ON TREES WITH MAXIMUM EXPONENTIAL HARMONIC AND SUM-CONNECTIVITY INDICES

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The harmonic index and sum-connectivity index are two of the best-known and most successful vertex-degree-based topological indices in mathematical chemistry. They are well correlated with the π-electronic energy of benzenoid hydrocarbons. The idea of introducing the exponential of vertex-degree-based topological indices was raised by Rada during the study of discrimination ability of these class of graph invariants. The aim of this research is to study the exponential of harmonic and sum-connectivity index over trees with fixed order and given maximum vertex degree.

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1. INTRODUCTION

Consider a graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For $a \in V(\Gamma)$, the open neighborhood $N_\Gamma(a)$ of a is the set $N_\Gamma(a) = \{b \in V(\Gamma) \mid ab \in E(\Gamma)\}$ and the degree $d_\Gamma(a)$ is the order of $N_\Gamma(a)$. By $\Delta(\Gamma)$, we mean the maximum degree of Γ. The distance $d_\Gamma(a, b)$ between the vertices $a, b \in V(\Gamma)$ is the length of a shortest $a - b$ path in Γ.

Topological indices are real numbers assigned to the chemical graph of a molecular compound which remain invariant under isomorphism of graph. They are applied in predicting the physico-chemical properties of chemical structures and considered as helpful measures in QSPR/QSAR investigations (see, for instance, [14]). Topological indices are divided into various classes among which *vertex-degree-based* (VDB) indices have an outstanding position. VDB indices can be formulated based on the degrees of vertices in graph and considered as useful tools in examining different properties of chemical structures containing viscosity, entropy, enthalpy of vaporization, gyrational radius, boiling point, etc. A large number of VDB indices have been put forward to date with varied degrees of applications in chemistry and other fields of science.

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The *harmonic index* and *sum-connectivity index* are among the best-known and thoroughly-investigated VDB indices in mathematical chemistry. The *harmonic index* was proposed by Fajtlowics [16] in 1987 within some conjectures generated by the computer program Graffiti. It is formulated by

$$H(\Gamma) = \sum_{ab\in E(\Gamma)} \frac{2}{d_\Gamma(a) + d_\Gamma(b)}.$$

The harmonic index correlates well with the π-electronic energy of benzenoid hydrocarbons. Also, it was shown that its correlations with physical and chemical properties is somewhat better than that of some other VDB indices like the Randić connectivity index [19].

The *sum-connectivity index* was introduced by Zhou and Trinajstić [28] in 2009. This index is formulated by

$$SC(\Gamma) = \sum_{ab\in E(\Gamma)} \frac{1}{\sqrt{d_\Gamma(a) + d_\Gamma(b)}}.$$

It was shown by Lučić et al. [17] that the sum-connectivity index is well correlated with the π-electronic energy of benzenoid hydrocarbons. Further results concerning mathematical properties and applications of the harmonic and sum-connectivity indices can be found in [1–3,9,13,17,20,21,24,26,27] and the references quoted therein.

A major issue in the investigation of topological indices is their discrimination power [11, 12]. In view of this, Rada [18] suggested the *exponential* of a VDB invariant in 2019. In particular, the exponential of the harmonic index and sum-connectivity indices are respectively defined as

$$e^{H}(\Gamma) = \sum_{ab\in E(\Gamma)} e^{\frac{2}{d_\Gamma(a)+d_\Gamma(b)}},$$

and

$$e^{SC}(\Gamma) = \sum_{ab\in E(\Gamma)} e^{\frac{1}{\sqrt{d_\Gamma(a)+d_\Gamma(b)}}}.$$

We refer the readers to [4–8, 10, 15, 22, 23, 25] for comprehensive, transparent information on exponential vertex-degree-based indices.

A famous topic in extremal graph theory is to investigate the extreme amounts of topological indices over trees with given graph parameters. In this paper, we aim to study the maximum values of the exponential of harmonic and sum-connectivity indices over trees with fixed order and maximum degree and characterize the maximal trees.

