EXPLICIT RECURRENCE CRITERIA FOR SYMMETRIC GRADIENT TYPE DIRICHLET FORMS SATISFYING A HAMZA TYPE CONDITION

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In this note, we present explicit conditions for symmetric gradient type Dirichlet forms to be recurrent. This type of Dirichlet form is typically strongly local and hence, associated to a diffusion. We consider the one dimensional case and the multidimensional case, as well as the case with reflecting boundary conditions. Our main achievement is that the explicit results are obtained under quite weak assumptions on the closability, hence, regularity of the underlying coefficients. Especially in dimension one, where a Hamza type condition is assumed, the construction of the sequence of functions $(u_n)_{n\in\mathbb{N}}$ in the Dirichlet space that determine recurrence works for quite general Dirichlet forms but is still explicit.

AMS 2010 Subject Classification: 31C25, 60J60, 60G17.

Key words: symmetric Dirichlet forms, diffusion processes, recurrence criteria.

1. INTRODUCTION

For all notions and results that may not be defined or cited in this introduction we refer to [1].

Let E be a locally compact separable metric space and μ be a σ-finite measure on the Borel σ-algebra of E. Consider a regular symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,\mu)$ with associated resolvent $(G_\alpha)_{\alpha>0}$. Then it is well-known that there exists a Hunt process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with lifetime ζ such that the resolvent $R_\alpha f(x) = E_x[\int_0^\infty e^{-\alpha t} f(X_t)\mathrm{d}t]$ of $\mathbb{M}$ is a q.c. μ-version of $G_\alpha f$ for all $\alpha > 0$, $f \in L^2(E,\mu)$, f bounded (see [1], Theorems 4.2.3 and 7.2.1). Here, E_x denotes the expectation w.r.t. P_x. Then $(\mathcal{E}, D(\mathcal{E}))$ is (called) *recurrent*, iff for any $f \in L^1(E,\mu)$ which is strictly positive, we have

$$E_x[\int_0^\infty f(X_t)\mathrm{d}t] = \infty \text{ for } \mu\text{-a.e. } x \in E. \tag{1}$$

The last is equivalent to the existence of a sequence $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \leq u_n \leq 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$ μ-a.e. and $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$ (cf. [1], Lemma 1.6.4 (ii) and Theorem 1.6.3) and ([7], Corollary 2.4 (iii)). In

particular, recurrence implies conservativeness, which means that $P_x(\zeta = \infty) = 1$ for quasi every $x \in E$ (see [7], [Corollary 2.4]). If the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is additionally *irreducible*, then some more refined recurrence statements than (1) can be made (cf. [2], Chapter 4.7 and [3]), and if the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ satisfies the absolute continuity condition (cf. [1], (4.2.9)), then the statements hold pointwise and not only for quasi every $x \in E$ (see [1], Problem 4.6.3).

A sufficient condition for recurrence determined by the μ-volume growth of balls is derived in ([8], Theorem 3) for energy forms, *i.e.* Dirichlet forms that are given by a carré du champs. The condition is very general, but in some cases not explicit, *i.e.* difficult to verify (cf. *e.g.* [6], Proposition 3.3. and Theorem 3.11) where the determination of the μ-volume of balls is tedious). Moreover, in [8] it is throughout assumed that the underlying Dirichlet form is irreducible, which we do not assume. In [7] and ([1], Theorem 1.6.7) the recurrence determining sequence $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ is explicitly constructed, but the underlying Dirichlet form fulfills stronger (explicit) assumptions than in [8], and moreover [7] and ([1], Theorem 1.6.7) concern only variants of the (a_{ij})-case.

This note concerns an intermediate approach, which resembles the construction in ([1], Theorem 1.6.7), but under quite more general assumptions. In particular, we consider reference measures that are different to the Lebesgue measure. The conditions for the existence of a recurrence determining sequence $(u_n)_{n\in\mathbb{N}}$ given here are (as in [1] and [8]) sufficient. However, in dimension one they turn out to be equivalent, whenever there exists a natural scale (see *e.g.* ([7], Lemma 3.1) and ([6], Theorem 3.11).

2. THE ONE DIMENSIONAL NON-REFLECTED CASE

Let $\varphi \in L^1_{loc}(\mathbb{R}, \mathrm{d}x)$ with $\varphi > 0$ $\mathrm{d}x$-a.e. and let μ be the σ-finite measure defined by $\mathrm{d}\mu = \varphi \mathrm{d}x$. In particular, it then follows that μ is σ-finite and μ has full support, *i.e.*

$$\int_V \varphi(x)\mathrm{d}x > 0 \qquad \text{for all } V \subset \mathbb{R},\ V \text{ non-empty and open.}$$

Let σ be a (measurable) function such that $\sigma > 0$ $\mathrm{d}x$-a.e. and such that $\sigma\varphi \in L^1_{loc}(\mathbb{R}, \mathrm{d}x)$. Consider the symmetric bilinear form

