A NOTE ON FOURTH POWER MEAN OF THE GENERAL TWO-TERM EXPONENTIAL SUMS

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Let q, m, n be any integer with $q \geq 3$, and λ a Dirichlet character mod q. An explicit formula for the fourth power mean

$$\sum_{\substack{m=1\\(m,q)=1}}^{q} \left|\sum_{a=1}^{q} \lambda(a) e\left(\frac{ma^3+na}{q}\right)\right|^4$$

is derived. Previously, only the prime modulus $q = p$ case was studied.

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1. INTRODUCTION AND MAIN RESULTS

Let $q \geq 3$ be a positive integer. For any integers m and n, the generalized two-term exponential sums $G(m, n, \lambda; q)$ are defined by

$$G(m,n,\lambda;q) = \sum_{a=1}^{q} \lambda(a) e\left(\frac{ma^3+na}{q}\right),$$

where $e(y) = e^{2\pi i y}$, λ denotes a Dirichlet character mod q.

H. Zhang and W. P. Zhang [11] studied the fourth power mean of sums $G(m, n, \lambda; q)$ in the prime modulus case $q = p$, with $\lambda = \lambda_0$ the principal character mod p, and derived the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid (p-1); \\ 2p^3 - 7p^2 & \text{if } 3|(p-1), \end{cases}$$

where $(n, p) = 1$.

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For the prime modulus case with $(3, p-1) = 1$, R. Duan and W. P. Zhang obtained in [5] an exact computational formula as

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-3p^2-3p-1 & \text{if } \lambda=\lambda_0; \\ 3p^3-8p^2 & \text{if } \lambda=\left(\frac{*}{p}\right); \\ 2p^3-7p^2 & \text{if } \lambda\neq\lambda_0, \left(\frac{*}{p}\right), \end{cases} \tag{1.1}$$

where p is an odd prime, $(n, p) = 1$ and $\left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

On the basis of [5], T. T. Wang [8] discussed the case of $p \equiv 1 \bmod 3$ and obtained

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-5p^2-15p+4dp-1 & \text{if } \lambda=\chi_0; \\ 2p^3-11p^2-2p\cdot M(\lambda,\psi) & \text{if } \lambda=\chi^3; \\ 3p^3-12p^2-2p\cdot M(\lambda,\psi) & \text{if } \lambda=\left(\frac{*}{p}\right); \\ 2p^3-5p^2 & \text{otherwise,} \end{cases}$$

where d and b are uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$, χ is a non-real character mod p, ψ is a third-order character mod p and $M(\lambda, \psi)$ is defined as

$$M(\lambda,\psi) = \left|\sum_{a=1}^{p-1}\lambda(a)\psi(a^3-1)\right|^2.$$

Various other related properties of $G(m, n, \lambda; q)$ can be found in [1]–[14], and the references therein.

Naturally, the above questions and results render us to consider the fourth power mean of the generalized two-term exponential sums

$$\sum_{\substack{m=1\\(m,q)=1}}^{q}\left|\sum_{a=1}^{q}\lambda(a)e\left(\frac{ma^3+na}{q}\right)\right|^4,$$

where $\sum_{\substack{m=1\\(m,q)=1}}^{q}$ denotes the summation over all integers $1 \le m \le q$ such that $(m, q) = 1$, and n is any integer with $(n, q) = 1$.

As far as we know, these questions and related results have never been addressed in the literature. Our interest is focused on finding an explicit formula for arbitrary modulus q. Until now, this goal can be achieved with a small step forward. Due to some technical reasons, our methods only work for the canonical representation of q being known. That is,

$$q = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \text{ with } (3, p_i-1) = 1\,(1 \le i \le k),$$

and λ denotes any primitive character mod q. We have

THEOREM 1.1. *Let $p>3$ be a prime with $(3,p-1)=1$, α be a positive integer with $\alpha\geq 2$, n be any integer with $(n,p)=1$. Then for any primitive character $\lambda \bmod p^\alpha$, we have*

$$\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^4=p^{2\alpha}\phi(p^\alpha)\left(\alpha+1-\frac{5}{p-1}\right).$$

Theorem 1.1 and (1.1) immediately imply the following corollary.