2. TREE

A *rooted tree* is a tree together with a special vertex chosen as the *root* of the tree. A *spider* is a tree with exactly one vertex of degree more than two. This vertex is known as the *center* of the spider. A *leg* of a spider is a path which connects its center to one of its leaves. A spider in which all legs have length one is a star. If needed, and by a slight abuse of language, we also consider a path graph to be a spider (with one or two leg).

For positive integers η and Δ, $\mathcal{T}_{\eta,\Delta}$ stands for the set of all trees having η vertices and maximum degree Δ.

LEMMA 2.1. *Let $\Psi \in \mathcal{T}_{\eta,\Delta}$, and let a be a vertex of Ψ with $d_\Psi(a) = \Delta$. If Ψ contains a vertex $b \neq a$ with $d_\Psi(b) \geq 3$, then there exists a tree $\Psi^{'} \in \mathcal{T}_{\eta,\Delta}$ such that $e^H(\Psi^{'}) > e^H(\Psi)$ and $e^{SC}(\Psi^{'}) > e^{SC}(\Psi)$.*

Proof. Consider Ψ as a tree rooted in a. We may assume that among all vertices $x \neq a$ with $d_\Psi(x) \geq 3$, the vertex b has maximum distance from a. Let $d_\Psi(b) = \ell$ and let $N_\Psi(b) = \{b_1, b_2, \ldots, b_\ell\}$, where b_ℓ is the neighbor of b that lies on the $b - a$ path in Ψ and $d_\Psi(b_\ell) = \alpha$. By our assumption on $d_\Psi(a, b)$, we have $d_\Psi(b_i) \in \{1, 2\}$ for each $1 \leq i \leq \ell - 1$. Based on the latter degrees, the following three cases can be considered.

Case 1: b is adjacent to at least two leaves. We may assume that $d_\Psi(b_1) = d_\Psi(b_2) = 1$. Define the tree $\Psi^{'} \in \mathcal{T}_{\eta,\Delta}$ as $\Psi' = (\Psi - b_1) + b_l b_2$. Recall that $\ell \geq 3$. Then

$$\begin{aligned}
e^H(\Psi) - e^H(\Psi^{'}) &= \sum_{3\leq i\leq \ell} e^{\frac{2}{d_\Psi(b_i)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(b_1)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(b_2)+d_\Psi(b)}} \\
&\quad - \sum_{3\leq i\leq \ell} e^{\frac{2}{d_{\Psi'}(b_i)+d_{\Psi'}(b)}} - e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(b_2)}} - e^{\frac{2}{d_{\Psi'}(b_2)+d_{\Psi'}(b)}} \\
&= \sum_{3\leq i\leq \ell} \left(e^{\frac{2}{d_\Psi(b_i)+\ell}} - e^{\frac{2}{d_\Psi(b_i)+\ell-1}}\right) + 2e^{\frac{2}{\ell+1}} - e^{\frac{2}{3}} - e^{\frac{2}{\ell+1}} \\
&< e^{\frac{2}{\ell+1}} - e^{\frac{2}{3}} < 0,
\end{aligned}$$

and

$$\begin{aligned}
e^{SC}(\Psi) - e^{SC}(\Psi^{'}) &= \sum_{3\leq i\leq \ell} e^{\frac{1}{\sqrt{d_\Psi(b_i)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(b_1)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(b_2)+d_\Psi(b)}}} \\
&\quad - \sum_{3\leq i\leq \ell} e^{\frac{1}{\sqrt{d_{\Psi'}(b_i)+d_{\Psi'}(b)}}} - e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(b_2)}}} - e^{\frac{1}{d_{\Psi'}(b_2)+d_{\Psi'}(b)}} \\
&= \sum_{3\leq i\leq \ell} \left(e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell}}} - e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell-1}}}\right) + 2e^{\frac{1}{\sqrt{\ell+1}}} - e^{\frac{1}{\sqrt{3}}} - e^{\frac{1}{\sqrt{\ell+1}}}
\end{aligned}$$