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}} \sigma(x) f'(x) g'(x) \mu(\mathrm{d}x), \qquad f, g \in C_0^\infty(\mathbb{R})$$

on $L^2(\mathbb{R}, \mu)$. Here, f' denotes the derivative of a function f and $C_0^\infty(\mathbb{R})$ denotes the set of all infinitely differentiable functions on $\mathbb{R}$ with compact support. Let

U be the largest open set in $\mathbb{R}$ such that

$$\frac{1}{\sigma\varphi} \in L^1_{loc}(U, \mathrm{d}x)$$

and assume

$$\mathrm{d}x(\mathbb{R} \setminus U) = 0. \tag{2}$$

Then $(\mathcal{E}, C_0^\infty(\mathbb{R}))$ is closable on $L^2(\mathbb{R}, d\mu)$ by ([1], Theorem 3.1.6). Denote the closure by $(\mathcal{E}, D(\mathcal{E}))$. We want to present sufficient conditions for the recurrence of the symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Here, we recall $(\mathcal{E}, D(\mathcal{E}))$ is called recurrent (in the sense of [1]) if there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \le u_n \le 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$ $\mathrm{d}x$-a.e. and

$$\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0.$$

Remark 2.1. Under quite weak regularity assumptions on σ and φ one can show using integration by parts that the generator L corresponding to $(\mathcal{E}, D(\mathcal{E}))$ (for the definition of generator see [1] or [4]) is given by

$$Lf = \frac{\sigma}{2} f' \; prime + \frac{1}{2}\Big(\sigma' + \sigma\frac{\varphi'}{\varphi}\Big) f',$$

i.e. for $f \in D(L) \subset D(\mathcal{E})$ we have

$$-\int_{\mathbb{R}} Lf \cdot g \mathrm{d}\mu = \mathcal{E}(f, g).$$

In particular (assuming again that everything is sufficiently regular) choosing

$$\varphi(x) = \frac{1}{\sigma(x)} e^{\int_0^x \frac{2b}{\sigma}(s)\mathrm{d}s}$$

for some (freely chosen) function b, we get

$$Lf = \frac{\sigma}{2} f' \; prime + bf'.$$

Therefore, our framework is suitable for the description of diffusion type operators in dimension one with concrete coefficients.

Since $U \subset \mathbb{R}$ is open, U is the disjoint union (we use the symbol $\cup$ to denote this) of countably (finite or infinite) many open intervals.

There are five possible cases that we summarize in the following theorem.

THEOREM 2.2. *$(\mathcal{E}, D(\mathcal{E}))$ is recurrent if one of the following conditions holds:*

(i) *$U = (-\infty, \infty)$ and*

$$\int_{-\infty}^0 \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \int_0^\infty \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \infty.$$

(ii) $U = (-\infty, a) \cup V \cup (b, \infty)$ *where* V *is some open set (so either* V *is empty if* $a = b$*, or* V *is non-empty if* $a < b$*) and*

$$\int_{-\infty}^{a-1} \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \int_{b+1}^{\infty} \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \infty.$$

(iii) $U = \bigcup_{n\in\mathbb{N}} I_{-n} \cup V \cup \bigcup_{n\in\mathbb{N}} I_n$, $I_n = (x_n, x_{n+1})$, $x_n < x_{n+1}$, $I_{-n} = (x_{-n}, x_{-n+1})$, $x_{-n} < x_{-n+1}$, $n \in \mathbb{N}$, V *some open set, and*

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)\mathrm{d}s \longrightarrow \infty$$
$$a_{-n} = b_n := \int_{d_{-n}}^{c_{-n}} \frac{1}{\sigma\varphi}(s)\mathrm{d}s \longrightarrow \infty$$

as $n \to \infty$*, where*

$$c_n = \frac{x_{n+1} + x_n}{2}, \quad n \in \mathbb{Z} \setminus \{0\},$$

and for $n \in \mathbb{N}$

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1} - c_n}{n} & x_{n+1} - c_n \le 1, \end{cases}$$

$$d_{-n} := \begin{cases} x_{-n} + \frac{1}{n} & c_{-n} - x_{-n} > 1, \\ x_{-n} + \frac{c_{-n} - x_{-n}}{n} & c_{-n} - x_{-n} \le 1. \end{cases}$$

(iv) $U = \bigcup_{n\in\mathbb{N}} I_{-n} \cup V \cup (b, \infty)$, $I_{-n} = (x_{-n}, x_{-n+1})$, $x_{-n} < x_{-n+1}$, $n \in \mathbb{N}$, V *some open set,*

$$\int_{b+1}^{\infty} \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \infty,$$

and

$$a_{-n} = b_n := \int_{d_{-n}}^{c_{-n}} \frac{1}{\sigma\varphi}(s)\mathrm{d}s \longrightarrow \infty$$

as $n \to \infty$ *where for* $n \in \mathbb{N}$

$$c_{-n} = \frac{x_{-n} + x_{-n+1}}{2}$$

and

$$d_{-n}(x) := \begin{cases} x_{-n} + \frac{1}{n} & c_{-n} - x_{-n} > 1, \\ x_{-n} + \frac{c_{-n} - x_{-n}}{n} & c_{-n} - x_{-n} \le 1. \end{cases}$$

