EXISTENCE OF LENGTH MINIMIZERS IN HOMOTOPY CLASSES OF LIPSCHITZ PATHS IN $\mathbb{H}^1$

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We show that for any purely 2-unrectifiable metric space M, for example the Heisenberg group $\mathbb{H}^1$ equipped with the Carnot–Carathéodory metric, every homotopy class $[\gamma]$ of Lipschitz paths contains a length minimizing representative γ_∞ that is unique up to reparametrization. The length minimizer γ_∞ is the core of the homotopy class $[\gamma]$ in the sense that the image of γ_∞ is a subset of the image of any path contained in $[\gamma]$. Furthermore, the existence of length minimizers guarantees that only the trivial class in the first Lipschitz homotopy group of M with a base point can be represented by a loop within each neighborhood of the base point. The results detailed here are used in Perry (Preprint, 2024) to define and prove properties of a universal Lipschitz path space over $\mathbb{H}^1$.

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1. INTRODUCTION

In this paper, we prove the following theorem.

THEOREM 1.1. *Let M be a purely 2-unrectifiable metric space, for example the Heisenberg group $\mathbb{H}^1$ endowed with the Carnot–Carathéodory metric. For any homotopy class $[\gamma]$ of Lipschitz paths in M, there exists a length minimizing Lipschitz path $\gamma_\infty \in [\gamma]$ where*

$$\ell(\gamma_\infty) = \inf\{\ell(\gamma) \mid \gamma \in [\gamma]\}.$$

Moreover, for any representative $\gamma \in [\gamma]$ in the class, $\operatorname{Im}(\gamma_\infty) \subset \operatorname{Im}(\gamma)$.

A length minimizer $\gamma_\infty \in [\gamma]$ can be thus thought of as the core of the homotopy class $[\gamma]$ where the extraneous branches of the paths in the class have been pruned. An immediate consequence is that for every point in a

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purely 2-unrectifiable metric space, only the trivial class in the first Lipschitz homotopy group can be represented by a loop within each neighborhood of the point (Corollary 3.11).

Studying metric spaces, in particular Heisenberg groups endowed with a Carnot–Carathéodory metric, through Lipschitz homotopies was introduced in [6] with the definition of Lipschitz homotopy groups. Since, Lipschitz homotopy groups have been calculated for various sub-Riemannian manifolds in [6–9], [14], and [18]. For an overview of sub-Riemannian geometry, see [13].

Results in [6], [14], and [18] concerning the Lipschitz homotopy groups of the Heisenberg group $\mathbb{H}^1$ and contact 3-manifolds rely heavily on these sub-Riemannian manifolds endowed with the Carnot–Carathéodory metric being purely 2-unrectifiable metric spaces in the sense of [3]. As is shown in Ambrosio and Kirchheim [3], the Heisenberg group $\mathbb{H}^1$ is purely 2-unrectifiable and, as is shown in [14], any contact 3-manifold endowed with a sub-Riemannian metric is purely 2-unrectifiable. We likewise make significant use of results of purely 2-unrectifiable metric spaces to conclude the existence of length minimizers in homotopy classes.

While we primarily center our attention on the Heisenberg group in discussion, the results in this paper apply to a number of other well-known examples. Filiform Carnot groups $J^k(\mathbb{R}, \mathbb{R})$ have two-dimensional horizontal spaces and are thus purely 2-unrectifiable. See [17] for theory on filiform Carnot groups and [11] for the unrectifiability assertion. Bourdon–Pajot spaces [5] and Laakso graphs [10] provide additional examples of purely 2-unrectifiable spaces.

The key step to the proof of Theorem 1.1 is, given a Lipschitz path γ in a homotopy class $[\gamma]$, determining a sequence of Lipschitz paths that are uniformly Lipschitz homotopic to γ and whose lengths converge to the infimum. Once such a sequence is obtained, Arzelà–Ascoli theorem yields a length minimizing path γ_∞ together with a homotopy to γ.

To find this sequence of paths, we apply a lemma (Lemma 3.8) which, given any homotopy from γ to any Lipschitz path β, constructs a homotopy with controlled Lipschitz constant from γ to a shorter path β'. The proof of Lemma 3.8 relies on a result of Wenger and Young in [18] concerning a factorization of Lipschitz maps with purely 2-unrectifiable target though a metric tree. Their result is stated in Theorem 2.5.

