MULTIPLICITY OF SOLUTIONS FOR SOME STEKLOV PROBLEMS INVOLVING THE $(p(x)$-$q(x))$-LAPLACE OPERATOR

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Communicated by Lucian Beznea

In this paper, we use the variational method to study some Steklov problems involving the $p(x)$-$q(x)$-Laplace operator. Specifically, in the first part of this paper, we combine the mountain pass theorem with Ekeland's variational principle to prove the existence of two nontrivial weak solutions. Furthermore, in the second part of this work, we use the symmetric version of the mountain pass theorem to prove the existence of an infinite number of solutions to such problems.

AMS 2020 Subject Classification: 35J20, 35A15, 46E35.

Key words: variational methods, symmetric mountain pass theorem, Steklov problems, Ekeland's variational principle.

1. INTRODUCTION

In recent years, the study of various variable exponent problems has received considerable attention due to their widespread use in applications in many fields, such as electro-rheological fluid modeling [23], image processing [9], and they raise many difficult mathematical issues. For more applications, we refer the interested readers to the overview papers [10, 17, 24].

The study of partial differential problems with variable exponents is a new and interesting topic. Recently, many researchers attracted their attention to the study of such problems; we refer, for example, to the papers [1, 2, 4–6, 13, 19, 26, 28], in which the authors have used different methods to obtain the existence and the multiplicity of solutions. Allaoui in [3] considered the following problem

$$(1)\qquad \begin{cases} (-\Delta)_{p(x)}u = \lambda\big(a(x)|u|^{q(x)-2}u + b(x)|u|^{r(x)-2}u\big) & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the operator $(-\Delta)_{\sigma(x)}$ is the $\sigma(x)$-Laplacian which is defined for a given positive continuous function σ on $\overline{\Omega}$, as follows:

$$(-\Delta)_{\sigma(x)}u = -\operatorname{div}\big(|\nabla u|^{\sigma(x)-2}\nabla u\big).$$

MATH. REPORTS **27(77)** (2025), *1-2*, 17–35
doi: 10.59277/mrar.2025.27.77.1.2.17

Under appropriate conditions and using the variational method combined with the mountain pass theorem, the author proves that if λ is small enough, then the problem (1) admits a nontrivial solution.

In a recent paper, Chammem et al. [8] have considered the following problem:

$$\begin{cases}(-\Delta)_{p(x)}u + a(x)|u|^{p(x)-2}u = f(x,u) & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} + b(x)|u|^{q(x)-2}u = g(x,u) & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial v}$ is the outer unit normal derivative. The functions a, b, p, q, g and f are assumed to satisfy some suitable assumptions. The authors proved the existence and the multiplicity of solutions by using variational methods, and mountain pass lemma combined with the Ekeland variational principle.

We note that the $p(x)$-Laplacian operator possesses more complicated nonlinearities than the well-known p-Laplacian, for example, it is inhomogeneous, and usually, it does not have the so-called first eigenvalue, since the infimum of its spectrum is zero. This causes many problems. For instance, some classical theories, and methods, such as the Lagrange multiplier theorem and the theory of Sobolev space, cannot be applied. Our goal in this paper is to continue this investigation to a more general operator, that is the $p(x)$-$q(x)$-Laplace operator. Precisely, we provide existing results for the following elliptic system of Steklov type:

$$(P_\lambda)\begin{cases}(-\Delta)_{p(x)}u+a(x)|u|^{p(x)-2}u+(-\Delta)_{q(x)}u+b(x)|u|^{q(x)-2}u = \lambda f(x,u) & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + |\nabla u|^{q(x)-2}\frac{\partial u}{\partial \nu} + g(x)|u|^{r(x)-2}u = h(x,u) & \text{on } \partial\Omega,\end{cases}$$

where $\lambda > 0$, a and b are bounded on Ω, g is a bounded on $\partial\Omega$, and f, h are Carathéodory functions. Recently, problems like (P_λ) are studied by many authors and by different methods. In the case when $p(x) \equiv p$ (a constant), interesting works can be found in [2, 4, 7, 20, 22]. In [7], Bonder and Rossi studied problem (P_λ) in the simple case when $a \equiv 1$, $b \equiv 0$, $f \equiv 0$ and $g(x,u) = g(u)$. By using mountain pass theorem, the authors prove the existence of nontrivial solution for such problem. After that, in [20], Martinez and Rossi were concerned with problem (P_λ) in the cases when $p(x) = q(x) \equiv p$, for $p > 1$ and the nonlinear terms f and g satisfy the Landesman–Lazer-type conditions. Also, Zhao et al. [29] considered the same problem of Martinez and Rossi [20] where the perturbation terms f and g, satisfy the Ambrosetti–Rabinowitz condition.

The rest of the present work is organized as follows. In Section 2, we recall some definitions and properties of the generalized Lebesgue and Sobolev spaces. In Section 3, using the mountain pass theorem, we present and prove the first main result of this paper. Section 4 is devoted to the proof of the second main result of this paper.

2. PRELIMINARIES

In order to apply the variational method to solve the question of the existence of solutions for problem (P_λ), we recall some definitions and properties about the generalized Lebesgue and Sobolev spaces. Throughout this paper, Ω is denoted as a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, $\mathbb{S}(\Omega)$ as the set of all measurable real functions defined on Ω, and $C_+(\overline{\Omega})$ is denoted by the following set:

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

Let us consider $p \in C_+(\overline{\Omega})$. The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \Big\{u \in \mathbb{S}(\Omega);\ \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty\Big\},$$

and equipped with the following norm

$$|u|_{L^{p(x)}(\Omega)} = \inf\Big\{\sigma > 0;\ \int_\Omega \Big|\frac{u(x)}{\sigma}\Big|^{p(x)} \mathrm{d}x \leq 1\Big\}.$$

By a similar way, $C_+(\partial\Omega)$, $L^{p(x)}(\partial\Omega)$ and $|u|_{L^{p(x)}(\partial\Omega)}$, can be defined by repalcing Ω with $\partial\Omega$ and $\mathrm{d}x$ with $d\sigma$, where $d\sigma$ is the surface measure on $\partial\Omega$.