(v) $U = (-\infty, a) \cup V \cup \bigcup_{n\in\mathbb{N}} I_n$, $I_n = (x_n, x_{n+1})$, $x_n < x_{n+1}$, $n \in \mathbb{N}$, V *some open set,*

$$\int_{-\infty}^{a-1} \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \infty,$$

and

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)\mathrm{d}s \longrightarrow \infty$$

as $n \to \infty$ *where for* $n \in \mathbb{N}$

$$c_n = \frac{x_{n+1} + x_n}{2}$$

and

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1}-c_n}{n} & x_{n+1} - c_n \le 1. \end{cases}$$

Remark 2.3. With the obvious modifications, Theorem 2.2 can be reformulated for Dirichlet forms that are given as the closure of

$$\frac{1}{2}\int_V \sigma(x)f'(x)g'(x)\mu(\mathrm{d}x), \qquad f, g \in C_0^\infty(V)$$

where V is an arbitrary open and connected set in $\mathbb{R}$. We omit this here to avoid trivial complications.

Proof of Theorem 2.2. (i) By ([1], Theorem 1.6.3), it suffices to find a sequence $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \le u_n \le 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$. Define sequences a_n and b_n by

$$a_n := \int_0^n \frac{1}{\sigma\varphi}(s)\mathrm{d}s, \qquad b_n := \int_{-n}^0 \frac{1}{\sigma\varphi}(s)\mathrm{d}s.$$

Since $1/\sigma\varphi \in L^1_{loc}(\mathbb{R}, \mathrm{d}x)$, these are well defined and converge to ∞. Let

$$u_n(x) := \begin{cases} 1 - \frac{1}{a_n}\int_0^x \frac{1}{\sigma\varphi}(t)\mathrm{d}t & x \in [0, n], \\ 1 - \frac{1}{b_n}\int_x^0 \frac{1}{\sigma\varphi}(t)\mathrm{d}t & x \in [-n, 0], \\ 0 & \text{elsewhere.} \end{cases}$$

For each $n \in \mathbb{N}$, u_n has compact support and is bounded. Moreover, since $1/\sigma\varphi \in L^1_{loc}(\mathbb{R}, \mathrm{d}x)$, $u_n(x)$ is differentiable at every Lebesgue point of $1/\sigma\varphi$, hence, $\mathrm{d}x$-a.e. Furthermore, we can easily check $u_n \nearrow 1$ $\mathrm{d}x$-a.e. as $n \to \infty$ and $u_n'(x)$ exists $\mathrm{d}x$-a.e. with

$$u_n'(x) = \begin{cases} -\frac{1}{a_n}\frac{1}{\sigma\varphi}(x) & x \in [0, n], \\ \frac{1}{b_n}\frac{1}{\sigma\varphi}(x) & x \in [-n, 0], \\ 0 & \text{elsewhere.} \end{cases}$$

Thus, it remains to show that $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$. Let η be a standard mollifier on $\mathbb{R}$. Set $\eta_\epsilon(x) = \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$ so that $\int_{\mathbb{R}} \eta_\epsilon \mathrm{d}x = 1$ and so that the support of η_ϵ is in $(-\epsilon, \epsilon)$. Then $\eta_\epsilon * u_n \in C_0^\infty(\mathbb{R})$ and $(\eta_\epsilon * u_n)' = \eta_\epsilon * u_n'$, $n \in \mathbb{N}$. We have

$$(\eta_\epsilon * u_n(x) - u_n(x)) = \int_{\mathbb{R}} [u_n(x-y) - u_n(x)]\eta_\epsilon(y)\mathrm{d}y,$$

$$|\eta_\epsilon * u_n(x) - u_n(x)|^2 \le \int_{\mathbb{R}} |u_n(x-y) - u_n(x)|^2 \eta_\epsilon(y)\mathrm{d}y,$$

$$\begin{aligned}\int_{\mathbb{R}}(\eta_\epsilon * u_n(x) - u_n(x))^2\varphi(x)\mathrm{d}x &\le \int_{\mathbb{R}}\int_{\mathbb{R}}|u_n(x-y)-u_n(x)|^2\eta_\epsilon(y)\mathrm{d}y\varphi(x)\mathrm{d}x\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}|u_n(x-y)-u_n(x)|^2\varphi(x)\mathrm{d}x\eta_\epsilon(y)\mathrm{d}y\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}|u_n(x-\epsilon y)-u_n(x)|^2\varphi(x)\mathrm{d}x\eta(y)\mathrm{d}y.\end{aligned}$$

If we let $g(y) := \int_{\mathbb{R}}|u_n(x-y)-u_n(x)|^2\varphi(x)\mathrm{d}x$, then $g(0) = 0$ and g is continuous and bounded. Thus, by the Dominated Convergence Theorem $\lim_{\epsilon\to 0}\eta_\epsilon * u_n = u_n$ in $L^2(\mathbb{R},\mu)$. Furthermore, for $0<\epsilon<1$

$$\begin{aligned}\mathcal{E}(\eta_\epsilon * u_n, \eta_\epsilon * u_n) &= \int_{\mathbb{R}}\Big(\eta_\epsilon * u_n'(x)\Big)^2\sigma(x)\varphi(x)\mathrm{d}x\\ &\le \|u_n'\|^2_{L^1(\mathbb{R},dx)}\int_{-n-1}^{n+1}\sigma(x)\varphi(x)\mathrm{d}x\end{aligned}$$

