Dedicated to Dr. Constantin VÂRSAN
on the occasion of his 70th birthday

LINEAR 2D HYBRID SYSTEMS OVER SPACES OF REGULATED FUNCTIONS

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We consider 2D generalized hybrid systems with coefficient matrices of bounded variation in the state equation and regulated matrix functions in the output equation, while the controls are regulated vector functions. The formulæ of the state and of the input-output map of these systems are obtained on the basis of a generalized 2D variation of parameters formula. The concepts of controllability and reachability are characterized in this framework by the means of a 2D generalized Gramian, and their relationship is emphasized.

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1. INTRODUCTION

An important class of 2D systems is represented by the hybrid models, their behaviour depending on two variables, one continuous and the other one discrete [4], [8]. They were used as models in the study of linear repetitive processes [1], [9] with practical applications in iterative learning control synthesis [6] or in long-wall coal cutting and metal rolling.

One aim of this paper is to extend the study of the hybrid systems to the general framework represented by the space of regulated functions. The topic of regulated functions is studied in a series of monographs or papers (e.g., [2], [3], [11]). We use Tvrdý's results [11], [12], concerning the properties of the Perron-Stieltjes integral with respect to regulated functions and the differential equation in this space. A class of 2D generalized hybrid linear systems is considered, with controls over the space of regulated functions and coefficient matrices of bounded variation in the state equation and regulated matrix functions in the output equation. A generalized 2D variation of parameters formula is obtained and formulæ for the state and the input-output map of these systems are derived. These results allows us to extend to this framework the concepts of controllability and reachability (see for instance [5]). Necessary and sufficient conditions of controllability and reachability are obtained by means of a 2D generalized Gramian and the relationship of these concepts is emphasized.

We shall use the following definitions and notation. A function $f : [a, b] \to \mathbf{R}$ which posseses finite side limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $\|f\| = |f(a)| + \mathrm{var}_a^b f$ also is a Banach space; the Banach space of n-vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n\times m}$ denotes the space of $n \times m$ matrices with entries in $BV(a, b)$. The set of functions $f : [a, b] \times \mathbf{Z} \Rightarrow \mathbf{R}$ such that $f(\cdot, k) \in G(a, b)$ $(BV(a, b))$, $\forall k \in \mathbf{Z}$, will be denoted $G_1(a, b)$ $(BV_1(a, b))$, and similar significances will have the above mentioned spaces with subscript 1 (G_1^n, BV_1^n, $BV_1^{n\times m}$).

A pair $D = (d, s)$ where $d = \{t_0, t_1, \ldots, t_m\}$ is a division of $[a, b]$ (i.e., $a = t_0 < t_1 < \ldots < t_m = b$) and $s = \{s_1, \ldots, s_m\}$ verifies $t_{j-1} \le s_j \le t_j$, $j = 1, \ldots, m$, is called a *partition* of $[a, b]$.

A function $\delta : [a, b] \to (0, +\infty)$ is called a *gauge* on $[a, b]$. Given a gauge δ, a partition (d, s) is said to be δ-*fine* if

$$[t_{j-1}, t_j] \subset (s_j - \delta(s_j), s_j + \delta(s_j)), \quad j = 1, \ldots, m.$$

Given the functions $f, g : [a, b] \to \mathbf{R}$ and a partition $D = (d, s)$ of $[a, b]$, consider the integral sum

$$S_D(f\Delta g) = \sum_{j=1}^{m} f(s_j)(g(t_j) - g(t_{j-1})).$$

Definition 1.1. The number $I \in \mathbf{R}$ is said to be the *Perron-Stieltjes* (*Kurzweil*) *integral of* f *with respect to* g *from* a *to* b, denoted $\int_a^b f\mathrm{d}g$ or $\int_a^b f(t)\mathrm{d}g(t)$, if for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that $|I - S_D(f\Delta g)| < \varepsilon$ for all δ-fine partitions D of $[a, b]$.