The generalized Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \inf\Big\{\sigma > 0;\ \int_\Omega \Big|\frac{u(x)}{\sigma}\Big|^{p(x)} + \Big|\frac{\nabla u(x)}{\sigma}\Big|^{p(x)} \mathrm{d}x \leq 1\Big\}.$$

For any $h \in C_+(\overline{\Omega})$, we denoted by h^+ and h^-, the following expressions

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \ \text{ and } \ h^- = \inf_{x\in\overline{\Omega}} h(x).$$

Note that, if $h \in C_+(\partial\Omega)$, then h^+ and h^- are the same as above by replacing $\overline{\Omega}$ with Ω. If β is bounded on Ω such that $\beta^- \geq 0$, then we can define the following equivalent norm on $W^{1,p(x)}(\Omega)$

$$\|u\|_\beta = \inf\Big\{\sigma > 0;\ \int_\Omega \Big(\beta(x)\Big|\frac{u(x)}{\sigma}\Big|^{p(x)} \mathrm{d}x + \Big|\frac{\nabla u(x)}{\sigma}\Big|^{p(x)}\Big) \mathrm{d}x \leq 1\Big\}.$$

PROPOSITION 2.1 ([11, 14]). *The following statements hold:*

(i) *The next sets* $(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)}$ *and* $(W^{1,p(x)}(\Omega), \|.\|$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(x) \in C_+(\overline{\Omega})$ is such that $q(x) < p^\star(x)$ for any $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, where*

$$p^\star(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

PROPOSITION 2.2 ([27]). *If $q(x) \in C_+(\partial\Omega)$ is such that $q(x) < p^\partial(x)$ for any $x \in \partial\Omega$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is continuous and compact, where*

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

PROPOSITION 2.3 ([14, 15]). *Let p' be such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in \overline{\Omega}$. Then for each $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\Big| \int_\Omega uv \mathrm{d}x \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-} \Big) |u|_{p(x)} |v|_{p'(x)}.$$

Next, we have

$$\rho_a(u) = \int_\Omega \big(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \big) \mathrm{d}x.$$

PROPOSITION 2.4 ([28]). *Let $u \in W^{1,p(x)}(\Omega)$. Then there exist positive constants κ_1 and κ_2, such that*

(i) *If $\rho_a(u) \geq 1$, then $\kappa_1 \|u\|_a^{p^-} \leq \rho_a(u) \leq \kappa_2 \|u\|_a^{p^+}$.*

(ii) *If $\rho_a(u) \leq 1$, then $\kappa_1 \|u\|_a^{p^+} \leq \rho_a(u) \leq \kappa_2 \|u\|_a^{p^-}$.*

(iii) *$\rho_a(u) \geq 1 (= 1, \geq 1) \Leftrightarrow \|u\|_a \geq 1 (= 1, \geq 1)$.*

For the simplicity, the space $W^{1,p(x)}(\Omega)$ is denoted by E. Associated to the problem (P_λ), we define the functional $J_\lambda : E \to \mathbb{R}$, by

$$J_\lambda(u) \quad = \quad \Lambda_{a,p}(u) + \Lambda_{b,q}(u) - I(u) - \lambda\Psi(u) + \Phi(u),$$

where

$$I(u) = \int_{\partial\Omega} H(x,u)\mathrm{d}\sigma, \quad \Psi(u) = \int_\Omega F(x,u)\mathrm{d}x, \quad \Phi(u) = \int_{\partial\Omega} \frac{g(x)|u|^{r(x)}}{r(x)} \mathrm{d}\sigma,$$

with $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$ and $H(x,t) = \int_0^t h(x,s)\mathrm{d}s$.

For a given nonnegative bounded function β and for $\sigma \in C_+(\Omega)$, we denote

$$\Lambda_{\beta,\sigma}(u) = \int_\Omega \frac{|\nabla u|^{\sigma(x)} + \beta(x)|u|^{\sigma(x)}}{\sigma(x)} \mathrm{d}x.$$

By standard arguments (see [18, 21]), we can prove that $\Lambda_{\beta,\sigma} \in C^1(E,\mathbb{R})$. Moreover, for all u and v in E, we have

$$<\Lambda'_{\beta,\sigma}(u), v>= \int_\Omega \big(|\nabla u|^{\sigma(x)-2}\nabla u\nabla v + \beta(x)|u|^{\sigma(x)-2}uv\big)\mathrm{d}x.$$

PROPOSITION 2.5 ([16]). 1. *The functional* $\Lambda_{\beta,\sigma} : E \to \mathbb{R}$ *is convex and sequentially weakly lower semi-continuous.*

2. *The mapping* $\Lambda'_{\beta,\sigma} : E \to E^\star$ *is a strictly monotone, bounded homeomorphism, and is of type* (S_+)*, namely, if* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty} <\Lambda'_{\beta,\sigma}(u_n), u_n - u> \leq 0,$$

then $u_n \to u$*.*

PROPOSITION 2.6 ([16]). 1. *The functionals* I *and* $\Psi : E \to \mathbb{R}$ *are sequentially weakly continuous in* $C^1(E,\mathbb{R})$*. Moreover, for all* $u, v \in E$*, we have*

$$<I'(u), v>= \int_{\partial\Omega} h(x,u)v\mathrm{d}\sigma \quad \textit{and} \quad <\Psi'(u), v>= \int_\Omega f(x,u)v\mathrm{d}x.$$

2. *The mappings* $I', \Psi' : E \to E^\star$ *are weakly-strongly continuous, namely, if* $u_n \rightharpoonup u$*, then we have* $I'(u_n) \to I'(u)$ *and* $\Psi'(u_n) \to \Psi'(u)$*.*

PROPOSITION 2.7 ([8]). 1. *The functional* $\Phi \in C^1(E,\mathbb{R})$ *and for all* $u,\ v \in E$*,*

$$<\Phi'(u), v>= \int_{\partial\Omega} g(x)|u|^{r(x)-2}uv\mathrm{d}\sigma.$$

2. *The mapping* $\Phi' : E \to E^\star$ *is weakly-strongly continuous.*

By Propositions 2.5, 2.6 and 2.7, it is easy to check that the functional $J_\lambda \in C^1(E,\mathbb{R})$. Moreover, for all $u,\ v \in E$, one has

$$\begin{aligned}<J'_\lambda(u), v> = &\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\,\mathrm{d}x - \int_{\partial\Omega} h(x,u)v\mathrm{d}\sigma \\ &+ \int_\Omega |\nabla u|^{q(x)-2}\nabla u\nabla v + b(x)|u|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega f(x,u)v\mathrm{d}x \\ &+ \int_{\partial\Omega} g(x)|u|^{r(x)-2}uv\mathrm{d}\sigma.\end{aligned}$$