Since $|\eta_\epsilon * u_n'(x)| \le \|u_n'\|_{L^1(\mathbb{R},\mathrm{d}x)} < \infty$, we have

$$\sup_{0<\epsilon<1}\mathcal{E}(\eta_\epsilon * u_n, \eta_\epsilon * u_n) < \infty.$$

Thus, from ([4], Lemma 2.12), $u_n \in D(\mathcal{E})$ for all $n\in\mathbb{N}$. Furthermore,

$$\begin{aligned}\mathcal{E}(u_n,u_n) &= \frac{1}{2}\int_{\mathbb{R}}u_n'(x)u_n'(x)\sigma(x)\varphi(x)\mathrm{d}x\\ &= \frac{1}{2}\Big[\frac{1}{a_n^2}\int_0^n\frac{1}{\sigma\varphi}(x)\mathrm{d}x + \frac{1}{b_n^2}\int_{-n}^0\frac{1}{\sigma\varphi}(x)\mathrm{d}x\Big]\\ &= \frac{1}{2}\Big[\frac{1}{a_n}+\frac{1}{b_n}\Big].\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathcal{E}(u_n,u_n) = 0$. *i.e.* $(\mathcal{E}, D(\mathcal{E}))$ is recurrent. (ii) Let

$$u_n(x) := \begin{cases}1 & x\in[a-1,b+1],\\ 1-\frac{1}{a_n}\int_{b+1}^x\frac{1}{\sigma\varphi}(t)\mathrm{d}t & x\in[b+1,b+1+n],\\ 1-\frac{1}{b_n}\int_x^{a-1}\frac{1}{\sigma\varphi}(t)\mathrm{d}t & x\in[a-1-n,a-1],\\ 0 & \text{elsewhere},\end{cases}$$

where $a_n = \int_{b+1}^{b+1+n}\frac{1}{\sigma\varphi}(s)\mathrm{d}s$, $b_n = \int_{a-1-n}^{a-1}\frac{1}{\sigma\varphi}(s)\mathrm{d}s$. Then $(u_n)_{n\in\mathbb{N}}$ satisfies the desired properties and determines recurrence.
(iii) Let

$$u_n(x) := \begin{cases}1 & x\in[c_{-n},c_n],\\ 1-\frac{1}{a_n}\int_{c_n}^x\frac{1}{\sigma\varphi}(t)\mathrm{d}t & x\in[c_n,d_n],\\ 1-\frac{1}{b_n}\int_x^{c_{-n}}\frac{1}{\sigma\varphi}(t)\mathrm{d}t & x\in[d_{-n},c_{-n}],\\ 0 & \text{elsewhere},\end{cases}$$

where $a_n = \int_{c_n}^{d_n}\frac{1}{\sigma\varphi}(s)\mathrm{d}s$, $b_n = \int_{d_{-n}}^{c_{-n}}\frac{1}{\sigma\varphi}(s)\mathrm{d}s$. Then $(u_n)_{n\in\mathbb{N}}$ satisfies the desired properties and determines recurrence.

(iv) and (v) are combinations of (ii) and (iii) and are proved by combining the proofs of (ii) and (iii). □

Remark 2.4. Note that we do not assume that $(\mathcal{E}, D(\mathcal{E}))$ is irreducible. As a non-trivial example consider the following: Let $S = \{x_i \in \mathbb{R} | \, i \in \mathbb{Z}\}$ with $x_i < x_{i+1}$ for all $i \in \mathbb{Z}$ and assume S does not have an accumulation point in $\mathbb{R}$. For $\alpha \geq 1$, define a function φ by

$$\varphi(x) = |x - x_i|^\alpha, x \in \Big[\frac{x_i + x_{i-1}}{2}, \frac{x_i + x_{i+1}}{2}\Big], i \in \mathbb{Z}.$$

Then $\varphi > 0$ on $\mathbb{R} \setminus S$, hence, dx-a.e. Assume $\sigma \equiv 1$, then, since $1/\varphi \in L^1_{loc}(\mathbb{R} \setminus S, \mathrm{d}x)$, (2) is also satisfied. Thus, the symmetric bilinear form $(\mathcal{E}, C_0^\infty(\mathbb{R}))$ defined by

$$\mathcal{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}} f'(x)g'(x)\mu(\mathrm{d}x), \qquad f, g \in C_0^\infty(\mathbb{R})$$

is closable on $L^2(\mathbb{R}, \mathrm{d}\mu)$ where $\mathrm{d}\mu = \varphi \mathrm{d}x$. Define the sequences a_n, b_n, c_n and d_n as in the Theorem 2.2 (iii), then

$$a_n = \int_{c_n}^{d_n} \frac{1}{\varphi}(s)\mathrm{d}s = \int_{c_n}^{d_n} \frac{1}{(x_{n+1} - s)^\alpha}\mathrm{d}s.$$