COROLLARY 1.2. *Let $q>3$ be an odd number with integer $q=NM$ and $(N,M)=1$, where we have the prime power decompositions $N=p_1p_2\cdots p_s$ and $M=p_{s+1}^{\alpha_1}p_{s+2}^{\alpha_2}\cdots p_{s+k}^{\alpha_k}$ with $3\nmid M$ and $(3,p_i-1)=1$ $(1\leq i\leq s+k)$. Let n be any integer with $(n,q)=1$, $(\frac{*}{p_i})$ denote the Legendre symbol $\bmod p_i$ $(1\leq i\leq s)$, λ_j denote any non-real primitive character $\bmod p_{s+j}^{\alpha_j}$ with $\alpha_j\geq 1$ $(1\leq j\leq k)$. Then for the primitive character $\lambda=(\frac{*}{p_1})(\frac{*}{p_2})\cdots(\frac{*}{p_s})\lambda_1\lambda_2\cdots\lambda_k$, we have*

$$\sum_{\substack{m=1\\(m,q)=1}}^{q}\left|\sum_{a=1}^{q}\lambda(a)e\left(\frac{ma^3+na}{q}\right)\right|^4=q^2\prod_{p|N}(3p-8)\prod_{p^\alpha\|M}\left(\phi(p^\alpha)\left(\alpha+1-\frac{5}{p-1}\right)\right),$$

where $p^\alpha\|q$ denotes that $p^\alpha|q$ and $p^{\alpha+1}\nmid q$.

2. SOME LEMMAS

To prove our theorem, we need the following several lemmas.

LEMMA 2.1. *Let $p>3$ be a prime with $(3,p-1)=1$, α be a positive integer with $\alpha\geq 2$, n be any integer n with $(n,p)=1$. If λ is a primitive character $\bmod p^\alpha$, we have*

$$\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2=p^{2\alpha}-2p^{2\alpha-1}.$$

Proof. By the properties of the reduced residue system $\bmod p^\alpha$, we have

$$\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2$$
$$=\sum_{a=1}^{p^\alpha}\sum_{b=1}^{p^\alpha}\lambda(a\bar{b})\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha}e\left(\frac{m(a^3-b^3)+n(a-b)}{p^\alpha}\right)$$

$$= \sum_{a=1}^{p^\alpha} \lambda(a) \sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha} \sum_{\substack{b=1\\(b,p)=1}}^{p^\alpha} e\left(\frac{mb^3(a^3-1)+nb(a-1)}{p^\alpha}\right)$$

$$= \sum_{a=1}^{p^\alpha} \lambda(a) \left(\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha} e\left(\frac{m(a^3-1)}{p^\alpha}\right)\right)\left(\sum_{\substack{b=1\\(b,p)=1}}^{p^\alpha} e\left(\frac{b(a-1)}{p^\alpha}\right)\right). \tag{2.1}$$

Note that the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q, & \text{if } q|n; \\ 0, & \text{if } q\nmid n, \end{cases} \tag{2.2}$$

if $(n,p)=1$ and $\alpha \geq 2$, we have

$$\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha} e\left(\frac{nm}{p^\alpha}\right) = \sum_{m=1}^{p^\alpha} e\left(\frac{nm}{p^\alpha}\right) - \sum_{m=1}^{p^{\alpha-1}} e\left(\frac{nm}{p^{\alpha-1}}\right) = 0. \tag{2.3}$$

According to (2.3), (2.1) is not 0 if and only if $p^{\alpha-1}|(a-1)$. From the properties of Gauss sums mod p^α, we have

$$\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha} \left| \sum_{a=1}^{p^\alpha} \lambda(a) e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2$$

$$= \phi^2(p^\alpha) + \sum_{r=1}^{p-1} \lambda(rp^{\alpha-1}+1) \left(\sum_{\substack{m=1\\(m,p)=1}}^{p^\alpha} e\left(\frac{3rm}{p}\right)\right)\left(\sum_{\substack{b=1\\(b,p)=1}}^{p^\alpha} e\left(\frac{rb}{p}\right)\right)$$

$$= \phi^2(p^\alpha) + p^{2(\alpha-1)} \sum_{r=1}^{p-1} \lambda(rp^{\alpha-1}+1)$$

$$= \phi^2(p^\alpha) + \frac{p^{2(\alpha-1)}}{\tau(\overline{\lambda})} \sum_{a=1}^{p^\alpha} \overline{\lambda}(a) \sum_{r=1}^{p-1} e\left(\frac{a(rp^{\alpha-1}+1)}{p^\alpha}\right)$$

$$= \phi^2(p^\alpha) + \frac{p^{2(\alpha-1)}}{\tau(\overline{\lambda})} \sum_{a=1}^{p^\alpha} \overline{\lambda}(a) e\left(\frac{a}{p^\alpha}\right) \sum_{r=1}^{p-1} e\left(\frac{ar}{p}\right)$$

$$= \phi^2(p^\alpha) - \frac{p^{2(\alpha-1)}}{\tau(\overline{\lambda})} \sum_{a=1}^{p^\alpha} \overline{\lambda}(a) e\left(\frac{a}{p^\alpha}\right) = p^{2\alpha} - 2p^{2\alpha-1}.$$

