POWERS OF CERTAIN THETA PRODUCTS

ZHONGHENG CHENG, JIANGYU WANG AND DONGXI YE

Communicated by Alexandru Zaharescu

In this work, we extend two modular identities due to Cooper and Ye [4] in weight aspect, and obtain two new Ramanujan–Mordell type formulas, which can be viewed as analogues of work of Chan and Cooper [2] in level aspect.

AMS 2020 Subject Classification: 11F03, 11F11, 11F27.

Key words: quadratic form, representation of integers, theta function.

1. INTRODUCTION

More than 100 years ago, in his seminal work [9, 10], Ramanujan recorded with no proofs an explicit decomposition of an even power of the modular theta function $\theta(\tau)^{2k}$, where

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{with } q = e^{2\pi i\tau} \text{ for } \tau \in \mathbb{H},$$

as a sum of (a linear combination of) Eisenstein series and some particular function whose Fourier coefficients were claimed to have lower growth order than that of Eisenstein series. More precisely, Ramanujan explicitly constructed a linear combination $\mathcal{E}_k(\tau)$ of Eisenstein series and a function $x(\tau)$ for which

$$\theta(\tau)^{2k} = \mathcal{E}_k(\tau) + \theta(\tau)^{2k} \sum_{1\le j\le (k-1)/4} c_{k,j}\, x(\tau)^j \tag{1}$$

for some rational coefficients $c_{k,j}$. Such a mysterious decomposition was later proved by Mordell [8] using the classical theory of modular forms. It can be interpreted by a well-known theorem stating that a holomorphic modular form (viewed as a vector over $\mathbb{C}$) of positive integral weight for some Fuchsian group Γ of the first kind can be written as a direct sum of a linear combination of Eisenstein series and a holomorphic modular form with a *positive* cuspidal divisor on $X(\Gamma)$, i.e., a cusp form. From this point of view, one may expect

Dongxi Ye was supported by the Guangdong Basic and Applied Basic Research Foundation (Grant No. 2023A1515010298).

MATH. REPORTS **25(75)** (2023), *4*, 577–587

doi: 10.59277/mrar.2023.25.75.4.577

and verify that the non-Eisenstein-series component of the decomposition of $\theta(\tau)^{2k}$ given by Ramanujan is indeed a cusp form. In addition, by some closer observations, one may note that $\theta(\tau)^{2k}$ is exactly the generating function of the numbers of representations of an integer by a sum of $2k$ squares, one of the most important arithmetic counting functions in number theory, and thereby, one of the novelties of Ramanujan's work is the implication of an "explicit" formula for this arithmetic counting function. These connections to arithmetic draw attention on Ramanujan's work and turn the findings of an explicit decomposition (now called Ramanujan–Mordell type formula) for a power of a (generalized) modular theta function into an interesting topic in relevant areas.

In [3], Cooper, Kane and the third author extended Ramanujan's work to the case for the theta product $\theta(\tau)\theta(p\tau)$ for $p \in \{3, 7, 11, 23\}$ and established a decomposition for $(\theta(\tau)\theta(p\tau))^k$ similarly. In [14, 11], the third author of the present work (with Singh in [11]) respectively established Ramanujan–Mordell type formulas for the cases for $(\theta(\tau)\theta(p\tau))^k$ for $p \in \{2, 4\}$ and $p \in \{5, 8\}$. While one may note that all cases treated in these previous works are related to the unary quadratic form $Q(x) = x^2$, non-unary cases are first treated in [2] by Chan and Cooper, in which they explicitly decomposed powers of the theta functions

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+an^2} \quad \text{for } a \in \{1, 2, 3, 6\}.$$

Following the previous papers mentioned above, it shall be interesting to consider theta products associated to non-unary quadratic form. For example, in [4], Cooper and the third author showed that

$$\begin{aligned}&\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)\left(\sum_{m,n=-\infty}^{\infty} q^{2\left(m^2+mn+2n^2\right)}\right)\\ &= -\frac{1}{18}E_2(\tau) - \frac{1}{9}E_2(2\tau) + \frac{7}{18}E_2(7\tau) + \frac{7}{9}E_2(14\tau)\\ &\quad + \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)\left(\sum_{m,n=-\infty}^{\infty} q^{5\left(m^2+mn+n^2\right)}\right)\\ &= -\frac{1}{16}E_2(\tau) - \frac{3}{16}E_2(3\tau) + \frac{5}{16}E_2(5\tau) + \frac{15}{16}E_2(15\tau)\\ &\quad + \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau),\end{aligned} \tag{3}$$

where $E_2(\tau)$ denotes the normalized Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

and $\eta(\tau)$ denotes the Dedekind eta function defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