(i) If $\alpha = 1$, then

$$\begin{aligned} a_n &= -\log(x_{n+1} - s)|_{c_n}^{d_n} \\ &= -\log(x_{n+1} - d_n) + \log(x_{n+1} - c_n). \end{aligned}$$

In this case, if $x_{n+1} - c_n > 1$, then $x_{n+1} - d_n = \frac{1}{n}$ and

$$\begin{aligned} a_n &= -\log\Big(\frac{1}{n}\Big) + \log(x_{n+1} - c_n) \\ &> \log n. \end{aligned}$$

And if $x_{n+1} - c_n \leq 1$, then $x_{n+1} - d_n = \frac{x_{n+1} - c_n}{n}$ and

$$\begin{aligned} a_n &= -\log\Big(\frac{x_{n+1} - c_n}{n}\Big) + \log(x_{n+1} - c_n) \\ &= \log n. \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = \infty$ if $\alpha = 1$.

(ii) If $\alpha > 1$, then

$$\begin{aligned} a_n &= \int_{c_n}^{d_n} \frac{1}{(x_{n+1} - s)^\alpha}\mathrm{d}s \\ &= \frac{-1}{1-\alpha}(x_{n+1} - s)^{1-\alpha}|_{c_n}^{d_n} \\ &= \frac{1}{\alpha - 1}\Big[(x_{n+1} - d_n)^{1-\alpha} - (x_{n+1} - c_n)^{1-\alpha}\Big]. \end{aligned}$$

In this case, if $x_{n+1} - c_n > 1$, then $x_{n+1} - d_n = \frac{1}{n}$ and

$$\begin{aligned} a_n &= \frac{1}{\alpha - 1}\Big[\big(\frac{1}{n}\big)^{1-\alpha} - (x_{n+1} - c_n)^{1-\alpha}\Big] \\ &> \frac{1}{\alpha - 1}(n^{\alpha-1} - 1). \end{aligned}$$

And if $x_{n+1} - c_n \leq 1$, then $x_{n+1} - d_n = \frac{x_{n+1}-c_n}{n}$ and

$$\begin{aligned} a_n &= \frac{1}{\alpha - 1}\Big[\big(\frac{x_{n+1} - c_n}{n}\big)^{1-\alpha} - (x_{n+1} - c_n)^{1-\alpha}\Big] \\ &= \frac{1}{\alpha - 1}\Big[(x_{n+1} - c_n)^{1-\alpha}\big\{\big(\frac{1}{n}\big)^{1-\alpha} - 1\big\}\Big] \\ &\geq \frac{1}{\alpha - 1}(n^{\alpha-1} - 1). \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = \infty$ if $\alpha \geq 1$. In the same way, $\lim_{n\to\infty} b_n = \infty$. Therefore, $(\mathcal{E}, D(\mathcal{E}))$ is recurrent, thus in particular conservative (cf. [1], Theorems 1.6.5 and 1.6.6). Since $(\mathcal{E}, D(\mathcal{E}))$ is also strongly local, the process associated to $(\mathcal{E}, D(\mathcal{E}))$ is a conservative diffusion (cf. [1]). Moreover, since by ([1], Example 3.3.2) $\mathrm{Cap}(\{x_i\}) = 0$, the sets (x_i, x_{i+1}) are all invariant, *i.e.*

$$p_t 1_{(x_i, x_{i+1})}(x) = 0, \quad x \notin (x_i, x_{i+1}),\ \forall i \in \mathbb{Z}$$

where p_t is the transition semigroup of (the process associated to) the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. But

$$\mathrm{d}\mu((x_i, x_{i+1})) \neq 0, \qquad \mathrm{d}\mu(\mathbb{R} \setminus (x_i, x_{i+1})) \neq 0 \quad \forall i \in \mathbb{Z}.$$

Therefore, $(\mathcal{E}, D(\mathcal{E}))$ is not irreducible (in the sense of [1]).

3. THE ONE DIMENSIONAL REFLECTED CASE

Let $I = [0, \infty)$ and $C_0^\infty(I) := \{f : I \to \mathbb{R} \mid \exists g \in C_0^\infty(\mathbb{R}) \text{ with } g = f \text{ on } I\}$. Let $\varphi \in L^1_{loc}(I, \mathrm{d}x)$ with $\varphi > 0$ $\mathrm{d}x$-a.e. Furthermore, assume σ be a (measurable) function such that $\sigma > 0$ $\mathrm{d}x$-a.e. and such that $\sigma\varphi \in L^1_{loc}(I, \mathrm{d}x)$. Consider the symmetric bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2}\int_I \sigma(x) f'(x) g'(x) \mu(\mathrm{d}x), \qquad f, g \in C_0^\infty(I)$$

on $L^2(I, \mu)$ where as before $\mu := \varphi \mathrm{d}x$. As in the Section 2, we suppose U is the largest open set in I such that

$$\frac{1}{\sigma\varphi} \in L^1_{loc}(U, \mathrm{d}x)$$

and assume

$$\mathrm{d}x(I \setminus U) = 0. \tag{3}$$

We suppose $(\mathcal{E}, C_0^\infty(I))$ is closable on $L^2(I,\mu)$. (By the results of ([9], Lemma 1.1), if there is some open set $\tilde{U} \subset I$ such that $1/\varphi \in L^1_{loc}(\tilde{U}, \mathrm{d}x)$ and $\mathrm{d}x(I \setminus (U \cap \tilde{U})) = 0$, then $(\mathcal{E}, C_0^\infty(I))$ is closable on $L^2(I,\mu)$). Denote the closure by $(\mathcal{E}, D(\mathcal{E}))$.