This proves Lemma 2.1. □

LEMMA 2.2. *Let $p>3$ be a prime with $(3,p-1)=1$, α be a positive integer with $\alpha \geq 2$, n be any integer n with $(n,p)=1$, β be an integer with*

$1 \le \beta \le \frac{1}{2}\alpha$. *If λ is a primitive character* mod p^α, *for any primitive character χ* mod p^β, *we have*

$$\left|\sum_{m=1}^{p^\alpha} \chi(m)\left|\sum_{a=1}^{p^\alpha} \lambda(a) e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 = \begin{cases} p^{4\alpha-\beta} & \text{if } \chi \text{ is a non-real character} \bmod p^\beta; \\ p^{4\alpha-2} & \text{if } \beta = 1 \text{ and } \chi = \chi_2 \text{ is the Legendre symbol} \bmod p. \end{cases}$$

Proof. By the properties of the reduced residue system mod p^α, we have

(2.4)

$$\begin{aligned}
&\sum_{m=1}^{p^\alpha} \chi(m)\left|\sum_{a=1}^{p^\alpha} \lambda(a) e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2 \\
&= \sum_{m=1}^{p^\alpha} \chi(m) \sum_{\substack{b=1\\(b,p)=1}}^{p^\alpha} \sum_{a=1}^{p^\alpha} \lambda(a) e\left(\frac{mb^3(a^3-1)+nb(a-1)}{p^\alpha}\right) \\
&= \chi^3(n) \sum_{m=1}^{p^\alpha} \chi(m) \sum_{\substack{b=1\\(b,p)=1}}^{p^\alpha} \sum_{a=1}^{p^\alpha} \lambda(a) e\left(\frac{mn^3b^3(a^3-1)+nb(a-1)}{p^\alpha}\right) \\
&= \chi^3(n) \sum_{a=1}^{p^\alpha} \lambda(a) \left(\sum_{m=1}^{p^\alpha} \chi(m) e\left(\frac{m(a^3-1)}{p^\alpha}\right)\right)\left(\sum_{b=1}^{p^\alpha} \overline{\chi}^3(b) e\left(\frac{b(a-1)}{p^\alpha}\right)\right).
\end{aligned}$$

Since χ is a primitive character mod p^β, we have

$$\begin{aligned}
\sum_{m=1}^{p^\alpha} \chi(m) e\left(\frac{m(a^3-1)}{p^\alpha}\right) &= \sum_{r=0}^{p^{\alpha-\beta}-1} \sum_{s=1}^{p^\beta} \chi(rp^\beta+s) e\left(\frac{(rp^\beta+s)(a^3-1)}{p^\alpha}\right) \\
&= \sum_{s=1}^{p^\beta} \chi(s) e\left(\frac{s(a^3-1)}{p^\alpha}\right) \sum_{r=0}^{p^{\alpha-\beta}-1} e\left(\frac{r(a^3-1)}{p^{\alpha-\beta}}\right).
\end{aligned}$$

According to (2.2), we know that if (2.4) is not 0, then $p^{\alpha-\beta} | (a^3-1)$ holds. If $a^3-1 = kp^\gamma$ with $(k,p)=1$ and $\alpha-\beta \le \gamma < \alpha$, we have

$$\begin{aligned}
\sum_{m=1}^{p^\alpha} \chi(m) e\left(\frac{m(a^3-1)}{p^\alpha}\right) &= p^{\alpha-\beta} \sum_{s=1}^{p^\beta} \chi(s) e\left(\frac{sk}{p^{\alpha-\gamma}}\right) \\
&= \begin{cases} p^{\alpha-\beta} \tau(\chi, p^\beta) \overline{\chi}\left(\frac{a^3-1}{p^{\alpha-\beta}}\right) & \text{if } \gamma = \alpha-\beta; \\ 0 & \text{if } \alpha-\beta < \gamma < \alpha, \end{cases}
\end{aligned}$$

where $\tau(\chi,p^\beta)=\sum_{a=1}^{p^\beta}\chi(a)e(\frac{a}{p^\beta})$. If $\gamma=\alpha$, similarly, we have

$$\sum_{m=1}^{p^\alpha}\chi(m)e\left(\frac{m(a^3-1)}{p^\alpha}\right)=p^{\alpha-\beta}\sum_{s=1}^{p^\beta}\chi(s)=0.$$

So we get

$$\sum_{m=1}^{p^\alpha}\chi(m)e\left(\frac{m(a^3-1)}{p^\alpha}\right)=\begin{cases}p^{\alpha-\beta}\tau(\chi,p^\beta)\overline{\chi}\big(\frac{a^3-1}{p^{\alpha-\beta}}\big) & \text{if } p^{\alpha-\beta}\|(a^3-1);\\ 0, & \text{otherwise.}\end{cases}\tag{2.5}$$