One can verify that both $\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$ and $\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$ are cusp forms. In this work, we extend (2) and (3) in weight aspect and establish Ramanujan–Mordell type decompositions for

$$\left\{ \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right) \left(\sum_{m,n=-\infty}^{\infty} q^{2\left(m^2+mn+2n^2\right)} \right) \right\}^k$$

and

$$\left\{ \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \right) \left(\sum_{m,n=-\infty}^{\infty} q^{5\left(m^2+mn+n^2\right)} \right) \right\}^k.$$

We now summarize the main results of this work in the following theorem as a closing for the present section.

THEOREM 1.1. *For $N = 14$ or 15, let $z_N = z_N(\tau)$ be a theta product defined by*

(4)
$$z_N(\tau) = \begin{cases} \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2} \right) \left(\sum_{m,n=-\infty}^{\infty} q^{2\left(m^2+mn+2n^2\right)} \right) & \text{if } N = 14, \\ \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \right) \left(\sum_{m,n=-\infty}^{\infty} q^{5\left(m^2+mn+n^2\right)} \right) & \text{if } N = 15. \end{cases}$$

Let $x_N = x_N(\tau)$ and $\mathcal{E}_N = \mathcal{E}_N(\tau)$ be defined by

$$x_N(\tau) = \begin{cases} \dfrac{\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)}{z_{14}(\tau)} & \text{if } N = 14, \\ \dfrac{\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)}{z_{15}(\tau)} & \text{if } N = 15 \end{cases}$$

and

$$\mathcal{E}_{k,N}(\tau)=\begin{cases}\dfrac{\substack{(-1)^kE_{2k}(\tau)+(-2)^kE_{2k}(2\tau)\\+7^kE_{2k}(7\tau)+14^kE_{2k}(14\tau)}}{(-1)^k+(-2)^k+7^k+14^k} & \text{if } N=14,\\[2ex] \dfrac{\substack{(-1)^kE_{2k}(\tau)+(-3)^kE_{2k}(3\tau)\\+5^kE_{2k}(5\tau)+15^kE_{2k}(15\tau)}}{(-1)^k+(-3)^k+5^k+15^k} & \text{if } N=15,\end{cases}$$

where

$$E_{2k}(\tau)=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$$

with B_{2k} the $2k$-th Bernoulli number and $\sigma_{2k-1}(n)=\sum_{d|n}d^{2k-1}$ is the normalized Eisenstein series of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$. Then there are rational constants $c_{k,j}$ depending only on k and j such that

$$z_N^k=\mathcal{E}_{k,N}+z_N^k\sum_{j=1}^{k}c_{k,j}x_N^j. \tag{5}$$

2. PRELIMINARIES AND OBSERVATIONS

In this section, we first recall some well-known conclusions on modular theta functions as preliminaries. We then outline the key observations that lead us to realize the existence of Theorem 1.1 and help us construct $\mathcal{E}_{k,N}(\tau)$.

2.1. Review of theta functions

Let $Q(\vec{x})$ be a positive definite quadratic form over $\mathbb{Z}$ of dimension $2k$ with k a positive integer for $\vec{x}=(x_1,\cdots,x_{2k})\in\mathbb{R}^{2k}$. The modular theta function $\theta(\tau;Q)$ associated to $Q(x)$ is defined by

$$\theta(\tau;Q)=\sum_{\vec{x}\in\mathbb{Z}^{2k}}q^{Q(\vec{x})}=\sum_{\vec{x}\in\mathbb{Z}^{2k}}q^{\frac{1}{2}\vec{x}A\vec{x}^t}$$

with $q=e^{2\pi i\tau}$ for $\tau\in\mathbb{H}$. It is well known that the theta function $\theta(\tau;Q)$ possesses the following modular transformations. See, e.g., [12, 13].

THEOREM 2.1. *Let $\theta(\tau;Q)$ be defined as above, and let A be the matrix associated to $Q(\vec{x})$ defined by*

$$A=\left(\frac{\partial^2Q}{\partial x_i\partial x_j}\right)_{1\le i,j\le 2k}.$$

Then the following transformation formulas hold.