$$< e^{\frac{1}{\sqrt{\ell+1}}} - e^{\frac{1}{\sqrt{3}}} < 0.$$

Case 2: b is adjacent to exactly one leaf. We may assume that b_1 is the leaf adjacent to b. Let $bc_1c_2\dots c_k$, be the path in Ψ where $b_2 = c_1$, $k \geq 2$, and c_k is a leaf. Let $\Psi^{'}$ be the tree obtained from Ψ by removing the vertex b_1 and the path $c_1c_2\dots c_k$ and attaching to b the path $c_1c_2\dots c_kb_1$. Then

$$\begin{aligned} e^H(\Psi) - e^H(\Psi^{'}) &= \sum_{2\leq i\leq \ell} e^{\frac{2}{d_\Psi(b_i)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(b_1)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(c_k)+d_\Psi(c_{k-1})}} \\ &\quad - \sum_{3\leq i\leq \ell} e^{\frac{2}{d_{\Psi'}(b_i)+d_{\Psi'}(b)}} - e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(c_k)}} - e^{\frac{2}{d_{\Psi'}(c_k)+d_{\Psi'}(c_{k-1})}} \\ &= \sum_{2\leq i\leq \ell} (e^{\frac{2}{d_\Psi(b_i)+\ell}} - e^{\frac{2}{d_\Psi(b_i)+\ell-1}}) + e^{\frac{2}{\ell+1}} - e^{\frac{2}{3}} - e^{\frac{2}{3}} - e^{\frac{2}{4}} \\ &< e^{\frac{2}{\ell+1}} - e^{\frac{2}{4}} \leq 0, \end{aligned}$$

and

$$\begin{aligned} e^{SC}(\Psi) - e^{SC}(\Psi^{'}) &= \sum_{2\leq i\leq \ell} e^{\frac{1}{\sqrt{d_\Psi(b_i)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(b_1)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(c_k)+d_\Psi(c_{k-1})}}} \\ &\quad - \sum_{2\leq i\leq \ell} e^{\frac{1}{\sqrt{d_{\Psi'}(b_i)+d_{\Psi'}(b)}}} - e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(c_k)}}} \\ &\quad - e^{\frac{1}{\sqrt{d_{\Psi'}(c_k)+d_{\Psi'}(c_{k-1})}}} \\ &= \sum_{2\leq i\leq \ell} (e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell}}} - e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell-1}}}) + e^{\frac{1}{\sqrt{\ell+1}}} + e^{\frac{1}{\sqrt{3}}} - e^{\frac{1}{\sqrt{3}}} - e^{\frac{1}{2}} \\ &< e^{\frac{1}{\sqrt{\ell+1}}} - e^{\frac{1}{2}} \leq 0. \end{aligned}$$

Case 3: None of the vertices adjacent to b is a leaf. Let $bc_1c_2\dots c_k$ and $bd_1d_2\dots d_s$ be the paths in Ψ such that $k, s \geq 2$, $b_1 = c_1$, $b_2 = d_1$, and c_k and d_s are leaves. Let $\Psi^{'}$ be the tree achieved from Ψ by removing the path $c_1\dots c_k$ and attaching the path $d_sc_1\dots c_k$. If $\ell \geq 5$, then

$$\begin{aligned} e^H(\Psi) - e^H(\Psi^{'}) &= \sum_{3\leq i\leq \ell} e^{\frac{2}{d_\Psi(b_i)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(b_1)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(d_s)+d_\Psi(d_{s-1})}} \\ &\quad - \sum_{3\leq i\leq \ell} e^{\frac{2}{d_{\Psi'}(b_i)+d_{\Psi'}(b)}} - e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(d_s)}} - e^{\frac{2}{d_{\Psi'}(d_s)+d_{\Psi'}(d_{s-1})}} \\ &= \sum_{3\leq i\leq \ell} (e^{\frac{2}{d_\Psi(b_i)+\ell}} - e^{\frac{2}{d_\Psi(b_i)+\ell-1}}) + e^{\frac{2}{\ell+2}} + e^{\frac{2}{3}} - 2e^{\frac{2}{4}} \\ &\leq e^{\frac{2}{7}} + e^{\frac{2}{3}} - 2e^{\frac{2}{4}} \approx -0.0189. \end{aligned}$$