Since $p>3$ is a prime with $(3,p-1)=1$, $\overline{\chi}^3$ is also a primitive character mod p^β. Then

$$\begin{aligned}\sum_{b=1}^{p^\alpha}\overline{\chi}^3(b)e\left(\frac{b(a-1)}{p^\alpha}\right)&=\sum_{r=0}^{p^{\alpha-\beta}-1}\sum_{s=1}^{p^\beta}\overline{\chi}^3(rp^\beta+s)e\left(\frac{(rp^\beta+s)(a-1)}{p^\alpha}\right)\\&=\sum_{s=1}^{p^\beta}\overline{\chi}^3(s)e\left(\frac{s(a-1)}{p^\alpha}\right)\sum_{r=0}^{p^{\alpha-\beta}-1}e\left(\frac{r(a-1)}{p^{\alpha-\beta}}\right)\\&=\begin{cases}p^{\alpha-\beta}\tau(\overline{\chi}^3,p^\beta)\chi^3\big(\frac{a-1}{p^{\alpha-\beta}}\big) & \text{if } p^{\alpha-\beta}\|(a-1);\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{2.6}$$

Noting that $2\beta\le\alpha$ and

$$\overline{\chi}(r^3p^{2(\alpha-\beta)}+3r^2p^{\alpha-\beta}+3r)=\overline{\chi}(3r),$$

from (2.4)–(2.6), we immediately deduce

$$\begin{aligned}&\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\\&=p^{2(\alpha-\beta)}\chi^3(n)\sum_{\substack{a=1\\p^{\alpha-\beta}\|(a-1)}}^{p^\alpha}\lambda(a)\tau(\chi,p^\beta)\overline{\chi}\left(\frac{a^3-1}{p^{\alpha-\beta}}\right)\tau(\overline{\chi}^3,p^\beta)\chi^3\left(\frac{a-1}{p^{\alpha-\beta}}\right)\\&=p^{2(\alpha-\beta)}\chi^3(n)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\chi^3(r)\overline{\chi}(r^3p^{2(\alpha-\beta)}+3r^2p^{\alpha-\beta}+3r)\\&\quad\times\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)\\&=p^{2(\alpha-\beta)}\chi^3(n)\overline{\chi}(3)\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\chi^2(r)\lambda(rp^{\alpha-\beta}+1).\end{aligned}\tag{2.7}$$

If χ is a non-real character mod p^β, from (2.7) and the properties of Gauss sums, we have

$$\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2$$

$$= p^{2(\alpha-\beta)}\chi^3(n)\overline{\chi}(3)\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\frac{\chi^2(r)}{\tau(\overline{\lambda},p^\alpha)}\sum_{b=1}^{p^\alpha}\overline{\lambda}(b)e\left(\frac{b(rp^{\alpha-\beta}+1)}{p^\alpha}\right)$$

$$= p^{2(\alpha-\beta)}\chi^3(n)\overline{\chi}(3)\frac{\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)}{\tau(\overline{\lambda},p^\alpha)}\sum_{b=1}^{p^\alpha}\overline{\lambda}(b)e\left(\frac{b}{p^\alpha}\right)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\chi^2(r)e\left(\frac{br}{p^\beta}\right)$$

$$= p^{2(\alpha-\beta)}\chi^3(n)\overline{\chi}(3)\frac{\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)\tau(\chi^2,p^\beta)}{\tau(\overline{\lambda},p^\alpha)}\sum_{b=1}^{p^\alpha}\overline{\lambda}(b)\overline{\chi}^2(b)e\left(\frac{b}{p^\alpha}\right)$$

$$= p^{2(\alpha-\beta)}\chi^3(n)\overline{\chi}(3)\frac{\tau(\chi,p^\beta)\tau(\overline{\chi}^3,p^\beta)\tau(\chi^2,p^\beta)\tau(\overline{\lambda}\overline{\chi}^2,p^\alpha)}{\tau(\overline{\lambda},p^\alpha)}.$$

Since χ, χ^2 and $\overline{\chi}^3$ are the primitive character mod p^β, $\overline{\lambda}$ and $\overline{\lambda}\overline{\chi}^2$ are the primitive character mod p^α, we have

$$\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 = p^{4\alpha-\beta}. \tag{2.8}$$