3. FIRST EXISTENCE RESULT AND ITS PROOF

In this section, we combine the mountain pass theorem [25] of Ambrosetti and Rabinowitz combined with Ekeland's variational principle in order to prove the existence of two nontrivial weak solutions of problem (P_λ). Note that $u \in E$ is said to be a weak solution of problem (P_λ), if for any $v \in E$, we have:

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \mathrm{d}x + \int_\Omega a(x)|u|^{p(x)-2}uv\mathrm{d}x + \int_\Omega |\nabla u|^{q(x)-2}\nabla u\nabla v\mathrm{d}x \\ + \int_\Omega b(x)|u|^{q(x)-2}uv\mathrm{d}x - \int_{\partial\Omega} h(x,u)v\mathrm{d}\sigma \\ + \int_{\partial\Omega} g(x)|u|^{r(x)-2}uv\mathrm{d}\sigma - \lambda\int_\Omega f(x,u)v\mathrm{d}x = 0.\end{aligned}$$

We recall now the mountain pass theorem, which we use to prove the first main result of this paper.

THEOREM 3.1 (Mountain Pass Theorem [25]). *Let X be a Banach space. Let $\varphi \in C^1(X,\mathbb{R})$, satisfying the following conditions:*

1. $\varphi(0) = 0$,
2. *φ satisfies the Palais–Smale condition, that is any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$, in $X^\star$ as $n \to \infty$, has a convergent subsequence.*
3. *There exist the positive constants r and ρ, such that if $\|u\| = r$, then, $\varphi(u) \geq \rho$,*
4. *There exist $e \in X$ with $\|e\| > r$ such that $\varphi(e) \leq 0$.*

Then, φ possesses a critical value $c \geq \rho$ which can be characterized as

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \varphi\big(\gamma(t)\big),$$

where,

$$\Gamma = \Big\{\gamma \in C\big([0,1],X\big) : \gamma(0) = 0, \gamma(1) = e\Big\}.$$

In order to state the first main result of this paper, we assume the following hypotheses:

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist $c_1 > 0$, $\alpha \in C_+(\overline{\Omega})$, such that

$$f(x,u) \leq c_1|u|^{\alpha(x)-1} \quad \text{for all } (x,u) \in \Omega \times \mathbb{R},$$

and

$$1 < \alpha(x) < p^\star(x) \quad \text{for all } x \in \overline{\Omega}, \tag{2}$$

where $p^\star(x)$ is defined in the statement of Proposition 2.1.

(H_2) $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist $c_2 > 0$, $\beta \in C_+(\partial\Omega)$, such that

$$h(x,u) \le c_2|u|^{\beta(x)-1} \quad \text{for all } (x,u) \in \partial\Omega \times \mathbb{R},$$

and

$$1 < \beta(x) < p^{\partial}(x) \quad \text{for all } x \in \partial\Omega, \tag{3}$$

where $p^{\partial}(x)$ is defined in the statement of Proposition 2.2.

(H_3) There exist $M_1 > 0$, $\theta_1 > p^+$ such that

$$0 < \theta_1 F(x,t) \le tf(x,t), \ |t| > M_1, \ x \in \Omega.$$

(H_4) There exist $M_2 > 0$, $\theta_2 > p^+$ such that

$$0 < \theta_2 H(x,t) \le th(x,t), \ |t| > M_2, \ x \in \partial\Omega.$$

(H_5) There exist $1 < \mu_1 < p^-$, such that

$$\liminf_{u\to 0} \frac{H(x,u) - \frac{g(x)|u|^{r(x)}}{r(x)}}{|u|^{\mu_1}} > 0 \quad \text{uniformly, for } x \in \partial\Omega.$$

(H_6) There exist $1 < \mu_2 < q^-$, such that

$$\liminf_{u\to 0} \frac{F(x,u)}{|u|^{\mu_2}} > 0 \quad \text{uniformly, for } x \in \Omega.$$

Now, we state the first main result of this paper.

THEOREM 3.2. *Suppose that hypotheses* (H_1)–(H_6) *hold. If* $p, q \in C_+(\overline{\Omega})$, *and* $r \in C_+(\partial\Omega)$ *are such that*

$$\max(p^+, q^+, r^+) < \min(\theta_1, \theta_2), \quad \textit{and} \ \ \alpha^- < \max(p^+, q^+) < \beta^-.$$

Then the problem (P_λ) *has at least two nontrivial weak solutions.*

To prove Theorem 3.2, we need to prove several lemmas.

LEMMA 3.3. *Assume that* $\alpha^- < p^+ < \beta^-$, *and* (H_1)–(H_2) *hold. Then for all* $\rho \in (0,1)$, *there exists* $\lambda^\star > 0$ *and* $m > 0$ *such that for all* $u \in E$ *with* $\|u\| = \rho$

$$J_\lambda(u) \ge m > 0 \ \ \textit{for all } \lambda \in (0, \lambda^\star).$$

Proof. Let $u \in E$. Then from hypotheses (H_1) and (H_2), we have

$$F(x,u) \le c_1 \frac{|u|^{\alpha(x)}}{\alpha(x)} \quad \text{for all } x \in \overline{\Omega} \tag{4}$$

and

$$H(x,u) \le c_2 \frac{|u|^{\beta(x)}}{\beta(x)} \quad \text{for all } x \in \partial\Omega. \tag{5}$$

Since $\alpha(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, then it follows by Proposition 2.1(ii) that $E \hookrightarrow L^{\alpha(x)}(\Omega)$. So, there exists $c_3 > 0$ such that

$$|u|_{L^{\alpha(x)}(\Omega)} \leq c_3\|u\| \quad \text{for all } u \in E. \tag{6}$$