Now let $\ell = 3$ or $\ell = 4$. Then

$$\begin{aligned}
e^{H}(\Psi) - e^{H}(\Psi') &< e^{\frac{2}{d_\Psi(b_2)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(b_1)+d_\Psi(b)}} + e^{\frac{2}{d_\Psi(d_s)+d_\Psi(d_{s-1})}} \\
&\quad - e^{\frac{2}{d_{\Psi'}(b_2)+d_{\Psi'}(b)}} - e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(d_s)}} - e^{\frac{2}{d_{\Psi'}(d_s)+d_{\Psi'}(d_{s-1})}} \\
&= 2e^{\frac{2}{\ell+2}} + e^{\frac{2}{3}} - e^{\frac{2}{\ell+1}} - 2e^{\frac{2}{4}}.
\end{aligned}$$

If $\ell = 4$, then

$$e^{H}(\Psi) - e^{H}(\Psi') < 2e^{\frac{2}{6}} + e^{\frac{2}{3}} - e^{\frac{2}{5}} - 2e^{\frac{2}{4}} \approx -0.0503,$$

and if $\ell = 3$, then

$$e^{H}(\Psi) - e^{H}(\Psi') < 2e^{\frac{2}{5}} + e^{\frac{2}{3}} - 3e^{\frac{2}{4}} \approx -0.0147.$$

Also if $\ell \geq 5$, then

$$\begin{aligned}
e^{SC}(\Psi) - e^{SC}(\Psi') &= \sum_{3\leq i\leq \ell} e^{\frac{1}{\sqrt{d_\Psi(b_i)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(b_1)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(d_s)+d_\Psi(d_{s-1})}}} \\
&\quad - \sum_{3\leq i\leq \ell} e^{\frac{1}{\sqrt{d_{\Psi'}(b_i)+d_{\Psi'}(b)}}} - e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(d_s)}}} \\
&\quad - e^{\frac{1}{\sqrt{d_{\Psi'}(d_s)+d_{\Psi'}(d_{s-1})}}} \\
&= \sum_{3\leq i\leq \ell} \left(e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell}}} - e^{\frac{1}{\sqrt{d_\Psi(b_i)+\ell-1}}}\right) + e^{\frac{1}{\sqrt{\ell+2}}} + e^{\frac{1}{\sqrt{3}}} - 2e^{\frac{1}{2}} \\
&\leq e^{\frac{1}{\sqrt{7}}} + e^{\frac{1}{\sqrt{3}}} - 2e^{\frac{1}{2}} \approx -0.0568.
\end{aligned}$$

Now let $\ell = 3$ or $\ell = 4$. Then

$$\begin{aligned}
e^{SC}(\Psi) - e^{SC}(\Psi') &< e^{\frac{1}{\sqrt{d_\Psi(b_2)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(b_1)+d_\Psi(b)}}} + e^{\frac{1}{\sqrt{d_\Psi(d_s)+d_\Psi(d_{s-1})}}} \\
&\quad - e^{\frac{1}{\sqrt{d_{\Psi'}(b_2)+d_{\Psi'}(b)}}} - e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(d_s)}}} - e^{\frac{1}{\sqrt{d_{\Psi'}(d_s)+d_{\Psi'}(d_{s-1})}}} \\
&= 2e^{\frac{1}{\sqrt{\ell+2}}} + e^{\frac{1}{\sqrt{3}}} - e^{\frac{1}{\sqrt{\ell+1}}} - 2e^{\frac{1}{2}}.
\end{aligned}$$

If $\ell = 4$, then

$$e^{SC}(\Psi) - e^{SC}(\Psi') < 2e^{\frac{1}{\sqrt{6}}} + e^{\frac{1}{\sqrt{3}}} - e^{\frac{1}{\sqrt{5}}} - 2e^{\frac{1}{2}} \approx -0.0717,$$

and if $\ell = 3$, then

$$e^{SC}(\Psi) - e^{SC}(\Psi') < 2e^{\frac{1}{\sqrt{5}}} + e^{\frac{1}{\sqrt{3}}} - 3e^{\frac{1}{2}} \approx -0.0369,$$

from which the proof is completed. $\square$

LEMMA 2.2. *If $\Psi \in \mathcal{T}_{\eta,\Delta}$ is a spider with $\Delta \geq 3$ such that Ψ has at least one leaf and one leg of length more than two, then there is a spider $\Psi' \in \mathcal{T}_{\eta,\Delta}$ such that $e^{H}(\Psi') > e^{H}(\Psi)$ and $e^{SC}(\Psi') > e^{SC}(\Psi)$.*