If $\beta=1$ and $\chi=\chi_2$ is the Legendre symbol mod p, we have

$$\sum_{m=1}^{p^\alpha}\chi_2(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2$$

$$= p^{2(\alpha-1)}\chi_2(n)\chi_2(3)\tau^2(\chi_2,p)\sum_{r=1}^{p-1}\lambda(rp^{\alpha-1}+1)$$

$$= p^{2(\alpha-1)}\chi_2(n)\chi_2(3)\tau^2(\chi_2,p)\left(\sum_{r=0}^{p-1}\lambda(rp^{\alpha-1}+1)-1\right)$$

$$= -p^{2(\alpha-1)}\chi_2(n)\chi_2(3)\tau^2(\chi_2,p),$$

where we have used the identity

$$\sum_{r=0}^{p-1}\lambda(rp^{\alpha-1}+u) = \frac{1}{\tau(\overline{\lambda})}\sum_{b=1}^{p^\alpha}\overline{\lambda}(b)e\left(\frac{bu}{p^\alpha}\right)\sum_{r=0}^{p-1}e\left(\frac{br}{p}\right) = 0. \tag{2.9}$$

So we have

$$\left|\sum_{m=1}^{p^\alpha}\chi_2(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2=p^{4\alpha-2}. \tag{2.10}$$

Now Lemma 2.2 follows from (2.8) and (2.10). □

LEMMA 2.3. *Let $p>3$ be a prime with $(3,p-1)=1$, α be a positive integer with $\alpha\geq 3$, n be any integer n with $(n,p)=1$, β be an integer with $\frac{1}{2}\alpha<\beta<\alpha$. If λ is a primitive character* mod p^α, *for any primitive character* χ mod p^β, *we have*

$$\sum_{\chi \bmod p^\beta}^{*}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2=p^{4\alpha-2\beta}\phi^2(p^\beta),$$

where $\sum^*_{\chi \bmod p^\beta}$ denotes the summation over all primitive characters of type mod p^β.

Proof. From the method of proving (2.7), the orthogonality of the characters mod p^β and the properties of primitive characters mod p^β, we have

$$\begin{aligned}
&\sum_{\chi \bmod p^\beta}^{*}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2\\
&=p^{4\alpha-2\beta}\sum_{\chi \bmod p^\beta}^{*}\left|\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\chi^3(r)\overline{\chi}(r^3p^{2(\alpha-\beta)}+3r^2p^{\alpha-\beta}+3r)\right|^2\\
&=p^{4\alpha-2\beta}\sum_{\chi \bmod p^\beta}^{*}\left|\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\chi}(p^{2(\alpha-\beta)}+3\overline{r}p^{\alpha-\beta}+3\overline{r}^2)\right|^2\\
&=p^{4\alpha-2\beta}\sum_{\chi \bmod p^\beta}\left|\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\chi}(p^{2(\alpha-\beta)}+3\overline{r}p^{\alpha-\beta}+3\overline{r}^2)\right|^2\\
&\quad-p^{4\alpha-2\beta}\sum_{\chi \bmod p^{\beta-1}}\left|\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\chi}(p^{2(\alpha-\beta)}+3\overline{r}p^{\alpha-\beta}+3\overline{r}^2)\right|^2\\
&=p^{4\alpha-2\beta}\phi^2(p^\beta)+p^{4\alpha-2\beta}\phi(p^\beta)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\ \sum_{\substack{s=1\\(s,p)=1\\rsp^{\alpha-\beta}+r+s\equiv 0 \bmod p^\beta}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1)
\end{aligned}$$

$$-p^{4\alpha-2\beta}\phi(p^{\beta-1})\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\sum_{\substack{s=1\\(s,p)=1\\s\equiv r \bmod p^{\beta-1}}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1)$$

(2.11)

$$-p^{4\alpha-2\beta}\phi(p^{\beta-1})\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\sum_{\substack{s=1\\(s,p)=1\\rsp^{\alpha-\beta}+r+s\equiv 0 \bmod p^{\beta-1}}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1),$$

where $\overline{r}$ denotes the multiplicative inverse of $r \mod p^\beta$ (that is $r\overline{r}\equiv 1 \bmod p^\beta$).

Then we show that the values of the above last three terms are all zero.