Given $f \in G(a, b)$ and $g \in G([a, b] \times [a, b])$ we define the differences $\Delta^+, \Delta^-, \Delta$ and $\Delta_s^+, \Delta_s^-, \Delta_s$ by $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$, $\Delta_s^+ g(t, s) = g(t, s+) - g(t, s)$, $\Delta_s^- g(t, s) = g(t, s) - g(t, s-)$; $\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a, b]$ and, similarly, for g we can define $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$ with respect to the argument t. We denote by $\sum\limits_t$ the sum $\sum\limits_{t\in\mathbf{D}}$ where $\mathbf{D} = \mathbf{D}^-(f) \cup \mathbf{D}^+(f) \cup \mathbf{D}^-(g) \cup \mathbf{D}^+(g)$.

Following [10] and [11], let us recall some basic properties of the Perron-Stieltjes integral. The existence theorem of the Perron-Stieltjes integral $\int_a^b f\mathrm{d}g$ for $f \in BV(a,b)$ and $g \in G(a,b)$, due to Tvrdý [11], is essential for our treatment.

THEOREM 1.2 ([11, Theorems 2.8 and 2.15]). *If $f \in G(a,b)$ and $g \in BV(a,b)$, then the Perron-Stieltjes integrals $\int_a^b f\mathrm{d}g$ and $\int_a^b g\mathrm{d}f$ exist and*

$$\int_a^b f\mathrm{d}g + \int_a^b g\mathrm{d}f = \tag{1.1}$$
$$= f(b)g(b) - f(a)g(a) + \sum_t [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)].$$

THEOREM 1.3 ([11, Proposition 2.16]). *If $\int_a^b f\mathrm{d}g$ exists, then the function $h(t) = \int_a^t f\mathrm{d}g$ is defined on $[a,b]$ and*

i) *if $g \in G(a,b)$ then $h \in G(a,b)$ and, for any $t \in [a,b]$*

$$\Delta^+ h(t) = f(t)\Delta^+ g(t), \quad \Delta^- h(t) = f(t)\Delta^- g(t); \tag{1.2}$$

ii) *if $g \in BV(a,b)$ and f is bounded on $[a,b]$, then $h \in BV(a,b)$.*

THEOREM 1.4 ([11, Theorem 2.19]). *Let f, g, h be such that h is bounded on $[a,b]$ and the integral $\int_a^b f\mathrm{d}g$ exists. Then the integral $\int_a^b h(t)f(t)\mathrm{d}g(t)$ exists if and only if the integral $\int_a^b h(t)\,\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]$ exists and, in this case,*

$$\int_a^b h(t)f(t)\mathrm{d}g(t) = \int_a^b h(t)\,\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]. \tag{1.3}$$

THEOREM 1.5 (Dirichlet formula, [7, Theorem I.4.32]). *If $h : [a,b] \times [a,b] \to \mathbf{R}$ is a bounded function and $\mathrm{var}_a^b h(s,\cdot) + \mathrm{var}_a^b h(\cdot\,,t) < \infty$, $\forall t,s \in [a,b]$, then*

$$\int_a^b \mathrm{d}g(t)\left(\int_a^t h(s,t)\mathrm{d}f(s)\right) = \int_a^b \left(\int_s^b \mathrm{d}g(t)h(s,t)\right)\mathrm{d}f(s) + \tag{1.4}$$
$$+ \sum_t [\Delta^- g(t)h(t,t)\Delta^- f(t) - \Delta^+ g(t)h(t,t)\Delta^+ f(t)]$$

for any $f, g \in BV(a,b)$.

2. GENERALIZED DIFFERENTIAL EQUATIONS

The symbol

$$\mathrm{d}x = \mathrm{d}[A]x + \mathrm{d}g, \tag{2.1}$$

where $A \in BV^{n\times n}$ and $g \in G^n(a,b)$, is said to be a *generalized linear differential equation* (GLDE) in the space of regulated functions.

Definition 2.1. A function $x : [a,b] \to \mathbf{R}^n$ is said to be a *solution* of GLDE (2.1) if

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + g(t) - g(t_0) \tag{2.2}$$

for any $t, t_0 \in [a,b]$. If x satisfies the initial condition

$$x(t_0) = x_0 \tag{2.3}$$

for given $t_0 \in [a,b]$ and $x_0 \in \mathbf{R}^n$, then x is called the *solution of the initial value problem* (2.1), (2.3).