1. *Suppose that N is the least positive integer such that NA^{-1} is integral and all the diagonal entries of NA^{-1} are even. For any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,
$$\theta\left(\frac{a\tau+b}{c\tau+d};Q\right) = \left(\frac{(-1)^k \det A}{d}\right)(c\tau+d)^k\theta(\tau;Q).$$

2. $$\theta(-1/\tau;Q) = (\det A)^{-1/2}(-i\tau)^k \sum_{\vec{x}\in\mathbb{Z}^{2k}} e^{2\pi i\tau\frac{1}{2}\vec{x}A^{-1}\vec{x}^t},$$

Setting $Q = x^2+xy+y^2$ or $x^2+xy+2y^2$, one can easily deduce the following from Theorem 2.1.

Corollary 2.1. *Let $Q_3 = x_1^2+x_1x_2+x_2^2$ and $Q_7 = x_1^2+x_2y_2+2y_2^2$. Then for $p\in\{3,7\}$,*

1. *$\theta(\tau;Q_p)$ is a holomorphic modular form of weight 1 for $\Gamma_0(p)$ with the quadratic character $\left(\frac{-p}{\cdot}\right)$;*

2. $$\theta\left(-\frac{1}{p\tau};Q_p\right) = p^{\frac{1}{2}}(-i\tau)\theta(\tau;Q_p).$$

Moreover, recall $z_N(\tau)$ for $N\in\{14,15\}$ be defined by (4). Then

1. *$z_{14}(\tau) = \theta(\tau;Q_7)\theta(2\tau;Q_7)$ and $z_{15}(\tau) = \theta(\tau;Q_3)\theta(5\tau;Q_3)$;*

2. *$z_N(\tau)$ is a holomorphic modular form of weight 2 for $\Gamma_0(N)$;*

3. *the following transformation formula holds*
$$z_N\left(-\frac{1}{N\tau}\right) = -N\tau^2 z_N(\tau).$$

2.2. Observations

In this subsection, we discuss some simple observations that lead us to construct an appropriate linear combination $\mathcal{E}_{k,N}(\tau)$ of Eisenstein series and sequentially establish Theorem 1.1.

A careful study of the Ramanujan–Mordell formula and all the previous related work motivates us to look at (2) and (3) in the following way:
$$z_{14}(\tau) = -\frac{1}{18}E_2(\tau) - \frac{1}{9}E_2(2\tau) + \frac{7}{18}E_2(7\tau) + \frac{7}{9}E_2(14\tau)$$

$$+ z_{14}(\tau)\frac{\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)}{z_{14}(\tau)},$$

$$z_{15}(\tau) = -\frac{1}{16}E_2(\tau) - \frac{3}{16}E_2(3\tau) + \frac{5}{16}E_2(5\tau) + \frac{15}{16}E_2(15\tau) + z_{15}(\tau)\frac{\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)}{z_{15}(\tau)}$$

which give us a hint that if Ramanujan–Mordell type decompositions exist for $z_{14}(\tau)^k$ and $z_{15}(\tau)^k$, then, respectively,

$$x_{14}(\tau) = \frac{\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)}{z_{14}(\tau)} \quad \text{and} \quad x_{15}(\tau) = \frac{\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)}{z_{15}(\tau)}$$

might be the ones fitting into the polynomial part like (1). In addition, it has been observed and discussed in [14] that a Ramanujan–Mordell type formula may be induced by a genus zero modular curve whose uniformizer over $\mathbb{P}^1(\mathbb{C})$ yields its polynomial part by some means. As such, if one may realize $x_N(\tau)$ as an uniformizer of some genus zero modular curve, through the associated modularity, one may construct the desired linear combination of Eisenstein series.

Following these thoughts, now let us first examine the behaviors of $x_{14}(\tau)$. Clearly, by Corollary 2.1 and well-known properties of the Dedekind eta function [6], the function $x_{14}(\tau)$ is a modular function of level $\Gamma_0(14)$ with simple zeros supported only at the four cusps $\{0, \frac{1}{2}, \frac{1}{7}, i\infty\}$ of $\Gamma_0(14)$, each of which is simple. To realize $x_{14}(\tau)$ as a uniformizer for some modular curve under $X(\Gamma_0(14))$ of finite degree, one can notice that all the four cusps mentioned above must be equivalent since an uniformizer serves as a degree one map. Since 14 is square free, the cusps 0, $\frac{1}{2}$ and $\frac{1}{7}$ are all equivalent to $i\infty$ via the Atkin–Lehner involutions of discriminants 14, 7 and 2, respectively, i.e.,

$$w_{14} = \begin{pmatrix} 0 & -1 \\ 14 & 0 \end{pmatrix}, \quad w_7 = \begin{pmatrix} 7 & 3 \\ 14 & 7 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 2 & 1 \\ 14 & 8 \end{pmatrix}.$$