The results in this paper are utilized in [15] to define a metric space called universal Lipschitz path space $\mathcal{P}_{\mathbb{H}^1}$ over $\mathbb{H}^1$ and prove requisite properties. For an arbitrary based metric space M, the universal Lipschitz path space $\mathcal{P}_M$ over M is a pseudo-metric space with pseudo-metric $d_{\mathcal{P}}$. The pseudo-metric $d_{\mathcal{P}}$ is defined as the infimum of lengths of paths in a particular homotopy class.

As is reported in [15], Corollary 3.11 implies that for a purely 2-unrectifiable space M, the pseudo-metric $d_{\mathcal{P}}$ is a metric. Moreover, for such a space M, the universal Lipschitz path space $\mathcal{P}_M$ is a Lipschitz simply connected length space which satisfies a unique lifting property.

2. BACKGROUND

2.1. Homotopy, geodesics, and metric trees

Convention 2.1. Throughout this paper, $I = [0,1]$ is the closed interval endowed with the Euclidean metric. All paths have domain I. For a metric space M and a path $\gamma : I \to M$, the length of the path γ is denoted $\ell(\gamma)$. For metric spaces A and M, the Lipschitz constant of a Lipschitz function $f : A \to M$ is denoted by $\mathrm{Lip}(f)$.

As we proceed, we endow $I \times I$ with the L^1 metric:

$$d^1\big((s,t),(s',t')\big) = |s-s'| + |t-t'| \quad \text{for } (s,t),(s',t') \in I \times I.$$

The metric d^1 is Lipschitz equivalent to the Euclidean metric on $I \times I$.

Definition 2.2. Let M be a metric space. Two Lipschitz paths $\gamma, \gamma' : I \to M$ are *homotopic rel endpoints* if the initial points $\gamma(0) = \gamma'(0)$ and end points $\gamma(1) = \gamma'(1)$ of the paths agree and there exists a Lipschitz map $H : I \times I \to M$ such that

$$H|_{I\times\{0\}} = \gamma, \quad H|_{I\times\{1\}} = \gamma', \quad H|_{\{0\}\times I} = \gamma(0), \quad \text{and} \quad H|_{\{1\}\times I} = \gamma(1).$$

The map H is a *homotopy* from γ to γ'. For a Lipschitz path γ, the class of all Lipschitz paths homotopic rel endpoints to γ is denoted $[\gamma]$ and is referred to as the *homotopy class* of γ.

The homotopy classes of loops based at a point $x_0 \in M$ are the elements of the first Lipschitz homotopy group $\pi_1^{\mathrm{Lip}}(M, x_0)$ of the metric space M. For the complete definition of Lipschitz homotopy groups and the initial study of $\pi_1^{\mathrm{Lip}}(\mathbb{H}^1)$, see [6]. Another example of studying first Lipschitz homotopy groups of purely 2-unrectifiable metric spaces can be found in [14] where contact 3-manifolds are considered.

Definition 2.3. Let (M,d) be a metric space. Let $x, y \in M$ and let $\eta : I \to M$ be a path from $\eta(0) = x$ to $\eta(1) = y$. The path η is *arc length parametrized* if for any $t, t' \in I$,

$$\ell(\eta|_{[t,t']}) = \ell(\eta)\ |t'-t|.$$

The path η is a *shortest path* from x to y if

$$\ell(\eta) = d(x, y).$$

The path η is a *geodesic* from x to y if for any $t, t' \in I$,

$$d\big(\eta(t), \eta(t')\big) = d(x, y)\ |t' - t|.$$

Every geodesic is a shortest path between its endpoints and is arc length parametrized. Thus, every geodesic is Lipschitz with Lipschitz constant equal to its length.

We primarily discuss geodesics with reference to metric trees. Metric trees were originally introduced in [16]. For a selection of results concerning metric trees, see [1], [2], and [12].

Definition 2.4. A non-empty metric space T is a *metric tree* if for any $x, x' \in T$, there exists a unique arc joining x and x' and there is a geodesic η from x to x'. A subset $T' \subset T$ of a metric tree is a *subtree* if T' is a metric tree with reference to the metric on T restricted to T'.

2.2. **Wenger and Young's factorization through a metric tree**

We make significant use of the work of Wenger and Young in [18] to show the existence of a length minimizing representative, in particular the following factorization.