THEOREM 2.2 ([10, Theorem III.2.10]). *Assume that for any* $t \in [a,b]$ *the matrix* $A \in BV^{n\times n}$ *is such that*

$$\det[I + \Delta^+ A(t)] \det[I - \Delta^- A(t)] \neq 0. \tag{2.4}$$

Then there exists a unique matrix valued function $U : [a,b] \times [a,b] \to \mathbf{R}^{n\times n}$ *satisfying the equation*

$$U(t,s) = I + \int_{s}^{t} \mathrm{d}[A(\tau)]U(\tau,s) \tag{2.5}$$

for any $(t,s) \in [a,b] \times [a,b]$*;* $U(t,s)$ *is called the fundamental matrix solution of the homogeneous equation*

$$\mathrm{d}x = \mathrm{d}[A]x \tag{2.6}$$

(*or the fundamental matrix of* A) *and for any* $\tau, t, s \in [a,b]$ *it has the following properties:*

$$U(t,s) = U(t,\tau)U(\tau,s); \tag{2.7}$$

$$U(t,t) = I; \tag{2.8}$$

$$U(t+,s) = [I + \Delta^+ A(t)]U(t,s), \quad U(t-,s) = [I - \Delta^- A(t)]U(t,s); \tag{2.9}$$
$$U(t,s+) = U(t,s)[I + \Delta^+ A(s)]^{-1}, \quad U(t,s-) = U(t,s)[I - \Delta^- A(s)]^{-1};$$

$$U(t,s)^{-1} = U(s,t); \tag{2.10}$$

there exists a constant $M > 0$ *such that*

$$|U(t,s)| + \mathrm{var}_a^b U(t,\cdot) + \mathrm{var}_a^b U(\cdot\,,s) + \mathrm{v}(U) < M, \tag{2.11}$$

where $\mathrm{v}(U)$ *is the twodimensional Vitali variation of* U *on* $[a,b] \times [a,b]$ ([10, *Definition* I.6.1]).

Some methods for computing the fundamental matrix $U(t,s)$ were provided in [7].

From [10, Theorem III.3.1] and [12, Proposition 2.5], we deduce

THEOREM 2.3 (variation-of-parameters formula). *If* $A \in BV^{n\times n}$ *satisfies condition* (2.4), *then the initial value problem* (2.1), (2.3) *has a unique solution given by*

$$x(t) = U(t,t_0)x_0 + g(t) - g(t_0) - \int_{t_0}^{t} \mathrm{d}_s[U(t,s)](g(s) - g(t_0)). \tag{2.12}$$

If $g \in G^n$ $(g \in BV^n)$ *then* $x \in G^n$ $(x \in BV^n)$.

3. 2D GENERALIZED HYBRID SYSTEMS

The linear spaces $X = G_1^n$, $U = G_1^m$ and $Y = G_1^p$ are called respectively the *state*, *input* and *output* spaces while $T = [a,b] \times \mathbf{Z}$ is the *time set.*

Definition 3.1. A 2D *generalized hybrid system* (2Dgh) is a quintuplet

$$\Sigma = (A_1(t,k), A_2(t,k), B(t,k), C(t,k), D(t,k)) \in$$
$$\in BV_1^{n\times n} \times BV_1^{n\times n} \times BV_1^{n\times m} \times G_1^{p\times n} \times G_1^{p\times m},$$

where $A_1(t,k)A_2(t,k) = A_2(t,k)A_1(t,k)$, $\forall (t,k) \in T$, with the state equation

$$\begin{aligned} \mathrm{d}x(t,k+1) = \mathrm{d}[A_1(t,k+1)]x(t,k+1) + A_2(t,k)\mathrm{d}x(t,k) - \\ -\mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k) + B(t,k)\mathrm{d}u(t,k) \end{aligned} \tag{3.1}$$

and the output equation

$$y(t,k) = C(t,k)x(t,k) + D(t,k)u(t,k). \tag{3.2}$$

Let $U(t,t_0;k)$ be the fundamental matrix of $A_1(t,k)$, $k \in \mathbf{Z}$, and $F(t;k,k_0)$ the discrete fundamental matrix of $A_2(t,k)$, $t \in [a,b]$, i.e.,

$$F(t;k,k_0) = \begin{cases} A_2(t,k-1)A_2(t,k-2)\cdots A_2(t,k_0) & \text{for } k > k_0, \\ I_n & \text{for } k = k_0. \end{cases}$$

Since $A_1(t,k)$ and $A_2(t,k)$ are commutative matrices for any $(t,k) \in T$, by the Peano-Baker type formula for U [7] and the definition of F, $U(t,t_0;k)$ and $F(t;k,k_0)$ are commutative matrices, too. We shall use the following notation: $\Delta^+ f(s,l) = f(s+,l) - f(s,l)$, $\Delta_s^+ U(t,s;k) = U(t,s+;k) - U(t,s;k)$ and we similarly define $\Delta^- f(s,l)$ and $\Delta_s^- U(t,s;k)$.