There are two possible cases that we summarize in the following theorem.

THEOREM 3.1. *$(\mathcal{E}, D(\mathcal{E}))$ is recurrent if one of the following conditions holds:*

(i) *$U = V \cup (a, \infty)$ where V is some open set (so either V is empty if $a = 0$, or V is non-empty if $a > 0$) and*

$$\int_{a+1}^{\infty} \frac{1}{\sigma\varphi}(s)\mathrm{d}s = \infty.$$

(ii) *$U = V \cup \bigcup_{n\in\mathbb{N}} I_n$, $I_n = (x_n, x_{n+1})$, $x_n < x_{n+1}$, $n \in \mathbb{N}$, V some open set, and*

$$a_n := \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)\mathrm{d}s \longrightarrow \infty$$

as $n \to \infty$, where for $n \in \mathbb{N}$

$$c_n = \frac{x_{n+1} + x_n}{2}$$

and

$$d_n := \begin{cases} x_{n+1} - \frac{1}{n} & x_{n+1} - c_n > 1, \\ x_{n+1} - \frac{x_{n+1} - c_n}{n} & x_{n+1} - c_n \le 1. \end{cases}$$

Proof. (i) Let

$$u_n(x) := \begin{cases} 1 & x \in [0, a+1], \\ 1 - \frac{1}{a_n}\int_{a+1}^{x} \frac{1}{\sigma\varphi}(t)\mathrm{d}t & x \in [a+1, a+1+n], \\ 0 & \text{elsewhere,} \end{cases}$$

where $a_n = \int_{a+1}^{a+1+n} \frac{1}{\sigma\varphi}(s)\mathrm{d}s$. Then $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \le u_n \le 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$.

(ii) Let

$$u_n(x) := \begin{cases} 1 & x \in [0, c_n], \\ 1 - \frac{1}{a_n}\int_{c_n}^{x} \frac{1}{\sigma\varphi}(t)\mathrm{d}t & x \in [c_n, d_n], \\ 0 & \text{elsewhere,} \end{cases}$$

where $a_n = \int_{c_n}^{d_n} \frac{1}{\sigma\varphi}(s)\mathrm{d}s$. Then $(u_n)_{n\in\mathbb{N}}$ satisfies the desired properties and determines recurrence. □

Example 3.2. If $\varphi(x) = x^{\delta-1}$ with $\delta > 0$ and $\sigma(x) \equiv 1$ on I, then clearly (3) is satisfied. In this case, the process associated to the regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ (cf. [1]) is the well-known Bessel process of dimension δ. We are going to find a sufficient condition on the dimension δ for recurrence. Since

$$\int_1^\infty \frac{1}{\varphi}(s)\mathrm{d}s = \int_1^\infty s^{1-\delta}\mathrm{d}s = \begin{cases} \lim_{n\to\infty} \log n & \delta = 2, \\ \lim_{n\to\infty} \frac{1}{2-\delta}[n^{2-\delta} - 1] & \delta \neq 2, \end{cases}$$

we see by Theorem 3.1 (i) with $a = 0$ that the Bessel processes of dimension δ is recurrent if $\delta \in (0, 2]$. Note that using ([8], Theorem 3) we obtain the same calculations up to a constant. However, in [8] the Dirichlet form is supposed to be irreducible throughout which we do not demand.

Remark 3.3. Of course Theorem 3.1 can be easily reformulated for Dirichlet forms defined on more general closed sets (cf. Remark 2.3).

4. THE MULTIDIMENSIONAL CASE

We have found sufficient conditions for recurrence of Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ with one dimensional state space. Now, we will extend our results to multidimensional Dirichlet forms of gradient type.