THEOREM 2.5 ([18], Theorem 5). *Let A be a quasi-convex metric space with quasi-convexity constant C and with $\pi_1^{\mathrm{Lip}}(A) = 0$. Let furthermore M be a purely 2-unrectifiable metric space. Then every Lipschitz map $f : A \to M$ factors through a metric tree T,*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & M, \\ & \underset{\psi}{\searrow} \quad \underset{\varphi}{\nearrow} & \\ & T & \end{array}$$

where $\mathrm{Lip}(\psi) = C \,\mathrm{Lip}(f)$ *and* $\mathrm{Lip}(\varphi) = 1$.

We include some details of their work presently. In the proof of Theorem 2.5 in [18], Wenger and Young define the following pseudo-metric on A:

$$d_f(a, a') := \inf\big\{\ell(f \circ c) \mid c \text{ is a Lipschitz path in } A \text{ from } a \text{ to } a'\big\}$$

where $a, a' \in A$. The metric tree is then defined as a quotient space $T := A/\sim$, where the equivalence relation is given by $a \sim a'$ if and only if $d_f(a, a') = 0$. The metric on T is then

$$d_T\big([a], [a']\big) := d_f(a, a').$$

The map ψ is the quotient map, $\psi(a) = [a]$. The original function f is constant on equivalence classes. As such, the map $\varphi([a]) = f(a)$ is well defined.

3. LENGTH MINIMIZERS OF HOMOTOPY CLASSES IN PURELY 2-UNRECTIFIABLE METRIC SPACES

3.1. Building a desirable homotopy

For the remainder of the paper, let M be a purely 2-unrectifiable metric space with metric d. Let γ and β be Lipschitz homotopic paths.

We use Theorem 2.5 and the definition of the metric tree to fashion a desirable Lipschitz homotopy from the path γ to a Lipschitz path β' whose length is less than or equal to the length of β and whose Lipschitz constant is bounded by the Lipschitz constant of γ. Furthermore, the desirable homotopy has Lipschitz constant equal to the Lipschitz constant of γ. This homotopy is used to show the existence of a length minimizer in each homotopy class in a purely 2-unrectifiable metric space.

Let $H : I \times I \to M$ be a homotopy from γ to β. So, $H|_{I\times\{0\}} = \gamma$ and $H|_{I\times\{1\}} = \beta$. Since $I \times I$ is a geodesic space and Lipschitz simply connected, Theorem 2.5 guarantees the Lipschitz map H factors through a metric tree T:

$$\begin{array}{ccc} I \times I & \xrightarrow{\quad H \quad} & M, \\ & \searrow^{\psi} \quad \nearrow_{\varphi} & \\ & T & \end{array}$$

where $\mathrm{Lip}(\psi) = \mathrm{Lip}(H)$ and $\mathrm{Lip}(\varphi) = 1$. Note that, since $H|_{\{0\}\times I} = \gamma(0)$ and $H|_{\{1\}\times I} = \gamma(1)$, the restricted maps $\psi|_{\{0\}\times I} = \psi(0,0)$ and $\psi|_{\{1\}\times I} = \psi(1,0)$ are constant.

Though the map ψ is $\mathrm{Lip}(H)$-Lipschitz, the restriction of the map ψ to $I \times \{0\}$ is at most $\mathrm{Lip}(\gamma)$-Lipschitz.

LEMMA 3.1. *The Lipschitz constant of the quotient map ψ restricted to $I\times\{0\}$ is bounded by the Lipschitz constant of the path γ, that is,* $\mathrm{Lip}(\psi|_{I\times\{0\}}) \leq \mathrm{Lip}(\gamma)$.

Proof. Let $t, t' \in I$ where $t < t'$. Using the definition of the metric on T,

$$\begin{aligned} d_T\big(\psi(t,0), \psi(t',0)\big) &= d_T\big([(t,0)], [(t',0)]\big) \\ &= d_H\big((t,0), (t',0)\big) \\ &= \inf\big\{\ell(H \circ c) \mid c \text{ is a path from } (t,0) \text{ to } (t',0)\big\}. \end{aligned}$$

Now, selecting the inclusion $c = (\mathbb{1}, 0) : [t, t'] \hookrightarrow I \times I$ which is a Lipschitz path from $(t, 0)$ to $(t', 0)$, yields that

$$d_T\big(\psi(t,0), \psi(t',0)\big) \leq \ell\left(H \circ (\mathbb{1}, 0) : [t, t'] \hookrightarrow M\right) = \ell(\gamma|_{[t,t']}).$$

Since γ is Lipschitz, we have the following string of inequalities:

$$d_T\big(\psi(t,0), \psi(t',0)\big) \leq \ell\big(\gamma|_{[t,t']}\big) \leq \mathrm{Lip}(\gamma)\ |t - t'|. \quad \square$$

When defining the new homotopy, our focus in the metric tree T is $T' := \mathrm{Im}(\psi|_{I\times\{0\}})$, the image of the restriction in Lemma 3.1, which is a subtree of T. Note that every element of the subtree T' can be written as $[(t, 0)]$ for some $t \in I$. The subtree T' has finite diameter bounded by the Lipschitz constant of the path γ, as is now shown.