Definition 3.2. A vector $x_0 \in X$ is called the *initial state* of Σ at the moment $(t_0,k_0) \in T$ if

$$x(t,k_0) = U(t,t_0;k_0)x_0 \quad \text{and} \quad x(t_0,k) = F(t_0;k,k_0)x_0 \tag{3.3}$$

$\forall(t,k) \in T$ with $(t,k) \geq (t_0,k_0)$.

PROPOSITION 3.3 (2D generalized variation of parameters formula). *If*

$$\det[(I-\Delta^- A_i(t,k))(I+\Delta^+ A_i(t,k))] \neq 0, \quad i=1,2,\ \forall t \in [a,b],\ k \in \mathbf{Z}, \tag{3.4}$$

then the solution of the generalized differential-difference equation

$$\mathrm{d}x(t,k+1) = \mathrm{d}[A_1(t,k+1)]x(t,k+1) + A_2(t,k)\mathrm{d}x(t,k) - \\ -\mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k) + \mathrm{d}f(t,k) \tag{3.5}$$

with the initial conditions (3.3) *is*

$$x(t,k) = U(t,t_0;k)F(t_0;k,k_0)x_0 + \\ + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)\mathrm{d}f(s,l) + \\ + \sum_{a \leq s < t} \Delta_s^+ U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)\Delta^+ f(s,l) - \\ - \sum_{a < s \leq t} \Delta_s^- U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)\Delta^- f(s,l). \tag{3.6}$$

Proof. We shall use the notation

$$\mathrm{d}g(t,k) = \mathrm{d}x(t,k) - \mathrm{d}[A_1(t,k)]x(t,k). \tag{3.7}$$

Equation (3.6) becomes

$$\mathrm{d}g(t,k+1) = A_2(t,k)\mathrm{d}g(t,k) + \mathrm{d}f(t,k). \tag{3.8}$$

Then

$$\mathrm{d}g(t,k_0+1) = A_2(t,k_0)\mathrm{d}g(t,k_0) + \mathrm{d}f(t,k_0) = \\ = F(t;k_0+1,k_0)\mathrm{d}g(t,k_0) + F(t;k_0+1,k_0+1)\mathrm{d}f(t,k_0).$$

Let us assume that

$$\mathrm{d}g(t,k) = F(t;k,k_0)\mathrm{d}g(t,k_0) + \sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{3.9}$$

Then, by (3.8), (3.9) and the definition of $F(t;k,k_0)$, we have

$$\mathrm{d}g(t,k+1) = A_2(t,k)F(t;k,k_0)\mathrm{d}g(t,k_0) + \\ + \sum_{l=k_0}^{k-1} A_2(t,k)F(t;k,l+1)\mathrm{d}f(t,l) + \mathrm{d}f(t,k) = \\ = F(t;k+1,k_0)\mathrm{d}g(t,k_0) + \sum_{l=k_0}^{k} F(t;k+1,l+1)\mathrm{d}f(t,l).$$

Hence (3.9) holds $\forall k > k_0$. Moreover, from (3.3), (3.7) and (2.6) we obtain

$$\mathrm{d}g(t,k_0) = \mathrm{d}x(t,k_0) - \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ = \mathrm{d}[U(t,t_0;k_0)]x_0 - \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ = \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0 - \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0 = 0,$$

hence, (3.9) becomes

$$\mathrm{d}g(t,k) = \sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{3.10}$$