Since λ is a primitive character mod p^α, λ^2 is also a primitive character mod p^α. At the same time, we have

$$\begin{aligned}p^\beta|(rsp^{\alpha-\beta}+r+s) \quad &\Leftrightarrow \quad p^\alpha|(rp^{\alpha-\beta}\cdot sp^{\alpha-\beta})+(rp^{\alpha-\beta}+sp^{\alpha-\beta})\\ &\Leftrightarrow \quad (rp^{\alpha-\beta}+1)(sp^{\alpha-\beta}+1)\equiv 1 \bmod p^\alpha.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\sum_{\substack{s=1\\(s,p)=1\\rsp^{\alpha-\beta}+r+s\equiv 0 \bmod p^{\beta}}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1)\\
&=\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\lambda^2(rp^{\alpha-\beta}+1)\\
&=\frac{1}{\tau(\overline{\lambda}^2)}\sum_{b=1}^{p^\alpha}\overline{\lambda}^2(b)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}e\left(\frac{b(rp^{\alpha-\beta}+1)}{p^\alpha}\right)\\
&=\frac{1}{\tau(\overline{\lambda}^2)}\sum_{b=1}^{p^\alpha}\overline{\lambda}^2(b)e\left(\frac{b}{p^\alpha}\right)\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}e\left(\frac{br}{p^\beta}\right)=0.\end{aligned}\tag{2.12}$$

From the properties of reduced residue system mod $p^{\beta-1}$, we have

$$\sum_{\substack{r=1\\(r,p)=1}}^{p^\beta}\sum_{\substack{s=1\\(s,p)=1\\s\equiv r \bmod p^{\beta-1}}}^{p^\beta}\lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1)$$

$$= \sum_{\substack{u=1\\(u,p)=1}}^{p^{\beta-1}} \sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \lambda((rp^{\beta-1}+u)p^{\alpha-\beta}+1)\overline{\lambda}((sp^{\beta-1}+u)p^{\alpha-\beta}+1)$$

$$= \sum_{\substack{u=1\\(u,p)=1}}^{p^{\beta-1}} \sum_{r=0}^{p-1} \lambda(rp^{\alpha-1}+up^{\alpha-\beta}+1) \sum_{s=0}^{p-1} \overline{\lambda}(sp^{\alpha-\beta}+up^{\alpha-\beta}+1),$$

and

$$\sum_{\substack{r=1\\(r,p)=1\\rsp^{\alpha-\beta}+r+s\equiv 0 \bmod p^{\beta-1}}}^{p^\beta} \sum_{\substack{s=1\\(s,p)=1}}^{p^\beta} \lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1)$$

$$= \sum_{\substack{u=1\\(u,p)=1\\uvp^{\alpha-\beta}+u+v\equiv 0 \bmod p^{\beta-1}}}^{p^{\beta-1}} \sum_{\substack{v=1\\(v,p)=1}}^{p^{\beta-1}} \sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \lambda((rp^{\beta-1}+u)p^{\alpha-\beta}+1)\overline{\lambda}((sp^{\beta-1}+v)p^{\alpha-\beta}+1)$$

$$= \sum_{\substack{u=1\\(u,p)=1\\uvp^{\alpha-\beta}+u+v\equiv 0 \bmod p^{\beta-1}}}^{p^{\beta-1}} \sum_{\substack{v=1\\(v,p)=1}}^{p^{\beta-1}} \sum_{r=0}^{p-1} \lambda(rp^{\alpha-1}+up^{\alpha-\beta}+1) \sum_{s=0}^{p-1} \overline{\lambda}(sp^{\alpha-1}+vp^{\alpha-\beta}+1).$$

Note that by the identity

$$\sum_{r=0}^{p-1} \lambda(rp^{\alpha-1}+up^{\alpha-\beta}+1)$$
$$= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p^\alpha} \overline{\lambda}(b) \sum_{r=0}^{p-1} e\left(\frac{b(rp^{\alpha-1}+up^{\alpha-\beta}+1)}{p^\alpha}\right)$$
$$= \frac{1}{\tau(\overline{\lambda})} \sum_{b=1}^{p^\alpha} \overline{\lambda}(b) e\left(\frac{b(up^{\alpha-\beta}+1)}{p^\alpha}\right) \sum_{r=0}^{p-1} e\left(\frac{br}{p}\right) = 0,$$

we have

$$\sum_{\substack{r=1\\(r,p)=1\\s\equiv r \bmod p^{\beta-1}}}^{p^\beta} \sum_{\substack{s=1\\(s,p)=1}}^{p^\beta} \lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1) = 0, \tag{2.13}$$

and

$$\sum_{\substack{r=1\\(r,p)=1\\rsp^{\alpha-\beta}+r+s\equiv 0 \bmod p^{\beta-1}}}^{p^\beta} \sum_{\substack{s=1\\(s,p)=1}}^{p^\beta} \lambda(rp^{\alpha-\beta}+1)\overline{\lambda}(sp^{\alpha-\beta}+1) = 0. \tag{2.14}$$

Combining with (2.11)–(2.14), Lemma 2.3 can be proved. □

LEMMA 2.4. *Let $p>3$ be a prime with $(3,p-1)=1$, α be a positive integer, n be any integer n with $(n,p)=1$, λ be any character* mod p^α. *For any primitive character χ* mod p^α, *we have*

$$\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2=p^{2\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1)\right|^2.$$