Moreover, $\beta(x) < p^\partial(x)$ for all $x \in \partial\Omega$, and according to Proposition 2.2, there exists a positive constant c_4 such that

$$|u|_{L^{\beta(x)}(\partial\Omega)} \leq c_4\|u\| \quad \text{for all } u \in E. \tag{7}$$

We fix ρ such that $0 < \rho < \min(1, \frac{1}{c_3}, \frac{1}{c_4})$, and let $u \in E$ with $\|u\| = \rho$. Then equations (6) and (7) imply

$$|u|_{L^{\alpha(x)}(\Omega)} < 1 \text{ and } |u|_{L^{\beta(x)}(\partial\Omega)} < 1. \tag{8}$$

So using equations (6), (7) and (8), we obtain

$$\begin{aligned}
J_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}\mathrm{d}x + \int_\Omega \frac{|\nabla u|^{q(x)} + b(x)|u|^{q(x)}}{q(x)}\mathrm{d}x \\
&\quad - \int_{\partial\Omega} H(x,u)\mathrm{d}\sigma - \lambda \int_\Omega F(x,u)\mathrm{d}x + \int_{\partial\Omega} \frac{g(x)|u|^{r(x)}}{r(x)}\mathrm{d}\sigma \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \int_{\partial\Omega} H(x,u)\mathrm{d}\sigma - \lambda \int_\Omega F(x,u)\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{c_1}{\beta^-}\int_{\partial\Omega} |u|^{\beta(x)}\mathrm{d}\sigma - \lambda\frac{c_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{c_1}{\beta^-}\max\big(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\big) \\
&\quad - \lambda\frac{c_2}{\alpha^-}\max\big(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\big) \\
&\geq \frac{\kappa_1}{p^+}\|u\|_a^{p^+} + \frac{\kappa_1'}{q^+}\|u\|_b^{q^+} - \frac{c_1}{\beta^-}|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)} - \lambda\frac{c_2}{\alpha^-}|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)} \\
&\geq \frac{\kappa_1\zeta_1}{p^+}\|u\|^{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\|u\|^{q^+} - \frac{c_1}{\beta^-}c_4^{\beta^-}\|u\|^{\beta^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\|u\|^{\alpha^-} \\
&\geq \Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big)\|u\|^{\max(p^+,q^+)} - \frac{c_1}{\beta^-}c_4^{\beta^-}\|u\|^{\beta^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\|u\|^{\alpha^-} \\
&= \rho^{\alpha^-}\Big(\Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big)\rho^{\max(p^+,q^+)-\alpha^-} - \frac{c_1}{\beta^-}c_4^{\beta^-}\rho^{\beta^- - \alpha^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\Big) \\
&= \rho^{\alpha^-}\big(\varphi(t) - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\big),
\end{aligned}$$

for some positive constants $\kappa_1, \zeta_1, \kappa_2$ and ζ_2, where $\varphi : (0,1) \to \mathbb{R}$ is defined by

$$\varphi(t) = \Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big)t^{\max(p^+,q^+)-\alpha^-} - \frac{c_1}{\beta^-}c_4^{\beta^-}t^{\beta^- - \alpha^-}.$$

A simple calculation shows that φ attains its maximum at t_0, which is given by

$$t_0 = \left(\frac{\left(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\right)\beta^-(\max(p^+,q^+) - \alpha^-)}{c_1 c_4^{\beta^-}(\beta^- - \alpha^-)} \right)^{\frac{1}{\beta^- - \max(p^+,q^+)}}.$$

If we put

$$\lambda^\star = \frac{\alpha^-\varphi(t_0)}{c_2 c_3^{\alpha^-}} \quad \text{and} \quad m = \rho^{\alpha^-}\left(\varphi(t_0) - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\right), \tag{9}$$

then, we can deduce that for each $u \in E$ with $\|u\| = \rho$, we have

$$J_\lambda(u) \geq m > 0.$$

The proof is now completed. $\square$

Lemma 3.4. *Under hypotheses* (H_3) *and* (H_4)*, the functional* J_λ *satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\} \subset E$ be a sequence such that

$$\left|J_\lambda(u_n)\right| < M_1, \text{ and } J_\lambda'(u_n) \to 0 \text{ in } E^\star \text{ as } n \to \infty, \tag{10}$$

for some positive constant M_1, where $E^\star$ is a dual space of E.

First, we show that $\{u_n\}$ is bounded in E. Indeed, assume by contradiction that $\{u_n\}$ is not bounded in E. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$.

From (10) and Proposition 2.4, we get

$$\begin{aligned}
M_1 > J_\lambda(u_n) &\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{1}{\theta_1}\int_{\partial\Omega} h(x,u_n)u_n \mathrm{d}\sigma \\
&\quad - \frac{\lambda}{\theta_2}\int_\Omega f(x,u_n)u_n \mathrm{d}x + \frac{1}{r^+}\int_{\partial\Omega} g(x)|u|^{r(x)}\mathrm{d}\sigma \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{1}{\theta}\Big(\int_{\partial\Omega} h(x,u_n)u_n \mathrm{d}\sigma + \lambda\int_\Omega f(x,u_n)u_n \mathrm{d}x\Big) \\
&\quad + \frac{1}{r^+}\int_{\partial\Omega} g(x)|u|^{r(x)}\mathrm{d}\sigma \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\theta}\Big)\rho_a(u) + \Big(\frac{1}{q^+} - \frac{1}{\theta}\Big)\rho_b(u) + \Big(\frac{1}{r^+} - \frac{1}{\theta}\Big)\int_{\partial\Omega} g(x)|u|^{r(x)}\mathrm{d}\sigma \\
&\quad + \frac{1}{\theta} < J_\lambda'(u_n), u_n > \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\theta}\Big)\kappa_1\|u_n\|_a^{p^-} + \frac{1}{\theta}\|J_\lambda'(u_n)\|\|u_n\|,
\end{aligned}$$

where $\theta = \min(\theta_1, \theta_2)$.