Equation (3.7) is equivalent to the generalized differential equation $\mathrm{d}x(t,k) = \mathrm{d}[A_1(t,k)]x(t,k) + \mathrm{d}g(t,k)$ with the solution

$$x(t,k) = U(t,t_0;k)x(t_0,k) - \\ - \int_{t_0}^{t} \mathrm{d}_s[U(t,s;k)] \int_{t_0}^{s} \mathrm{d}g(\tau,k) + \int_{t_0}^{t} \mathrm{d}g(s,k) \tag{3.11}$$

given by Theorem 2.3. By Theorem 1.3, (3.11) can be written as

$$x(t,k) = U(t,t_0;k)x(t_0,k) + \int_{t_0}^{t} U(t,s;k)\mathrm{d}\int_{t_0}^{s} \mathrm{d}g(\tau,k) + \\ + \sum_{a\le s<t} \Delta_s^+ U(t,s;k)\Delta^+ \int_{t_0}^{s} \mathrm{d}g(\tau,k) - \sum_{a<s\le t} \Delta_s^- U(t,s;k)\Delta^- \int_{t_0}^{s} \mathrm{d}g(\tau,k). \tag{3.12}$$

Replace (3.10) in (3.12). We obtain (3.6) from (3.12) on account of the equation

$$\int_{t_0}^{t} \mathrm{d}g(s,k) = \sum_{l=k_0}^{k-1} \int_{t_0}^{t} F(s;k,l+1)\mathrm{d}f(s,l),$$

equations (3.3) and Theorems 1.3, 1.4 and 1.5. □

Proposition 3.4. *If* (3.4) *holds, then the state of the system at the moment* $(t,k) \in T$ *determined by the initial state* x_0 *at the moment* $(t_0,k_0) \in T$

and the control $u : [t_0, t] \times \{k_0, k_0 + 1, \ldots, k - 1\} \to \mathbf{R}^m$ *is*

$$x(t,k) = U(t,t_0;k)F(t_0;k,k_0)x_0+ \tag{3.13}$$

$$+\int_{t_0}^{t}\sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l)+$$

$$+\sum_{a\le s<t}\Delta_s^{+}U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^{+}u(s,l)-$$

$$-\sum_{a<s\le t}\Delta_s^{-}U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^{-}u(s,l).$$

Proof. The state equation (3.1) can be obtained from (3.3) by replacing $f(t,k) = \int_{t_0}^{t} B(s,k)\mathrm{d}u(s,k)$. Then (3.13) follows from (3.6) and (1.2). □

Now, we replace the state $x(t,k)$ given by (3.13) into the output equation of Σ (3.2) and deduce

THEOREM 3.5. *Under the hypothesis* (3.4) *the input-output map of the* 2Dgh *system* Σ (3.1), (3.2) *is*

$$y(t,k) = C(t,k)U(t,t_0;k)F(t_0;k,k_0)x_0+ \tag{3.14}$$

$$+\int_{t_0}^{t}\sum_{l=k_0}^{k-1} C(t,k)U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l)+D(t,k)u(t,k)+$$

$$+\sum_{a\le s<t}C(t,k)\Delta_s^{+}U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^{+}u(s,l)-$$

$$-\sum_{a<s\le t}C(t,k)\Delta_s^{-}U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^{-}u(s,l).$$

COROLLARY 3.6. *If* $u \in G_1^m$ $(u \in BV_1^m)$ *then* $x \in G_1^n$ *and* $y \in G_1^p$ $(x \in BV_1^n$ *and* $y \in BV_1^p)$.

Proof. We apply Theorems 2.3 and 3.5 and Proposition 3.4. □

Definition 3.7. The *space of admissible controls* is the set

$$\mathcal{U} = \{u \in G_1^m(a,b) \mid D_t^{+}(A_i(\cdot\,,k)\cap D_t^{+}(u(\cdot\,,k)) = \emptyset,$$
$$D_t^{-}(A_i(\cdot\,,k))\cap D_t^{-}(u(\cdot\,,k)) = \emptyset,\ i=1,2,\ \forall k\in\mathbf{Z}\}.$$

COROLLARY 3.8. *If $u \in \mathcal{U}$ then the state and the output of the system Σ are given by the formulæ*

$$x(t,k) = U(t,t_0;k)F(t_0;k,k_0)x_0 + \tag{3.15}$$
$$+ \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l)$$

and

$$y(t,k) = C(t,k)U(t,t_0;k)F(t_0;k,k_0)x_0 + \tag{3.16}$$
$$+ \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k)U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l) + D(t,k)u(t,k).$$

