \big(v, f\big)_{L^2(\mathcal{X})} = \sum_{\widetilde{\alpha}\in\widetilde{\Gamma}} \big(v\,,\,\mathcal{G}_{\widetilde{\alpha}}\big)_{L^2(\mathcal{X})} \big(\mathcal{G}_{\widetilde{\alpha}}\,,\,f\big)_{L^2(\mathcal{X})}$$

with the series converging for the $\ell^2(\widetilde{\Gamma})$ norm.

The above Remark implies that:

$$\big(v\,,\,\mathcal{G}_{\widetilde{\alpha}}\big)_{L^2(\mathcal{X})} = \widehat{[\overline{v}\,g_\alpha]}_{-\alpha^*} = \overline{\widehat{[v\,g_\alpha]}_{\alpha^*}},\ \big(\mathcal{G}_{\widetilde{\alpha}}\,,\,f\big)_{L^2(\mathcal{X})} = \widehat{[f\,g_\alpha]}_{\alpha^*}$$

and Parseval Theorem implies that for any $N \in \mathbb{N}$:

$$\lim_{M\nearrow\infty} \sum_{|\alpha|\leq N} \sum_{|\alpha^*|\leq M} \big(v\,,\,\mathcal{G}_{\widetilde{\alpha}}\big)_{L^2(\mathcal{X})} \big(\mathcal{G}_{\widetilde{\alpha}}\,,\,f\big)_{L^2(\mathcal{X})} =$$

$$= \sum_{|\alpha|\leq N} \int_{\tau_{-\alpha}(-1,1)^d} dx\, \overline{[v\,g_\alpha](x)}[f\,g_\alpha](x) = \sum_{|\alpha|\leq N} \int_{\mathcal{X}} dx\, \overline{[v\,g_\alpha](x)}[f\,g_\alpha](x)$$

$$\underset{N\nearrow\infty}{\longrightarrow} \big(v\,,\,f\big)_{L^2(\mathcal{X})},$$

(Dominated Convergence Theorem and the quadratic Γ-partition of unity condition). □

Matrices of Weyl operators with Hörmander symbols.

Main Theorem

Given some $p \in \mathbb{R}$ and some $\Phi \in \mathscr{S}'(\Xi)$, the following two statements are equivalent:
(i) Φ belongs to $S^p_0(\mathcal{X}^*, \mathcal{X})$.
(ii) For any $(n_1, n_2) \in \mathbb{N}^2$ there exists a constant $C(\Phi) > 0$ such that the $\widetilde{\Gamma}$-indexed matrix of $\mathfrak{Op}(\Phi)$ in the frame $\{\mathcal{G}_{\widetilde{\gamma}}\}_{\widetilde{\gamma}\in\widetilde{\Gamma}}$ has the following behavior:

$$\sup_{(\widetilde{\alpha},\widetilde{\beta})\in\widetilde{\Gamma}^2} < \alpha - \beta >^{n_1} < \alpha^* - \beta^* >^{n_2} < \alpha^* + \beta^* >^{-p} \left| \mathbf{M}[\mathfrak{Op}(\Phi)]_{\widetilde{\alpha},\widetilde{\beta}} \right| \leq C(\Phi)$$

Moreover, the constant $C(\Phi)$ only depends on some norm $\nu^{p,0}_{m_1,m_2}(\Phi)$.

Definition

We denote by $\mathscr{M}^p_{\widetilde{\Gamma}}$ the family of $\widetilde{\Gamma} \times \widetilde{\Gamma}$-indexed complex matrices verifying the estimations in the above Theorem.

Proof of the Main Theorem. Direct implication.

Suppose that $\Phi \in S^p_0(\mathcal{X}^*, \mathcal{X})$:

$$\mathbf{M}[\mathfrak{Op}(\Phi)]_{\widetilde{\alpha},\widetilde{\beta}} = (2\pi)^{-d} \int_{\mathcal{X}} dx \int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\eta \times$$

$$\times\, e^{-\frac{i}{2\pi}<\alpha^*,x-\alpha>} g(x-\alpha)\, e^{i<\eta,x-y>}\, \Phi\big((x+y)/2,\eta\big)\, e^{\frac{i}{2\pi}<\beta^*,y-\beta>} g(y-\beta).$$

We make the change of variables:

$$\mathcal{X} \times \mathcal{X} \ni (x,y) \mapsto (z,v) := \big((x+y-\alpha-\beta)/2, x-y-\alpha+\beta\big) \in \mathcal{X} \times \mathcal{X}$$

and the following similar bijective change of indices:

$$\begin{array}{lcl} \Gamma \times \Gamma \ni (\alpha,\beta) & \mapsto & (\mu,\nu) := \big(\alpha+\beta, \alpha-\beta\big) \in [\Gamma]^2_\circ, \\ \Gamma_* \times \Gamma_* \ni (\alpha^*,\beta^*) & \mapsto & (\mu^*,\nu^*) := \big(\alpha^*+\beta^*, \beta^*-\alpha^*\big) \in [\Gamma_*]^2_\circ, \end{array}$$

where $[\Gamma]^2_\circ := \{(\mu,\nu) \in \Gamma \times \Gamma\,,\, (-1)^{\mu+\nu} = 1\}$ and similarly for Γ_*.