Since $\theta > p^+$, then by letting n tends to infinity in the above inequality, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E which is a reflexive Banach space. Therefore, there exists a subsequence, again denoted by $\{u_n\}$ and $u \in E$ such that

$$u_n \rightharpoonup u \text{ in } E.$$

To finish the proof, we need to prove that $u_n \to u$ strongly in E. Using the fact that $J'_\lambda(u_n) \to 0$ and u_n is bounded in E, we have

$$\lim_{n\to\infty} < J'_\lambda(u_n), u_n - u >= 0. \tag{11}$$

On the other hand, by the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\partial\Omega} g(x)|u_n|^{r(x)-2}u_n(u_n-u)d\sigma \\
&\qquad\leq \|g\|_\infty \int_{\partial\Omega} g(x)|u_n|^{r(x)-1}|u_n-u|\mathrm{d}\sigma \\
&\qquad\leq \|g\|_\infty ||u_n|^{r(x)-1}|_{L^{\frac{r(x)}{r(x)-1}}(\partial\Omega)} |u_n-u|_{L^{r(x)}(\partial\Omega)}\mathrm{d}\sigma \\
&\qquad\leq \|g\|_\infty |u_n-u|_{L^{r(x)}(\partial\Omega)} |u_n|^{r(x)-1}_{L^{r(x)}(\partial\Omega)} \\
&\qquad\leq c_5\|g\|_\infty |u_n-u|_{L^{r(x)}(\partial\Omega)} \max\left(\|u_n\|^{r^+-1}, \|u_n\|^{r^--1}\right).
\end{aligned}$$

Since $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, we deduce that E is compactly embedded in $L^{r(x)}(\partial\Omega)$. So, $u_n \to u$ in $L^{r(x)}(\partial\Omega)$. Then,

$$\lim_{n\to\infty} \int_{\partial\Omega} g(x)|u_n|^{r(x)-2}u_n(u_n-u)\mathrm{d}\sigma = 0. \tag{12}$$

Now, from (H_1) and the Hölder inequality, one has

$$\begin{aligned}
\int_{\partial\Omega} h(x,u_n)(u_n-u)\mathrm{d}\sigma &\leq c_2 \int_{\partial\Omega} |u_n|^{\beta(x)-1}|u_n-u|\mathrm{d}\sigma \\
&\leq c_2||u_n|^{\beta(x)-1}|_{L^{\frac{\beta(x)}{\beta(x)-1}}(\partial\Omega)} |u_n-u|_{L^{\beta(x)}(\partial\Omega)}\mathrm{d}\sigma \\
&\leq c_2 |u_n-u|_{L^{\beta(x)}(\partial\Omega)} |u_n|^{\beta(x)-1}_{L^{\beta(x)}(\partial\Omega)} \\
&\leq c_6c_2 |u_n-u|_{L^{\beta(x)}(\partial\Omega)} \max\left(\|u_n\|^{\beta^+-1}, \|u_n\|^{\beta^--1}\right).
\end{aligned}$$

Since $\beta(x) < p^\partial(x)$ for all $x \in \partial\Omega$, we deduce that E is compactly embedded in $L^{\beta(x)}(\partial\Omega)$. So, $u_n \to u$ in $L^{\beta(x)}(\partial\Omega)$. Then,

$$\lim_{n\to\infty} \int_{\partial\Omega} h(x,u_n)(u_n-u)\mathrm{d}\sigma = 0. \tag{13}$$

With similar arguments, we can obtain that

$$\lim_{n\to\infty} \int_{\Omega} f(x,u_n)(u_n-u)\mathrm{d}x = 0. \tag{14}$$

By combining (11), (12), (13) with (14), we obtain

$$\lim_{n\to\infty} < \Lambda'_{a,p}(u_n), u_n - u > + \lim_{n\to\infty} < \Lambda'_{b,q}(u_n), u_n - u >= 0. \tag{15}$$

Since $a \in L^\infty(\Omega),$ and $a^- = ess \inf_{x\in\Omega} a(x) \geq 0$, then we have

$$\begin{aligned}
&< \Lambda'_{a,p}(u_n) - \Lambda'_{a,p}(u), u_n - u > \\
&\qquad = \int_\Omega \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\big)(\nabla u_n - \nabla u)\mathrm{d}x.
\end{aligned}$$

Using the following elementary inequality which holds for all $s,\ t \in \mathbb{R}^N$

$$\big(|s|^{p-2}s - |t|^{p-2}t, s-t\big) \geq \begin{cases} C_p|s-t|^p & \text{for all } p \geq 2, \\ C_p\frac{|s-t|^2}{(|s|+|t|)^{2-p}} & \text{for all } p \leq 2, \end{cases} \tag{16}$$

where C_p is a positive constant and $(.,.)$ is the standard scalar product in $\mathbb{R}^N$, and using equations (15) and (16), we obtain

$$\lim_{n\to\infty} < \Lambda'_{a,p}(u_n) - \Lambda'_{a,p}(u), u_n - u >= \lim_{n\to\infty} < \Lambda'_{b,q}(u_n) - \Lambda'_{b,q}(u), u_n - u >= 0.$$

Finally, Proposition 2.5 implies that $\{u_n\}$ converges strongly to u in E. This completes the proof. $\square$

Lemma 3.5. *Under hypotheses (H_5) and (H_6), there exists $e \in E$ such that $J_\lambda(te) < 0$ for all $t > 0$ small enough.*

Proof. Let $u \in E$ such that $\|u\| < 1$ small enough. From hypotheses (H_5) and (H_6), there exist two positive constants $c_7,\ c_8 > 0$ such that

$$H(x,u) - \frac{g(x)|u|^{r(x)}}{r(x)} \geq c_7|u|^{\mu_1} \text{ and } \lambda F(x,u) \geq c_8|u|^{\mu_2}. \tag{17}$$

Let $e \in E$ be such that

$$\int_{\partial\Omega} |e|^{\mu_1}\mathrm{d}\sigma \int_\Omega |e|^{\mu_2}\mathrm{d}x \neq 0.$$