Remark 3.9. The continuous-discrete 2D systems [8] with state equation

$$\frac{\partial x}{\partial t}(t,k+1) = \tilde{A}_1(t,k+1)x(t,k+1) +$$
$$+\tilde{A}_2(t,k)\frac{\partial x}{\partial t}(t,k) - \tilde{A}_1(t,k)\tilde{A}_2(t,k)x(t,k) + \tilde{B}(t,k)\tilde{u}(t,k)$$

are special cases of 2Dgh (3.1) with absolutely continuous matrices $A_i(t,k) = \int_a^t \tilde{A}_i(s,k)\mathrm{d}s$, $i = 1,2$, and controls $u(t,k) = \int_a^t \tilde{u}(s,k)\mathrm{d}s$.

4. CONTROLLABILITY AND REACHABILITY OF 2Dgh SYSTEMS

The system Σ with the state equation (3.1) will be denoted $\Sigma = (A_1, A_2, B)$. We assume that (3.4) holds.

The triple $(x,t,k) \in \mathbf{R}^n \times [a,b] \times \mathbf{Z}$ is a *phase* of Σ if x is the state of Σ at the moment (t,k) (i.e., if (3.6) holds with $x(t,k) = x$). For $(t,k),(s,l) \in T$, $(s,l) < (t,k)$ means $s \leq t$, $l \leq k$ and $(s,l) \neq (t,k)$. If (3.13) holds for $(t_0,k_0) < (t,k)$, one says that the control u transfers the phase (x_0,t_0,k_0) to the phase (x,t,k). We denote by $\mathcal{T}$ the set $\mathcal{T} = [t_0,t] \times [k_0+1,\ldots,k\} \subset T$.

Definition 4.1. A state $x \in \mathbf{R}^n$ is said to be *G-reachable/$\mathcal{U}$-reachable on $\mathcal{T}$* if there exists a control $u : \mathcal{T} \to \mathbf{R}^m$, $u \in G_1^m/u \in \mathcal{U}$, which transfers the phase $(0,t_0,k_0)$ to (x,t,k).

A state $x \in \mathbf{R}^n$ is said to be *G-controllable/$\mathcal{U}$-controllable* on $\mathcal{T}$ if there exists a control $u : \mathcal{T} \to \mathbf{R}^m$, $u \in G_1^m/u \in \mathcal{U}$, which transfers the phase (x,t_0,k_0) to $(0,t,k)$.

By replacing $x(t,k)$ by x and x_0 by 0 in (3.13) and (3.15), respectively, we obtain

PROPOSITION 4.2. *The state $x \in \mathbf{R}^n$ is G-reachable/$\mathcal{U}$-reachable iff $\exists u \in G_1^m/u \in \mathcal{U}$ such that*

$$x = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l) + \tag{4.1}$$

$$+ \sum_{a \le s < t} \Delta_s^+ U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)u(s,l) -$$

$$- \sum_{a < s \le t} \Delta_s^- U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)\Delta^- u(s,l)$$

and

$$x = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l). \tag{4.2}$$

Similarly, for $x(t,k) = 0$ and $x_0 = x$ in (3.13)/(3.15), we obtain

PROPOSITION 4.3. *A state $x \in \mathbf{R}^n$ is G-controllable/$\mathcal{U}$-controllable if $\exists u \in G_1^m/u \in \mathcal{U}$ such that*

$$U(t,t_0;k)F(t_0;k,k_0)x_0 = \tag{4.3}$$

$$= - \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l) -$$

$$- \sum_{a \le s < t} \Delta_s^+ U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)\Delta^+ u(s,l) +$$

$$+ \sum_{a < s \le t} \Delta_s^- U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)\Delta^- u(s,l)$$

and

$$U(t,t_0;k)F(t_0;k,k_0)x_0 = \tag{4.4}$$

$$= - \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)B(s,l)\mathrm{d}u(s,l).$$

Definition 4.4. If $\forall x \in \mathbf{R}^n$ is G-reachable/$\mathcal{U}$-reachable on $\mathcal{T}$ one says that the system Σ is *completely G-reachable/completely $\mathcal{U}$-reachable* on $\mathcal{T}$.