Proof. Note that χ is a primitive character mod p^α, from the properties of Gauss sums we have

$$\begin{aligned}&\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\\&=\sum_{a=1}^{p^\alpha}\lambda(a)\sum_{b=1}^{p^\alpha}\overline{\lambda}(b)\sum_{m=1}^{p^\alpha}\chi(m)e\left(\frac{m(a^3-b^3)+n(a-b)}{p^\alpha}\right)\\&=\sum_{a=1}^{p^\alpha}\lambda(a)\sum_{b=1}^{p^\alpha}\overline{\chi}^3(b)e\left(\frac{nb(a-1)}{p^\alpha}\right)\sum_{m=1}^{p^\alpha}\chi(m)e\left(\frac{m(a^3-1)}{p^\alpha}\right)\\&=\chi^3(n)\tau(\overline{\chi}^3)\tau(\chi)\sum_{a=1}^{p^\alpha}\lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1).\end{aligned}$$

From this formula, we immediately deduce Lemma 2.4. □

LEMMA 2.5. *Let $p>2$ be a prime and $\alpha\geq 2$ be a positive integer. If λ is any primitive character* mod p^α, *we have*

$$\sum_{\substack{a=1\\(a(a-1),p)=1}}^{p^\alpha}\ \sum_{\substack{b=1\\(b(b-1),p)=1\\(ab-1)(b-a)\equiv 0 \bmod p^\alpha}}^{p^\alpha}\lambda(a)\overline{\lambda}(b)=p^{\alpha-1}(p-3).$$

Proof. It is clear that

$$\sum_{\substack{a=1\\(a(a-1),p)=1}}^{p^\alpha}\ \sum_{\substack{b=1\\(b(b-1),p)=1\\(ab-1)(b-a)\equiv 0 \bmod p^\alpha}}^{p^\alpha}\lambda(a)\overline{\lambda}(b)=\sum_{\substack{a=1\\(a(a+1),p)=1}}^{p^\alpha}\ \sum_{\substack{b=1\\(b(b+1),p)=1\\(ab+a+b)(b-a)\equiv 0 \bmod p^\alpha}}^{p^\alpha}\lambda(a+1)\overline{\lambda}(b+1).$$

From the second case of Lemma 1.4 in [6], we have

$$\sum_{\substack{a=1\\(a(a+1),p)=1}}^{p^\alpha}\ \sum_{\substack{b=1\\(b(b+1),p)=1\\(ab+a+b)(b-a)\equiv 0 \bmod p^\alpha}}^{p^\alpha}\lambda(a+1)\overline{\lambda}(b+1)=p^{\alpha-1}(p-3).$$

This immediately implies Lemma 2.5. □

3. PROOF OF THEOREM 1.1

In this section, we complete the proof of Theorem 1.1. For any integer n with $(n,p)=1$, from the orthogonality of the Dirichlet characters mod p^α, we have

$$\sum_{\chi \bmod p^\alpha}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 = \phi(p^\alpha)\sum_{m=1}^{p^\alpha}\chi_0(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^4, \tag{3.1}$$

where χ_0 is the principal character mod p^α.

On the other hand, if $\alpha \geq 2$ and λ is a primitive character mod p^α, then from Lemmas 2.1–2.4 and the properties of primitive characters mod p^α, we also have

$$\begin{aligned}
&\sum_{\chi \bmod p^\alpha}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 \\
&= \sum_{\beta=1}^{\alpha-1}\sideset{}{^*}\sum_{\chi \bmod p^\beta}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 \\
&\quad + \sideset{}{^*}\sum_{\chi \bmod p^\alpha}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 \\
&\quad + \left|\sum_{m=1}^{p^\alpha}\chi_0(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2 \\
&= (p^{2\alpha}-2p^{2\alpha-1})^2 + p^{4\alpha-2} + p^{4\alpha-1}(p-3) + \sum_{2\leq\beta\leq\frac{1}{2}\alpha}\left(\phi(p^\beta)-\phi(p^{\beta-1})\right)p^{4\alpha-\beta} \\
&\quad + \sum_{\frac{1}{2}\alpha<\beta\leq\alpha-1} p^{4\alpha-2\beta}\phi^2(p^\beta) + p^{2\alpha}\sideset{}{^*}\sum_{\chi \bmod p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1)\right|^2 \\
&= \alpha p^{2\alpha}\phi^2(p^\alpha) - 3p^{3\alpha-1}\phi(p^\alpha) + p^{2\alpha}\sum_{\chi \bmod p^\alpha}\left|\sum_{a=1}^{p^\alpha}\lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1)\right|^2
\end{aligned}$$