Also, by Corollary 2.1, one can easily check that the modular function $x_{14}(\tau)$ is invariant under w_{14}, w_7 and w_2, and thus, $x_{14}(\tau)$ can be viewed as a modular function for $\langle \Gamma_0(14), w_{14}, w_7, w_2 \rangle$. The involutionality of w_{14}, w_7 and w_2 indicates that modding out the action of them, $X(\Gamma_0(14))$ yields a modular curve $X_0(14)/\langle w_2, w_7, w_{14} \rangle$ of finite degree, on which $x_{14}(\tau)$ has a simple zero only at the cusp $i\infty$. This shows that $x_{14}(\tau)$ is indeed an uniformizer for $X_0(14)/\langle w_2, w_7, w_{14} \rangle$ over $\mathbb{P}^1(\mathbb{C})$, whose genus-zeroness can also be directly computed using the Riemann–Hurwitz formula [5]. Up to this point, we may be convinced that one may establish a Ramanujan–Mordell type formula for $z_{14}(\tau)^k$ induced by $X_0(14)/\langle w_2, w_7, w_{14} \rangle$ as long as one is able to find a linear combination $\mathcal{E}_{k,14}(\tau)$ of Eisenstein series so that $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ is a modular function on $X_0(14)/\langle w_2, w_7, w_{14}\rangle$. Similar arguments indicate that a Ramanujan–Mordell type formula for $z_{15}(\tau)^k$ should be induced by $X_0(15)/\langle w_3, w_5, w_{15}\rangle$, where

$$w_{15} = \begin{pmatrix} 0 & -1 \\ 15 & 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 3 & 1 \\ 15 & 6 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 5 & -2 \\ 15 & -5 \end{pmatrix}.$$

2.3. **Construction of $\mathcal{E}_{k,14}(\tau)$**

In this subsection, we discuss how $\mathcal{E}_{k,14}(\tau)$ is constructed as an illustration. For $k \geq 2$, it is known that the space of Eisenstein series of weight $2k$ for $\Gamma_0(14)$ is spanned by

$$\{E_{2k}(\tau), E_{2k}(2\tau), E_{2k}(7\tau), E_{2k}(14\tau)\}.$$

Since $X(\Gamma_0(14))$ is a covering of $X_0(14)/\langle w_2, w_7, w_{14}\rangle$, we may suppose that

$$\mathcal{E}_{k,14}(\tau) = a_1 E_{2k}(\tau) + a_2 E_{2k}(2\tau) + a_3 E_{2k}(7\tau) + a_4 E_{2k}(14\tau).$$

Then to have $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ to be a meromorphic function on $X_0(14)/\langle w_2, w_7, w_{14}\rangle$, one must have

$$\left.\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}\right| w_{14} = \left.\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}\right| w_7 = \left.\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}\right| w_2 = \frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k},$$

where

$$f(\tau)|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\left(\frac{a\tau+b}{c\tau+d}\right)$$

for a modular function $f(\tau)$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

By the linear independence and modularity of $E_{2k}(\tau)$, $E_{2k}(2\tau)$, $E_{2k}(7\tau)$, $E_{2k}(14\tau)$, and Corollary 2.1, some routine computations show that

$$a_2 = 2^k a_1, \quad a_3 = (-7)^k a_1, \quad a_4 = (-14)^k a_1.$$

Finally, we take

$$a_1 = \frac{(-1)^k}{(-1)^k + (-2)^k + 7^k + 14^k}$$

in order to normalize $\mathcal{E}_{k,14}(\tau)$ to have a constant term of 1.