Proof. Denote by a the center of Ψ and assume that ab_1, $ac_1c_2\dots c_l$ are two legs of Ψ such that $l\geq 3$. Define $\Psi^{'}\in\mathcal{T}_{\eta,\Delta}$ as $\Psi'=(\Psi\setminus c_lc_{l-1})+c_lb_1$. Then

$$\begin{aligned}e^{H}(\Psi)-e^{H}(\Psi^{'})&=e^{\frac{2}{d_\Psi(b_1)+d_\Psi(b)}}+e^{\frac{2}{d_\Psi(c_l)+d_\Psi(c_{l-1})}}+e^{\frac{2}{d_\Psi(c_{l-1})+d_\Psi(c_{l-2})}}\\&\quad-e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(b)}}-e^{\frac{2}{d_{\Psi'}(b_1)+d_{\Psi'}(c_l)}}-e^{\frac{2}{d_{\Psi'}(c_{l-1})+d_{\Psi'}(c_{l-2})}}\\&=e^{\frac{2}{\Delta+1}}+e^{\frac{2}{4}}-e^{\frac{2}{\Delta+2}}-e^{\frac{2}{3}}.\end{aligned}$$

Let $f(\sigma)=e^{\frac{2}{\sigma+1}}-e^{\frac{2}{\sigma+2}}$. Then

$$f'(\sigma)=\frac{2e^{\frac{2}{\sigma+2}}}{(\sigma+2)^2}-\frac{2e^{\frac{2}{\sigma+1}}}{(\sigma+1)^2}=\frac{2e^{\frac{2}{\sigma+2}}(\sigma+1)^2-2e^{\frac{2}{\sigma+1}}(\sigma+2)^2}{(\sigma^2+3\sigma+2)^2}.$$

Thus, $f(\sigma)$ is a decreasing function, when $\sigma\geq 3$. Therefore

$$e^{H}(\Psi)-e^{H}(\Psi^{'})\leq 2e^{\frac{1}{2}}-e^{\frac{2}{5}}-e^{\frac{2}{3}}\approx -0.1421.$$

Also

$$\begin{aligned}e^{SC}(\Psi)-e^{SC}(\Psi^{'})&=e^{\frac{1}{\sqrt{d_\Psi(b_1)+d_\Psi(b)}}}+e^{\frac{1}{\sqrt{d_\Psi(c_l)+d_\Psi(c_{l-1})}}}+e^{\frac{1}{\sqrt{d_\Psi(c_{l-1})+d_\Psi(c_{l-2})}}}\\&\quad-e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(b)}}}-e^{\frac{1}{\sqrt{d_{\Psi'}(b_1)+d_{\Psi'}(c_l)}}}\\&\quad-e^{\frac{1}{\sqrt{d_{\Psi'}(c_{l-1})+d_{\Psi'}(c_{l-2})}}}\\&=e^{\frac{1}{\sqrt{\Delta+1}}}+e^{\frac{1}{2}}-e^{\frac{1}{\sqrt{\Delta+2}}}-e^{\frac{1}{\sqrt{3}}}.\end{aligned}$$

Let $f(\sigma)=e^{\frac{1}{\sqrt{\sigma+1}}}-e^{\frac{1}{\sqrt{\sigma+2}}}$. Then

$$\begin{aligned}f'(\sigma)&=\frac{e^{\frac{1}{\sqrt{\sigma+2}}}}{2\sqrt{(\sigma+2)^3}}-\frac{e^{\frac{1}{\sqrt{\sigma+1}}}}{2\sqrt{(\sigma+1)^3}}\\&=\frac{2\sqrt{(\sigma+1)^3}e^{\frac{1}{\sqrt{\sigma+2}}}-2\sqrt{(\sigma+2)^3}e^{\frac{1}{\sqrt{\sigma+1}}}}{2\sqrt{(\sigma^2+3\sigma+2)^3}}.\end{aligned}$$