(3.2)

$$- p^{2\alpha} \sum_{\chi \bmod p^{\alpha-1}} \left| \sum_{a=1}^{p^\alpha} \lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1) \right|^2 .$$

By the orthogonality of the characters mod p^α and Lemma 2.5, we have

$$\sum_{\chi \bmod p^{\alpha}} \left| \sum_{a=1}^{p^\alpha} \lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1) \right|^2$$

$$= \phi(p^\alpha) \sum_{\substack{a=1 \\ (a(a^3-1),p)=1}}^{p^\alpha} \sum_{\substack{b=1 \\ (b(b^3-1),p)=1 \\ (a^3-1)(b-1)^3 \equiv (b^3-1)(a-1)^3 \bmod p^\alpha}}^{p^\alpha} \lambda(a)\overline{\lambda}(b)$$

$$= \phi(p^\alpha) \sum_{\substack{a=1 \\ (a(a^3-1),p)=1}}^{p^\alpha} \sum_{\substack{b=1 \\ (b(b^3-1),p)=1 \\ (a-1)(b-1)(ab-1)(b-a) \equiv 0 \bmod p^\alpha}}^{p^\alpha} \lambda(a)\overline{\lambda}(b)$$

$$= \phi(p^\alpha) \sum_{\substack{a=1 \\ (a(a-1),p)=1}}^{p^\alpha} \sum_{\substack{b=1 \\ (b(b-1),p)=1 \\ (ab-1)(b-a) \equiv 0 \bmod p^\alpha}}^{p^\alpha} \lambda(a)\overline{\lambda}(b) = \phi(p^\alpha)p^{\alpha-1}(p-3), \tag{3.3}$$

where the simple fact that $(a^3-1,p)=1$ if and only if $(a-1,p)=1$ is used.

Recalling (2.9), we have

$$\sum_{\chi \bmod p^{\alpha-1}} \left| \sum_{a=1}^{p^\alpha} \lambda(a)\overline{\chi}(a^3-1)\chi^3(a-1) \right|^2$$

$$= \phi(p^{\alpha-1}) \sum_{\substack{a=1 \\ (a(a^3-1),p)=1}}^{p^\alpha} \sum_{\substack{b=1 \\ (b(b^3-1),p)=1 \\ (a-1)(b-1)(ab-1)(b-a) \equiv 0 \bmod p^{\alpha-1}}}^{p^\alpha} \lambda(a)\overline{\lambda}(b)$$

$$= \phi(p^{\alpha-1}) \sum_{r=0}^{p-1}\sum_{s=0}^{p-1} \sum_{\substack{u=1 \\ (u(u^3-1),p)=1}}^{p^{\alpha-1}} \sum_{\substack{v=1 \\ (v(v^3-1),p)=1 \\ (u-v)(uv-1) \equiv 0 \bmod p^{\alpha-1}}}^{p^{\alpha-1}} \lambda(rp^{\alpha-1}+u)\overline{\lambda}(sp^{\alpha-1}+v)$$

$$= \phi(p^{\alpha-1}) \sum_{\substack{u=1 \\ (u(u^3-1),p)=1}}^{p^{\alpha-1}} \sum_{\substack{v=1 \\ (v(v^3-1),p)=1 \\ (u-v)(uv-1) \equiv 0 \bmod p^{\alpha-1}}}^{p^{\alpha-1}} \sum_{r=0}^{p-1} \lambda(rp^{\alpha-1}+u) \sum_{s=0}^{p-1} \overline{\lambda}(sp^{\alpha-1}+v)$$

$$= 0. \tag{3.4}$$

Combining with (3.2)–(3.4), we obtain

$$\sum_{\chi \bmod p^\alpha}\left|\sum_{m=1}^{p^\alpha}\chi(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^2\right|^2$$
$$= \alpha p^{2\alpha}\phi^2(p^\alpha) - 3p^{3\alpha-1}\phi(p^\alpha) + p^{2\alpha}\left(\phi(p^\alpha)p^{\alpha-1}(p-3)\right)$$
$$= p^{2\alpha}\phi^2(p^\alpha)\left(\alpha+1-\frac{5}{p-1}\right). \tag{3.5}$$

From (3.1) and (3.5), we deduce the identity

$$\sum_{m=1}^{p^\alpha}\chi_0(m)\left|\sum_{a=1}^{p^\alpha}\lambda(a)e\left(\frac{ma^3+na}{p^\alpha}\right)\right|^4 = p^{2\alpha}\phi(p^\alpha)\left(\alpha+1-\frac{5}{p-1}\right).$$

This completes the proof of Theorem 1.1.

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