Proof of Theorem 1.1. In the course of discussing $x_{14}(\tau)$ and constructing $\mathcal{E}_{k,14}(\tau)$, one has already seen that $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ is a modular function on

$$X_0(14)/\langle w_2, w_7, w_{14}\rangle$$

with poles only supported at location of the zero of $z_{14}(\tau)$ of order at most k. Since $x_{14}(\tau)$ has a simple zero only at the cusp $i\infty$, $x_{14}(\tau)$ has a simple pole only at the location of the zero of $z_{14}(\tau)$ in $X_0(14)/\langle w_2, w_7, w_{14}\rangle$. Therefore, $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ must be a polynomial of degree at most k with constant term of 1 in $x_{14}(\tau)$ since $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ is 1 at $i\infty$.

Finally, the rationality of the coefficients of such a polynomial follows from that of $\frac{\mathcal{E}_{k,14}(\tau)}{z_{14}(\tau)^k}$ and $x_{14}(\tau)$. This completes the proof for the case $N = 14$. The treatment of the case $N = 15$ is essentially the same, and we omit the details. $\square$

Example 2.1. When $k = 2$, Theorem 1.1 yields that

$$\begin{aligned}
&\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)^2 \left(\sum_{m,n=-\infty}^{\infty} q^{2\left(m^2+mn+2n^2\right)}\right)^2 \\
&= \frac{1}{250}E_4(\tau) + \frac{2}{125}E_4(2\tau) + \frac{49}{250}E_4(7\tau) + \frac{98}{125}E_4(14\tau) \\
&\quad + \frac{76}{25}\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}\right)\left(\sum_{m,n=-\infty}^{\infty} q^{2\left(m^2+mn+2n^2\right)}\right)\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) \\
&\quad + \frac{12}{25}(\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau))^2
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2 \left(\sum_{m,n=-\infty}^{\infty} q^{5\left(m^2+mn+n^2\right)}\right)^2 \\
&= \frac{1}{260}E_4(\tau) + \frac{9}{260}E_4(3\tau) + \frac{5}{52}E_4(5\tau) + \frac{45}{52}E_4(15\tau) \\
&\quad + \frac{144}{13}\left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)\left(\sum_{m,n=-\infty}^{\infty} q^{5\left(m^2+mn+n^2\right)}\right)\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \\
&\quad - \frac{360}{13}(\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))^2.
\end{aligned}$$

To the best of our knowledge, these two identities has not appeared in any literature.

Corollary 2.2. *For $N \in \{14, 15\}$, denote by $r(n;k,N)$ the number of representations of a positive integer n by the quadratic form*

$$\begin{cases} \displaystyle\sum_{j=1}^{k}(x_{2j-1}^2 + x_{2j-1}x_{2j} + 2x_{2j}^2) + 2\sum_{j=k+1}^{2k}\left(x_{2j-1}^2 + x_{2j-1}x_{2j} + 2x_{2j}^2\right) & \text{if } N = 14, \\ \displaystyle\sum_{j=1}^{k}(x_{2j-1}^2 + x_{2j-1}x_{2j} + x_{2j}^2) + 5\sum_{j=k+1}^{2k}\left(x_{2j-1}^2 + x_{2j-1}x_{2j} + x_{2j}^2\right) & \text{if } N = 15. \end{cases}$$

Then for any positive integer k, as $n \to \infty$,

$$r(n;k,N) \sim -\frac{4k}{B_{2k}} \times \begin{cases} \dfrac{\substack{(-1)^k\sigma_{2k-1}(n)+(-2)^k\sigma_{2k-1}(n/2) \\ +7^k\sigma_{2k-1}(n/7)+14^k\sigma_{2k-1}(n/14)}}{(-1)^k + (-2)^k + 7^k + 14^k} & \text{if } N = 14, \\ \dfrac{\substack{(-1)^k\sigma_{2k-1}(n)+(-3)^k\sigma_{2k-1}(n/3) \\ +5^k\sigma_{2k-1}(n/5)+15^k\sigma_{2k-1}(n/15)}}{(-1)^k + (-3)^k + 5^k + 15^k} & \text{if } N = 15. \end{cases}$$

Proof. Let z_N and x_N be defined as in Theorem 1.1. It is not hard to see that $z_N^k \sum_{j=1}^k c_{k,j} x_N^j$ given in Theorem 1.1 is a cusp form of weight $2k$ whose coefficients are known (see, e.g., [7]) to have growth order $O(n^k)$. Clearly, $\sigma_{2k-1}(n) \asymp n^{2k-1}$. Together with these, Theorem 1.1 directly implies the corollary. $\square$

3. REMARK

In [1], Alaca and Williams derive a formula analogous to (2) and (3) for

$$z_6 = z_6(\tau) := \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2 \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2(m^2+mn+n^2)}\right)^2,$$

namely,

$$\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2 \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2(m^2+mn+n^2)}\right)^2$$

$$= \frac{E_4(\tau) + 4E_4(2\tau) + 9E_4(3\tau) + 36E_4(6\tau)}{50} + \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2.$$

By the discussion given in Section 2, it is very natural for one to question if a Ramanujan–Mordell type formula exists for $z_6(\tau)^k$ for k any positive integer. The answer is positive and is given below.