From (17) and for $t > 0$ small enough, one has

(18)

$$\begin{aligned}
J_\lambda(te) &= \int_\Omega \frac{t^{p(x)}}{p(x)}\big[|\nabla e|^{p(x)} + a(x)|e|^{p(x)}\big]\mathrm{d}x + \int_\Omega \frac{t^{q(x)}}{q(x)}\big[|\nabla e|^{q(x)} + b(x)|e|^{q(x)}\big]\mathrm{d}x \\
&\quad - \int_{\partial\Omega}\Big[H(x,te) - \frac{g(x)}{r(x)}|te|^{r(x)}\Big]\mathrm{d}\sigma - \lambda\int_\Omega F(x,te)\mathrm{d}x \\
&\leq \frac{t^{p^-}}{p^-}\rho_a(e) + \frac{t^{q^-}}{q^-}\rho_b(e) - c_7t^{\mu_1}\int_{\partial\Omega}|e|^{\mu_1}\mathrm{d}\sigma - c_8t^{\mu_2}\int_\Omega |e|^{\mu_2}\mathrm{d}x \\
&\leq t^{\mu_1}\Big(t^{p^- - \mu_1}\frac{\rho_a(e)}{p^-} - c_7\int_{\partial\Omega}|e|^{\mu_1}\mathrm{d}\sigma\Big) + t^{\mu_2}\Big(t^{q^- - \mu_2}\frac{\rho_b(e)}{q^-} - c_8\int_\Omega |e|^{\mu_2}\mathrm{d}x\Big).
\end{aligned}$$

From (18), we have $J_\lambda(te) < 0$ for all $0 < t < \min(t_1, t_2)$ where

$$t_1 = \Big(\frac{p^- c_7 \int_{\partial\Omega} |e|^{\mu_1} \mathrm{d}\sigma}{\rho_a(e)}\Big)^{\frac{1}{p^- - \mu_1}} \quad \text{and} \quad t_2 = \Big(\frac{q^- c_8 \int_{\Omega} |e|^{\mu_2} \mathrm{d}x}{\rho_b(e)}\Big)^{\frac{1}{q^- - \mu_2}}. \quad \square$$

Proof of Theorem 3.2. We begin by noting that from conditions (H_3)–(H_4), there exist two positive constants l_1, $l_2 > 0$, such that

$$F(x,s) \le l_1 |s|^{\theta_1} \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}, \tag{19}$$

and

$$H(x,s) \le l_2 |s|^{\theta_2} \quad \text{for all } (x,s) \in \partial\Omega \times \mathbb{R}. \tag{20}$$

Let $u \in E$ and $t > 1$, then from equations (19) and (20), one has

$$\begin{aligned} J_\lambda(tu) &= \int_\Omega \frac{t^{p(x)}}{p(x)} \big[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\big]\mathrm{d}x + \int_\Omega \frac{t^{q(x)}}{q(x)} \big[|\nabla u|^{q(x)} + b(x)|u|^{q(x)}\big]\mathrm{d}x \\ &\quad - \int_{\partial\Omega} H(x,tu)\mathrm{d}\sigma - \lambda \int_\Omega F(x,tu)\mathrm{d}x + \int_{\partial\Omega} \frac{t^{r(x)}}{r(x)} \big[g(x)|u|^{r(x)}\big]\mathrm{d}\sigma \\ &\le \frac{t^{p^+}}{p^-} \int_\Omega \big[|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\big]\mathrm{d}x + \frac{t^{q^+}}{q^-} \int_\Omega \big[|\nabla u|^{q(x)} + b(x)|u|^{q(x)}\big]\mathrm{d}x \\ &\quad - l_2 t^{\theta_2} \int_{\partial\Omega} |u|^{\theta_2}\mathrm{d}\sigma - \lambda l_1 t^{\theta_1} \int_\Omega |u|^{\theta_1}\mathrm{d}x + \frac{t^{r^+}}{r^-}\|g\|_\infty \int_{\partial\Omega} |u|^{r(x)}\mathrm{d}\sigma. \end{aligned}$$

Next, since $\min(\theta_1, \theta_2) > \max(p^+, q^+, r^+)$, then for each $\lambda > 0$, we have $J_\lambda(tu) \to -\infty$ as $t \to \infty$.

It follows that there exists $t_0 > 0$ large enough such that $e = t_0 u$ satisfies $\|e\| > \rho$ and $J_\lambda(e) < 0$. Hence, condition (4) of Theorem 3.1 is satisfied. Moreover, Lemma 3.3 implies that condition (3) of Theorem 3.1 is also satisfied. On the other hand, condition (2) of Theorem 3.1 is a direct consequence of Lemma 3.4. Finally, since $J_\lambda(0) = 0$, then Theorem 3.1 implies that there exists $u_1 \in E$ which is a nontrivial weak solution of problem (P_λ).

Now, we apply Ekeland's variational principle [12] to get the second solution of problem (P_λ). We note that from Lemma 3.3, we have

$$\inf_{u \in \partial B_\rho(0)} J_\lambda(u) > 0,$$

where

$$B_\rho(0) = \big\{\omega \in E;\ \|\omega\| < \rho\big\}.$$

On the other hand, by Lemma 3.5, there exists $e \in E$ such that $J_\lambda(te) < 0$ for $t > 0$ small enough.

Since for all $u \in B_\rho(0)$, we have

$$J_\lambda(u) \ge \frac{\kappa_1}{p^+}\|u\|^{p^+} - \frac{c_1}{\beta^-} c_4^{\beta^-} \|u\|^{\beta^-} - \lambda \frac{c_2}{\alpha^-} c_3^{\alpha^-} \|u\|^{\alpha^-}.$$

So, we deduce that

$$-\infty \leq \underline{c} := \inf_{u \in \overline{B_\rho(0)}} J_\lambda(u) < 0.$$

Let ϵ be such that

$$0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J_\lambda(u) - \inf_{u \in B_\rho(0)} J_\lambda(u).$$

Then using the above information, the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. So, by applying Ekeland's variational principle, we can find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq J_\lambda(u_\epsilon) \leq \underline{c} + \epsilon \\ 0 < J_\lambda(u) - J_\lambda(u_\epsilon) + \epsilon.\|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{u \in \overline{B_\rho(0)}} J_\lambda(u) + \epsilon \leq \inf_{u \in B_\rho(0)} J_\lambda(u) + \epsilon < \inf_{u \in \partial B_\rho(0)} J_\lambda(u),$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $K_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$K_\lambda(u) = J_\lambda(u) + \epsilon\|u - u_\epsilon\|.$$

It is clear that u_ϵ is a minimum point of K_λ and thus

$$\frac{K_\lambda(u_\epsilon + tv) - K_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_\rho(0)$. That is