With these new indices we consider the corresponding matrix

$$\widetilde{\mathbb{M}}[\mathfrak{Op}(\Phi)]_{\widetilde{\mu}(\widetilde{\alpha},\widetilde{\beta}),\widetilde{\nu}(\widetilde{\alpha},\widetilde{\beta})} := \mathbb{M}[\mathfrak{Op}(\Phi)]_{\widetilde{\alpha},\widetilde{\beta}},$$

Proof of the Main Theorem. Direct implication 2.

$$\widetilde{\mathbb{M}}[\mathfrak{Op}(\Phi)]_{\widetilde{\mu},\widetilde{\nu}} = (2\pi)^{-d} e^{\frac{i}{4\pi}<\mu^*,\nu>} \int_{\mathcal{X}} dz\, e^{\frac{i}{2\pi}<\nu^*,z>} \int_{\mathcal{X}^*} d\zeta\, e^{i<\zeta,\nu>} \times$$
$$\times \int_{\mathcal{X}} dv\, e^{i<\zeta,v>} g(z+v/2) g(z-v/2)\, \Phi\big(z+\mu, \zeta+(1/4\pi)\mu^*\big),$$

We use the integral by parts of oscillatory factors, noticing that:

$$\forall m_1 \in \mathbb{N}: \quad e^{\frac{i}{2\pi}<\nu^*,z>} = \left[\left(\frac{1-\Delta_z}{<\nu^*>^2}\right)^{m_1} e^{\frac{i}{2\pi}<\nu^*,z>}\right] \rightsquigarrow <\nu^*>^{-2m_1}$$
$$\forall m_2 \in \mathbb{N}: \quad e^{i<\zeta,\nu>} = \left[\left(\frac{1-\Delta_\zeta}{<\nu>^2}\right)^{m_2} e^{i<\zeta,\nu>}\right] \rightsquigarrow <\nu>^{-2m_2}$$
$$\forall m_3 \in \mathbb{N}: \quad e^{i<\zeta,v>} = \left[\left(\frac{1-\Delta_v}{<\zeta>^2}\right)^{m_3} e^{i<\zeta,v>}\right] \rightsquigarrow <\zeta>^{-2m_3}.$$

Moreover $\big|z_j \pm (v_j/2)\big| \leq 1$ implies that $|z_j| \leq 1$ and $|v_j| \leq 2$.

Taking $2m_3 \geq |p| + d + 1$ and using the inequality:

$$<\zeta+(1/4\pi)\mu^*>^p <\zeta>^{-|p|} \leq 2^{|p/2|} <(1/4\pi)\mu^*>^p .$$

Proof of the Main Theorem. Inverse implication.

We start from the identities:
$$\begin{cases} \mathfrak{K}[\Phi] = (2\pi)^{-d/2}\big[\big(\mathbf{1}_{\mathcal{X}}\otimes\mathcal{F}^{-}\big)\,\Phi\big]\circ\Upsilon \\ \mathfrak{K}[\Phi] = \sum_{(\widetilde{\alpha},\widetilde{\beta})\in\widetilde{\Gamma}^2}\mathbf{M}[\mathfrak{Op}(\Phi)]_{\widetilde{\alpha},\widetilde{\beta}}\,\mathcal{G}_{\widetilde{\alpha}}\otimes\mathcal{G}_{\widetilde{\beta}} \end{cases}$$

and obtain that
$$\Phi = (2\pi)^{d/2}\big[\big(\mathbf{1}_{\mathcal{X}}\otimes\mathcal{F}\big)\Big[\sum_{(\widetilde{\alpha},\widetilde{\beta})\in\widetilde{\Gamma}^2}\mathbf{M}[\mathfrak{Op}(\Phi)]_{\widetilde{\alpha},\widetilde{\beta}}\Big(\mathcal{G}_{\widetilde{\alpha}}\otimes\mathcal{G}_{\widetilde{\beta}}\Big)\circ\Upsilon^{-1}\Big]$$

.

The rapid off-diagonel decay of the matrix elements and a procedure very similar to the previous one allow to control the convergence of the integrals and series in the above formula and obtain the conclusion (i). □

Matrix calculus

The rapid off-diagonal decay of the matrices in $\mathscr{M}^p_{\tilde{\Gamma}}$ allows to extend the usual matrix product to pairs of elements from $\mathscr{M}^p_{\tilde{\Gamma}}$.

Proposition.

For $\Phi \in S^p(\Xi)$ and $\Psi \in S^q(\Xi)$ the following equality is true:

$$\mathbf{M}[\mathfrak{Op}(\Phi)]\,\mathbf{M}[\mathfrak{Op}(\Psi)] = \mathbf{M}[\mathfrak{Op}(\Phi\sharp\Psi)].$$

One easily obtains now:

- the Composition Property of the Moyal product (a corollary of our Main Theorem, using the above observation),
- the Calderon-Vaillancourt Theorem (a direct application of the Schur-Holmgren criterion),

Thank you for your attention.

Happy Anniversary Professor Jaiani !