Let $d \geq 2$ and $\varphi \in L^1_{loc}(\mathbb{R}^d, \mathrm{d}x)$ such that $\varphi > 0$ $\mathrm{d}x$-a.e. Let further a_{ij} be measurable functions such that $a_{ij} = a_{ji}$, $1 \leq i, j \leq d$ and such that for each compact set $K \subset \mathbb{R}^d$, there exists $c_K > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq c_K \sum_{i=1}^d \xi_i^2, \qquad \forall x \in K, \quad \forall \xi \in \mathbb{R}^d. \tag{4}$$

Assume that the bilinear form

$$\mathcal{E}(f, g) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x)\partial_i f(x)\partial_j g(x)\varphi(x)\mathrm{d}x, \quad f, g \in C_0^\infty(\mathbb{R}^d)$$

is well-defined, positive and closable on $L^2(\mathbb{R}^d, \mu)$ where $\mathrm{d}\mu = \varphi\mathrm{d}x$. Let S be the $(d-1)$-dimensional surface measure. Define

$$\langle A(x)x, x\rangle := \sum_{i,j=1}^d a_{ij}(x)x_i x_j, \quad x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

and

$$\psi(r) := \int_{\partial\overline{B}_r} \frac{\langle A(x)x, x\rangle}{|x|^2}\varphi(x)S(\mathrm{d}x),$$

where $|\cdot|$ denotes the Euclidean norm, $B_r := \{x \in \mathbb{R}^d | |x| < r\}$ and $\overline{B}$ denotes the closure of a set $B \subset \mathbb{R}^d$. Assume $\psi(r) \in (0, \infty)$ dr-a.e. and let $U \subset [0, \infty)$ be

the largest open set such that

$$\frac{1}{\psi} \in L^1_{loc}(U, dr).$$

We suppose $\mathrm{d}r([0,\infty)\setminus U) = 0$.

Remark 4.1. Suppose that we also have the lower bound in (4), *i.e.* (a_{ij}) is locally strictly elliptic. Then $\psi(r) > 0$ for almost every $r > 0$ follows easily from ([5], Theorem, page 38) since $\varphi > 0$ $\mathrm{d}x$-a.e. Hence, our assumption $\psi(r) \in (0,\infty)$ for a.e. r is satisfied whenever we also have the lower bound in (4) and $\varphi \in L^1(\partial\overline{B}_r, dS)$ for a.e. r. The latter is for instance satisfied if $\varphi \in H^{1,1}_{loc}(\mathbb{R}^d, \mathrm{d}x)$.

Again we will present sufficient conditions for the recurrence of $(\mathcal{E}, D(\mathcal{E}))$, *i.e.* for the existence of $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \le u_n \le 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$ and $\lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$.

THEOREM 4.2. *$(\mathcal{E}, D(\mathcal{E}))$ is recurrent if one of the following conditions holds:*

(i) *$U = V \cup (a,\infty)$ where V is some open set (so either V is empty if $a = 0$, or V is non-empty if $a > 0$) and*

$$a_n := \int_{a+1}^{a+1+n} \frac{1}{\psi}(r)\mathrm{d}r \longrightarrow \infty$$

as $n \to \infty$.

(ii) *$U = V \cup \bigcup_{n\in\mathbb{N}} I_n$, $I_n = (r_n, r_{n+1})$, $r_n < r_{n+1}$, $n \in \mathbb{N}$, V some open set, and*

$$a_n := \int_{c_n}^{d_n} \frac{1}{\psi}(r)\mathrm{d}r \longrightarrow \infty$$

as $n \to \infty$, where for $n \in \mathbb{N}$

$$c_n = \frac{r_{n+1} + r_n}{2}$$

and

$$d_n := \begin{cases} r_{n+1} - \frac{1}{n} & r_{n+1} - c_n > 1, \\ r_{n+1} - \frac{r_{n+1}-c_n}{n} & r_{n+1} - c_n \le 1. \end{cases}$$

Proof. (i) Let

$$u_n(x) := \begin{cases} 1 & |x| \in [0, a+1], \\ 1 - \frac{1}{a_n}\int_{a+1}^{|x|} \frac{1}{\psi}(t)\mathrm{d}t & |x| \in [a+1, a+1+n], \\ 0 & \text{elsewhere}, \end{cases}$$

where $a_n = \int_{a+1}^{a+1+n} \frac{1}{\psi}(r)\mathrm{d}r$. Then $(u_n)_{n\in\mathbb{N}} \subset D(\mathcal{E})$ with $0 \le u_n \le 1$, $n \in \mathbb{N}$, $u_n \nearrow 1$ as $n \to \infty$. Since

$$\begin{aligned}\mathcal{E}(u_n,u_n) &= \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u_n(x)\partial_j u_n(x)\varphi(x)\mathrm{d}x \\ &= \int_{\overline{B}_{a+1+n}\setminus B_{a+1}} \sum_{i,j=1}^{d} a_{ij}(x)x_i x_j \frac{1}{|x|^2}\frac{1}{a_n^2}\frac{1}{\psi(|x|)^2}\varphi(x)\mathrm{d}x \\ &= \frac{1}{a_n^2}\int_{a+1}^{a+1+n}\Big[\frac{1}{r^2}\sum_{i,j=1}^{d}\int_{\partial\overline{B}_r} a_{ij}(x)x_i x_j\varphi(x)S(\mathrm{d}x)\Big]\frac{1}{\psi(r)^2}\mathrm{d}r \\ &= \frac{1}{a_n^2}\int_{a+1}^{a+1+n}\frac{1}{\psi(r)}\mathrm{d}r = \frac{1}{a_n},\end{aligned}$$

it follows that $\lim_{n\to\infty}\mathcal{E}(u_n,u_n) = 0$.