Thus, $f(\sigma)$ is a decreasing function, when $\sigma\geq 3$. Therefore

$$e^{SC}(\Psi)-e^{SC}(\Psi^{'})\leq 2e^{\frac{1}{2}}-e^{\frac{1}{\sqrt{5}}}-e^{\frac{1}{\sqrt{3}}}\approx -0.0478.$$

So, the claims are valid. □

At this point, we present the main results of this paper.

THEOREM 2.3. *For* $\Psi\in\mathcal{T}_{\eta,\Delta}$,

$$e^{H}(\Psi)\leq \Delta(e^{\frac{2}{\Delta+2}}+e^{\frac{2}{3}})+(\eta-2\Delta-1)e^{\frac{1}{2}},$$

when $\Delta \leq \frac{\eta-1}{2}$, *and*

$$e^H(\Psi) \leq (\eta-\Delta-1)(e^{\frac{2}{\Delta+2}} + e^{\frac{2}{3}}) + (2\Delta-\eta+1)e^{\frac{2}{\Delta+1}},$$

when $\Delta > \frac{\eta-1}{2}$. *The equality case occurs iff* Ψ *is a spider in which all legs are of length at most two or all legs are of length at least two.*

Proof. We assume that $\Psi_1 \in \mathcal{T}_{\eta,\Delta}$ such that $e^H(\Psi_1) \geq e^H(\Psi)$ for all $\Psi \in \mathcal{T}_{\eta,\Delta}$. Rooted Ψ_1 at a such that $d_{\Psi_1}(a) = \Delta$. First, consider the case where $\Delta = 2$. In this case,

$$e^H(\Psi) = e^H(P_\eta) = (\eta-3)e^{\frac{1}{2}} + 2e^{\frac{2}{3}}.$$

Now assume that $\Delta \geq 3$. Then by Lemma 2.1, Ψ_1 is a spider with center a and by Lemma 2.2, all legs of Ψ_1 either are of length at least two or are of length at most two. If all legs of Ψ_1 have length at least two, then $\Delta \leq \frac{\eta-1}{2}$ and

$$e^H(\Psi_1) = \Delta e^{\frac{2}{\Delta+2}} + (\eta-2\Delta-1)e^{\frac{1}{2}} + \Delta e^{\frac{2}{3}}.$$

Now let all legs of Ψ_1 be of length at most two. If all legs of Ψ_1 are of length one, then $e^H(\Psi) = \Delta e^{\frac{2}{\Delta+1}}$. Otherwise, $2\Delta+1-\eta$ leaves are adjacent to a and we have

$$e^H(\Psi_1) = (\eta-\Delta-1)e^{\frac{2}{\Delta+2}} + (2\Delta-\eta+1)e^{\frac{2}{\Delta+1}} + (\eta-\Delta-1)e^{\frac{2}{3}}.$$

This completes the proof. $\square$

The proof of the subsequent theorem is analogous to that of Theorem 2.3, hence it is not presented.

THEOREM 2.4. *For* $\Psi \in \mathcal{T}_{\eta,\Delta}$,

$$e^{SC}(\Psi) \leq \Delta(e^{\frac{1}{\sqrt{\Delta+2}}} + e^{\frac{1}{\sqrt{3}}}) + (\eta-2\Delta-1)e^{\frac{1}{2}},$$

when $\Delta \leq \frac{\eta-1}{2}$, *and*

$$e^{SC}(\Psi) \leq (\eta-\Delta-1)(e^{\frac{1}{\sqrt{\Delta+2}}} + e^{\frac{1}{\sqrt{3}}}) + (2\Delta-\eta+1)e^{\frac{1}{\sqrt{\Delta+1}}},$$

when $\Delta > \frac{\eta-1}{2}$. *The equality case occurs iff* Ψ *is a spider in which all legs are of length at most two or all legs are of length at least two.*

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