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

By letting $t \to 0$, we obtain

$$< J'_\lambda(u_\epsilon), v > + \epsilon\|v\| \geq 0,$$

which implies that

$$\|J'_\lambda(u_\epsilon)\| \leq \epsilon.$$

If we take $\epsilon = \frac{1}{n}$ and $v_n = u_{\frac{1}{n}}$, then we can see that $\{v_n\} \subset B_\rho(0)$ such that

$$J_\lambda(v_n) \to \underline{c} \text{ and } J'_\lambda(v_n) = 0. \tag{21}$$

Since J_λ satisfies the Palais–Smale condition on E, we conclude from Lemma 3.4, the existence of a subsequence still denoted by $\{v_n\}$ and $u_2 \in E$ such that $\{v_n\}$ strongly converges to u_2 in E. And so, u_2 is a weak solution for problem (P_λ). On the other hand, from equation (21), it follows that

$$J_\lambda(u_2) = \underline{c} < 0 < J_\lambda(u_1).$$

That is u_1 and u_2 are two distinct nontrivial solution for problem (P_λ). □

4. SECOND MAIN RESULT AND ITS PROOF

In this section, we state and prove the second main result of this paper. Precisely, we prove that under appropriate conditions, problem (P_λ) admits infinitely many nontrivial solutions. Our main tools are based on the Z_2 symmetric version of the mountain pass theorem (that is for even function), which we collect in the following theorem.

THEOREM 4.1. *Let X be an infinite dimensional real Banach space. Let $\varphi \in C^1(X, \mathbb{R})$, satisfying the following conditions:*

1. *φ is an even functional such that $\varphi(0) = 0$.*
2. *φ satisfies the (PS)-condition.*
3. *There exist two positive constants r and ρ, such that if $\|u\| = r$, then, $\varphi(u) \geq \rho$.*
4. *For each finite dimensional subspace $X_1 \subset X$, the set $\{u \in X_1, \varphi(u) \geq 0\}$ is bounded in X.*
   Then φ has an unbounded sequence of critical values.

We assume the following hypothesis.
(H_1') $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$f(x, u) \geq c_1' |u|^{\alpha(x)-1} \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

for some positive constant c_1', where $\alpha \in C_+(\overline{\Omega})$ is such that

$$\max(p^+, q^+, r^+) < \alpha^- \leq \alpha(x) < p^\star(x) \quad \text{for all } x \in \overline{\Omega}, \tag{22}$$

and

$$\beta^- > \max(p^+, q^+). \tag{23}$$

The second main result of this paper is the following.

THEOREM 4.2. *Under hypotheses (H_1'), (H_3) and (H_4), if in addition the functions $F(x, u)$ and $H(x, u)$ are even with respect to u, then the problem (P_λ) has infinitely many nontrivial weak solutions in E.*

In order to prove Theorem 4.2, we need to prove the following two lemmas.

LEMMA 4.3. *Assume that the hypotheses of Theorem 4.2 are satisfied. Then, for any $\lambda > 0$ there exist $\rho, m > 0$ such that $J_\lambda \geq m > 0$ for any $u \in E$ with $\|u\| = \rho$.*

Proof. Using equations (6), (7) and (8), we obtain

$$\begin{aligned}
J_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}\mathrm{d}x + \int_\Omega \frac{|\nabla u|^{q(x)} + b(x)|u|^{q(x)}}{q(x)}\mathrm{d}x \\
&\qquad - \int_{\partial\Omega} H(x,u)\mathrm{d}\sigma - \lambda \int_\Omega F(x,u)\mathrm{d}x + \int_{\partial\Omega} \frac{g(x)|u|^{r(x)}}{r(x)}\mathrm{d}\sigma \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \int_{\partial\Omega} H(x,u)\mathrm{d}\sigma - \lambda \int_\Omega F(x,u)\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{c_1}{\beta^-}\int_{\partial\Omega} |u|^{\beta(x)}\mathrm{d}\sigma - \lambda\frac{c_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}\mathrm{d}x \\
&\geq \frac{1}{p^+}\rho_a(u) + \frac{1}{q^+}\rho_b(u) - \frac{c_1}{\beta^-}\max\big(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\big) \\
&\qquad - \lambda\frac{c_2}{\alpha^-}\max\big(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\big) \\
&\geq \frac{\kappa_1}{p^+}\|u\|_a^{p^+} + \frac{\kappa_1'}{q^+}\|u\|_b^{q^+} - \frac{c_1}{\beta^-}|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)} - \lambda\frac{c_2}{\alpha^-}|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)} \\
&\geq \frac{\kappa_1\zeta_1}{p^+}\|u\|^{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\|u\|^{q^+} - \frac{c_1}{\beta^-}c_4^{\beta^-}\|u\|^{\beta^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\|u\|^{\alpha^-} \\
&\geq \Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big)\|u\|^{\max(p^+,q^+)} - \frac{c_1}{\beta^-}c_4^{\beta^-}\|u\|^{\beta^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}\|u\|^{\alpha^-},
\end{aligned}$$

for some positive constants $\kappa_1, \zeta_1, \kappa_2$ and ζ_2. We define

$$\varphi_\lambda(t) = \Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big)t^{\max(p^+,q^+)} - \frac{c_1}{\beta^-}c_4^{\beta^-}t^{\beta^-} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}t^{\alpha^-}.$$

We have that $\alpha^- > \max(p^+, q^+)$ and $\beta^- > \max(p^+, q^+)$. Then

$$\varphi_\lambda(t) = t^{\max(p^+,q^+)}\Phi_\lambda(t),$$

where

$$\Phi_\lambda(t) = \Big(\frac{\kappa_1\zeta_1}{p^+} + \frac{\kappa_1'\zeta_1'}{q^+}\Big) - \frac{c_1}{\beta^-}c_4^{\beta^-}t^{\beta^- - \max(p^+,q^+)} - \lambda\frac{c_2}{\alpha^-}c_3^{\alpha^-}t^{\alpha^- - \max(p^+,q^+)}.$$

Because $\alpha^- > \max(p^+, q^+)$ and $\beta^- > \max(p^+, q^+)$, we have that $\varphi_\lambda(t) > 0$ for t small enough, $t > 0$.