THEOREM 3.1. *Let z_6 and x_6 be defined by*

$$z_6 = z_6(\tau) = \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2 \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{2(m^2+mn+n^2)}\right)^2,$$

$$x_6 = x_6(\tau) = \frac{\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2}{z_6}.$$

Let k be a positive integer. Then there are rational numbers $c_{k,j}$ depending only on k and j such that

$$z_6^k = \frac{E_{4k}(\tau) + 2^k E_{4k}(2\tau) + 3^k E_{4k}(3\tau) + 6^k E_{4k}(6\tau)}{1 + 2^k + 3^k + 6^k} + z_6^k \sum_{1\le j\le k} c_{k,j} x_6^j.$$

We leave the details of computations and proof to the reader.

Acknowledgments. The authors would like to thank the anonymous referee for his/her useful comments, corrections and suggestions.

REFERENCES

[1] S. Alaca and K.S. Williams, *Evaluation of the convolution sums $\sum_{l+6m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+3m=n}\sigma(l)\sigma(m)$*. J. Number Theory **124** (2007), 491–510.

[2] H.H. Chan and S. Cooper, *Powers of theta functions.* Pacific J. Math. **235** (2008), *1*, 1–14.

[3] S. Cooper, B. Kane, and D. Ye, *Analogues of the Ramanujan–Mordell Theorem.* J. Math. Anal. Appl. **446** (2017), 568–579.

[4] S. Cooper and D. Ye, *Level 14 and 15 analogues of Ramanujan's theories of elliptic functions to alternative bases.* Trans. Amer. Math. Soc. **368** (2016), 7883–7910.

[5] P.G. Kluit, *On the normalizer of $\Gamma_0(N)$*. In: *Modular Functions of One Variable, V* (Bonn, 1976), 239–246, Lecture Notes in Math. **601**, Springer-Verlag, New York, 1977.

[6] C. Meyer, *Über einige Anwendungen Dedekindscher Summen.* J. Reine Angew. Math. **198** (1957), 143–203.

[7] T. Miyake, *Modular Forms.* Springer Monogr. Math., Springer, Berlin, 2005.

[8] L.J. Mordell, *On the representation of numbers as the sum of $2r$ squares.* Quart. J. Pure Appl. Math. **48** (1917), 93–104.

[9] S. Ramanujan, *On certain arithmetical functions.* Trans. Cambridge Philos. Soc. **22** (1916), 159–184.

[10] S. Ramanujan, *Collected papers of Srinivasa Ramanujan.* G. H. Hardy et al. (Eds.), AMS Chelsea Publishing, Providence, RI, 2000.

[11] A.K. Singh and D. Ye, *Ramanujan–Mordell type formulas associated to certain quadratic forms of discriminant* 20^k *or* 32^k. Indian J. Pure Appl. Math. **51** (2020), *3*, 1083–1096.

[12] B. Schoeneberg, *Elliptic Modular Functions. An Introduction.* Die Grundlehren der mathematischen Wissenschaften **203**, Springer, New York, 1974.

[13] X. Wang and D. Pei, *Modular forms with integral and half-integral weights.* Science Press, Beijing, 2011.

[14] D. Ye, *Representations of integers by certain* $2k$*-ary quadratic forms.* J. Number Theory **179** (2017), 50–64.

Received May 9, 2021

University of Amsterdam
Korteweg-de Vries Institute for Mathematics
P.O. Box 94248, 1090 GE Amsterdam
The Netherlands
zhongheng.cheng@student.uva.nl

Sun Yat-sen University
School of Mathematics
Guangzhou 510275, Guangdong
People's Republic of China
wangjiangyu28_2@126.com

Sun Yat-sen University
School of Mathematics (Zhuhai)
Zhuhai 519082, Guangdong
People's Republic of China
yedx3@mail.sysu.edu.cn