If $\forall x \in \mathbf{R}^n$ is *G-controllable/$\mathcal{U}$-controllable* on $\mathcal{T}$ one says that the system Σ is *completely G-controllable/$\mathcal{U}$-controllable* on $\mathcal{T}$.

We consider a matrix, called the 2D *reachability Gramian* of the system Σ and defined by a Perron-Stieltjes integral as

(4.5)
$$\mathcal{R}(\mathcal{T}) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l{+}1)B(s,l)B(s,l)^{\mathrm{T}}F(s;k,l{+}1)^{\mathrm{T}}U(t,s;k)^{\mathrm{T}}\mathrm{d}s.$$

THEOREM 4.5. *There exists a control $u \in \mathcal{U}$ which transfers the phase (t_0, k_0, x_0) to the phase (t, k, x) if and only if*

$$x - U(t,t_0;k)F(t_0;k,k_0)x_0 \in \mathcal{R}(\mathcal{T}). \tag{4.6}$$

The *proof* is omitted.

COROLLARY 4.6. *The set of states of Σ which are $\mathcal{U}$-reachable on $\mathcal{T}$ is the linear space $X_{\mathcal{T}} = \operatorname{Im}\mathcal{R}(\mathcal{T})$.*

Proof. With $x_0 = 0$ in (4.6), we deduce that x is $\mathcal{U}$-reachable on $\mathcal{T}$ iff $x \in \operatorname{Im}\mathcal{R}(\mathcal{T})$. □

Similarly, by replacing x by 0 and x_0 by x in (4.6), we get

COROLLARY 4.7. *A state x is $\mathcal{U}$-controllable on $\mathcal{T}$ iff*

$$U(t,t_0;k)F(t_0;k,k_0)x \in \operatorname{Im}\mathcal{R}(\mathcal{T}). \tag{4.7}$$

THEOREM 4.8. *A system $\Sigma = (A_1, A_2, B)$ is completely $\mathcal{U}$-reachable on $\mathcal{T}$ iff*

$$\operatorname{rank}\mathcal{R}(\mathcal{T}) = n. \tag{4.8}$$

Proof. By Corollary 4.6, Σ is completely $\mathcal{U}$-reachable on $\mathcal{T}$ iff $\operatorname{Im}\mathcal{R}(T) = \mathbf{R}^n$, i.e., iff (4.7) holds. □

THEOREM 4.9. *If the matrix $A_2(t_0, l)$ is nonsingular $\forall l \in \{k_0, k_0+1, \dots, k-1\}$, then the system Σ is completely $\mathcal{U}$-controllable on $\mathcal{T}$ iff* (4.8) *holds.*

Proof. The matrix $U(t,t_0;k)$ is nonsingular by Theorem 2.2 (see (2.10)) and since $A_2(t_0,l)$ is nonsingular, $F(t_0;k,k_0)$ is nonsingular, too. Then condition (4.7) is equivalent to $x \in \operatorname{Im}(F(t_0;k,k_0)^{-1}U(t_0,t;k)\mathcal{R}(\mathcal{T})$. It follows that Σ is completely $\mathcal{U}$-controllable on $\mathcal{T}$ iff $\operatorname{rank} F(t_0;k,k_0)^{-1}U(t_0,t;k)\mathcal{R}(\mathcal{T}) = n$, condition which is equivalent to (4.8). □

By Theorems 4.8 and 4.9 we get (in accordance with the discrete character of Σ with respect to k) the result below.

THEOREM 4.10. *If the system Σ is completely $\mathcal{U}$-reachable on $\mathcal{T}$, then Σ is completely $\mathcal{U}$-controllable on $\mathcal{T}$.*

If Σ is completely $\mathcal{U}$-controllable on $\mathcal{T}$ and the matrix $A_2(t_0, l)$ is non-singular $\forall l \in \{k_0, k_0+1, \ldots, k-1\}$, then Σ is completely $\mathcal{U}$-reachable.

Conclusion. We studied the state space representation for a class of time-varying 2D hybrid systems in the general case of spaces over the set of regulated functions, and some results concerning controllability and reachability of these systems were obtained. This study can be continued by analyzing for this class other important concepts as stability, observability, the problem of realisations, etc.

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