We deduce that for any $\lambda > 0$, we can choose $\rho, m > 0$ such that $J_\lambda(u) \geq m > 0$ for all $u \in E$ with $\|u\| = \rho$. This completes the proof. □

Lemma 4.4. *Let $E_1 \subset E$ be a finite dimensional subspace. Then, the set*

$$\Sigma = \big\{u \in E_1;\ J_\lambda(u) \geq 0\big\}$$

is bounded in E.

Proof. First, by Proposition 2.4, we deduce that there exist two positive constants K_1 and K_2 such that

$$\rho_a(u) \leq K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) \text{ and } \rho_b(u) \leq K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big). \tag{24}$$

Now, by condition (H_1') and equation (24), we obtain that for any $u \in E$

$$\begin{aligned} J_\lambda(u) &\leq \frac{1}{p^-}\rho_a(u) + \frac{1}{q^-}\rho_b(u) + \frac{\|g\|_\infty}{r^-}\int_{\partial\Omega}|u|^{r(x)}\mathrm{d}\sigma - \lambda\int_\Omega F(x,u)\mathrm{d}x \\ &\leq \frac{1}{p^-}K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{1}{q^-}K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big) \\ &\quad + \frac{\|g\|_\infty}{r^-}\int_{\partial\Omega}|u|^{r(x)}\mathrm{d}\sigma - \lambda\frac{c_1'}{\alpha^+}\int_\Omega|u|^{\alpha(x)}\mathrm{d}x. \end{aligned}$$

Let $u \in E$. We have

$$\int_{\partial\Omega}|u|^{r(x)}\mathrm{d}\sigma \leq |u|^{r^-}_{L^{r(x)}(\partial\Omega)} + |u|^{r^+}_{L^{r(x)}(\partial\Omega)}. \tag{25}$$

Moreover, the fact that E is continuously embedded in $L^{r(x)}(\partial\Omega)$ assures that there exists a positive constant K_3 such that

$$|u|_{L^{r(x)}(\partial\Omega)} \leq K_3\|u\|. \tag{26}$$

The inequalities (25) and (26) show that there exists $K_4 > 0$ such that

$$\int_{\partial\Omega}|u|^{r(x)}\mathrm{d}\sigma \leq K_4\big(\|u\|^{r^+} + \|u\|^{r^-}\big). \tag{27}$$

So, we can write

$$\begin{aligned} J_\lambda(u) \leq{}& \frac{K_1}{p^-}\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{K_2}{q^-}\big(\|u\|^{q^+} + \|u\|^{q^-}\big) + \frac{K_4}{r^-}\big(\|u\|^{r^+} + \|u\|^{r^-}\big) \\ &- \lambda\frac{c_1'}{\alpha^+}\int_\Omega|u|^{\alpha(x)}\mathrm{d}x. \end{aligned}$$

Put $\Omega = \Omega_< \cup \Omega_\geq$ where

$$\Omega_< = \big\{x \in \Omega; |u(x)| < 1\big\} \text{ and } \Omega_\geq = \big\{x \in \Omega; |u(x)| \geq 1\big\}.$$

Then, we obtain

$$\begin{aligned} J_\lambda(u) \leq{}& \frac{1}{p^-}K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{1}{q^-}K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big) \\ &+ \frac{\|g\|_\infty}{r^-}K_4\big(\|u\|^{r^+} + \|u\|^{r^-}\big) - \lambda\frac{c_1'}{\alpha^+}\int_\Omega|u|^{\alpha(x)}\mathrm{d}x \\ \leq{}& \frac{1}{p^-}K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{1}{q^-}K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big) \\ &+ \frac{\|g\|_\infty}{r^-}K_4\big(\|u\|^{r^+} + \|u\|^{r^-}\big) - \lambda\frac{c_1'}{\alpha^+}\int_{\Omega_\geq}|u|^{\alpha-}\mathrm{d}x \end{aligned}$$

$$\leq \frac{1}{p^-}K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{1}{q^-}K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big)$$
$$+ \frac{\|g\|_\infty}{r^-}K_4\big(\|u\|^{r^+} + \|u\|^{r^-}\big) - \lambda\frac{c_1'}{\alpha^+}\int_\Omega |u|^{\alpha-}\mathrm{d}x + \lambda\frac{c_1'}{\alpha^+}\int_{\Omega_<} |u|^{\alpha-}\mathrm{d}x.$$

Moreover, there exists a positive constant $K_5(\lambda)$ such that, for any $u \in E$, we have

$$\lambda\frac{c_1'}{\alpha^+}\int_{\Omega_<} |u|^{\alpha-}\mathrm{d}x \leq K_5(\lambda).$$

Since the functional $|.|_{\alpha^-} : E \to \mathbb{R}$ defined by

$$|u|_{\alpha^-} = \Big(\int_\Omega |u|^{\alpha-}\mathrm{d}x\Big)^{\frac{1}{\alpha-}}$$

is a norm in E and in the finite dimensional subspace E_1 the norms $|.|_{\alpha^-}$ and $\|.\|$ are equivalent, we deduce that there exists $K_6(E_1) > 0$ such that

$$\|u\| \leq K_6(E_1)|.|_{\alpha^-} \quad \text{for all } u \in E_1.$$

Therefore, there exists a constant $K_7(\lambda) > 0$ such that

$$J_\lambda(u) \leq \frac{1}{p^-}K_1\big(\|u\|^{p^+} + \|u\|^{p^-}\big) + \frac{1}{q^-}K_2\big(\|u\|^{q^+} + \|u\|^{q^-}\big)$$
$$+ \frac{\|g\|_\infty}{r^-}K_4\big(\|u\|^{r^+} + \|u\|^{r^-}\big) - K_7(\lambda)\|u\|^{\alpha^-} + K_5(\lambda).$$

Since $\alpha^- > \max(p^+, r^+, q^+)$ then Σ is bounded. Hence, J_λ has an unbounded sequence of critical values in E. This ends the proof. □

Proof of Theorem 4.2. Since the functions $F(x, u)$ and $H(x, u)$ are even with respect to u, then it is clear that J_λ is an even functional, moreover, we have $J_\lambda(0) = 0$. Lemmas 4.3, 3.4, and 4.4 imply that Theorem 4.1 can be applied to the functional J_λ. So, we conclude that problem (1) has infinitely many weak solutions in E. □

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Received 17 March 2024

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