(ii) Let

$$u_n(x) := \begin{cases} 1 & |x| \in [0,c_n], \\ 1 - \frac{1}{a_n}\int_{c_n}^{|x|}\frac{1}{\psi}(t)\mathrm{d}t & |x| \in [c_n,d_n], \\ 0 & \text{elsewhere}, \end{cases}$$

where $a_n = \int_{c_n}^{d_n}\frac{1}{\psi}(r)\mathrm{d}r$. Then $(u_n)_{n\in\mathbb{N}}$ satisfies the desired properties and determines recurrence. □

Finally, let us consider a case which is possibly easier to calculate than the previous one. Let $\varphi \in L^1_{loc}(\mathbb{R}^d,\mathrm{d}x)$, $\varphi > 0$ $\mathrm{d}x$-a.e. and a_{ij} be measurable functions such that $a_{ij} = a_{ji}$, $1 \le i,j \le d$. We assume that the bilinear form

$$\mathcal{E}(f,g) = \sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}(x)\partial_i f(x)\partial_j g(x)\mu(\mathrm{d}x), \quad f,g \in C_0^\infty(\mathbb{R}^d)$$

is well-defined, positive and closable on $L^2(\mathbb{R}^d,\mu)$ where $\mu := \varphi\mathrm{d}x$.

THEOREM 4.3. *Suppose that there is some compact set $K \subset \mathbb{R}^d$ and a function ϕ, strictly positive a.e., with*

$$\|A(x)\|\varphi(x) \le \phi(|x|) \qquad \forall x \in \mathbb{R}^d \setminus K$$

where $\|A(x)\| = \big[\sum_{i,j=1}^d a_{ij}(x)^2\big]^{\frac{1}{2}}$ and $\phi(|\cdot|) \in L^1_{loc}(\mathbb{R}^d\setminus K,\mathrm{d}x)$. Let

$$a_n := \int_{\overline{B}_n\setminus B_\rho}\frac{|y|^{2-2d}}{\phi(|y|)}\mathrm{d}y$$

where $\rho > 0$ is such that $K \subset B_\rho$ and $n > \rho$. If a_n is finite for any $n > \rho$ and tends to ∞ as $n \to \infty$, then $(\mathcal{E}, D(\mathcal{E}))$ is recurrent.

Proof. Define

$$\psi_n(r) := \begin{cases} 1 & r \in [0,\rho], \\ 1 - \frac{1}{a_n}\int_{\overline{B}_r \setminus B_\rho} \frac{|y|^{2-2d}}{\phi(|y|)}\mathrm{d}y & r \in [\rho, n], \\ 0 & \text{elsewhere.} \end{cases}$$

Set $u_n(x) := \psi_n(|x|)$, $n \in \mathbb{N}$ sufficiently large. Then u_n is rotationally invariant, differentiable $\mathrm{d}x$-a.e. and in $D(\mathcal{E})$. Since $\frac{|y|^{2-2d}}{\phi(|y|)}$ is rotationally invariant, we have

$$\int_{\overline{B}_r \setminus B_\rho} \frac{|y|^{2-2d}}{\phi(|y|)}\mathrm{d}y = \gamma \int_\rho^n \frac{r^{1-d}}{\phi(r)}\mathrm{d}r,$$

with $\gamma := d \cdot vol_d(B_1)$. Then it is easy to see that $\partial_i u_n(x) = -\frac{\gamma}{a_n}\frac{|x|^{1-d}}{\phi(|x|)}\frac{x_i}{|x|}$, $1 \le i \le d$.

Consequently,

$$\begin{aligned} \mathcal{E}(u_n, u_n) &= \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u_n(x)\partial_j u_n(x)\mu(\mathrm{d}x) \\ &= \sum_{i,j=1}^d \int_{\overline{B}_n \setminus B_\rho} \frac{\gamma^2}{a_n^2} a_{ij}(x) x_i x_j |x|^{-2d} \frac{1}{\phi^2(|x|)}\varphi(x)\mathrm{d}x \\ &\le \frac{\gamma^2}{a_n^2}\int_{\overline{B}_n \setminus B_\rho} |x|^{2-2d}\frac{\|A(x)\|}{\phi^2(|x|)}\varphi(x)\mathrm{d}x = \frac{\gamma^2}{a_n}. \end{aligned}$$

converges to zero as $n \to \infty$. Therefore, $(\mathcal{E}, D(\mathcal{E}))$ is recurrent. □

Remark 4.4. Of course, Theorems 4.2 and 4.3 can be easily reformulated for more general open sets and in the reflected case (cf. Remarks 2.3 and 3.3, and Section 3).

Acknowledgments. We are grateful to the referee for the comments on the manuscript, which we feel have resulted in an improved paper. The research of Minjung Gim was supported by NRF (National Research Foundation of Korea) Grant funded by the Korean Government (NRF-2012-Global Ph.D. Fellowship Program). The research of the corresponding author Gerald Trutnau was supported by Seoul National University Research Grant 0450-20110022 and Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (MEST)(2012006987).

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Received 5 March 2013

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