Lemma 3.2. *The subtree T' has diameter bounded by the Lipschitz constant of the path γ, that is,* $\mathrm{diam}(T') \leq \mathrm{Lip}(\gamma)$.

Proof.

$$\begin{aligned}
\mathrm{diam}(T') &= \sup_{[(t,0)],[(t',0)]\in T'} d_T\big([(t,0)], [(t',0)]\big) \\
&= \sup_{t,t'\in I} d_H\big((t,0), (t',0)\big) \\
&= \sup_{t,t'\in I} \inf_{c} \ \ell(H \circ c) \\
&\leq \sup_{t,t'\in I} \ell\left(H \circ (\mathbb{1}, 0) : [t, t'] \to M\right) \\
&= \sup_{t,t'\in I} \ell(\gamma|_{[t,t']}) \\
&\leq \sup_{t,t'\in I} \mathrm{Lip}(\gamma)\ |t - t'| \\
&= \mathrm{Lip}(\gamma). \quad \square
\end{aligned}$$

Now, let $\eta : I \to T$ be the geodesic in T from $\psi(0, 0)$ to $\psi(1, 0)$. Since η is a geodesic, for any $t, t' \in I$,

$$d_T\big(\eta(t), \eta(t')\big) = d_T\big(\psi(0,0), \psi(1,0)\big)|t - t'|. \tag{1}$$

Define a new path $\beta' : I \to M$ by $\beta'(t) = \varphi \circ \eta(t)$. As we now show, the length of β' is bounded above by the length of the path β, the image of β' is a subset of the image of γ, and the Lipschitz constant for β' is bounded above by the Lipschitz constant of the initial path γ.

Lemma 3.3. *The length of the path β' is bounded by the length of the path β, that is,* $\ell(\beta') \leq \ell(\beta)$.

Proof. If $\psi(0,0) = \psi(1,0)$, then the geodesic η is a constant path, as is the path β'. The desired result is then immediate.

Assume $\psi(0,0) \neq \psi(1,0)$. Then the geodesic η joining these distinct points is non-constant and injective. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} = 1$ be a partition of the interval I. Then, as is argued below, there is a partition $0 = t_0^* < t_1^* < t_2^* < \cdots < t_{n+1}^* = 1$ such that $\beta'(t_i) = \beta(t_i^*)$.

Now η is a geodesic from $\psi(0,0) = \psi(0,1)$ to $\psi(1,0) = \psi(1,1)$ and the map $\psi|_{I\times\{1\}}$ is a path in the metric tree T with the same initial and terminal points as η. Thus, the image of η is a subset of the image of $\psi|_{I\times\{1\}}$. Furthermore, for each $i = 1, \ldots, n$, there is a time $t_i^* \in I$ such that

$$\psi(t_i^*, 1) = \eta(t_i) \text{ and } \psi(t,1) \neq \eta(t_i) \text{ for all } t > t_i^*,$$

that is, t_i^* is the last time the path $\psi|_{I\times\{1\}}$ visits the point $\eta(t_i)$. Thus,

$$\beta'(t_i) = \varphi\big(\eta(t_i)\big) = \varphi\big(\psi(t_i^*, 1)\big) = \beta(t_i^*).$$

Let $i < j$. Suppose $t_i^* \geq t_j^*$. If $t_i^* = t_j^*$, then $\eta(t_i) = \eta(t_j)$, contradicting that the geodesic η is injective. Assume $t_i^* > t_j^*$. Then, the restricted path $\psi|_{[t_i^*,1]\times\{1\}}$ begins at $\eta(t_i)$ and ends at $\eta(1)$ and therefore travels through the point $\eta(t_j)$, contradicting that t_j^* is the last time that $\psi|_{I\times\{1\}}$ visits that point $\eta(t_j)$. Therefore, $t_i^* < t_j^*$ for all $i < j$. We thus have attained the desired partition.

So, for each partition $0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} = 1$, there exists a partition $0 = t_0^* < t_1^* < t_2^* < \cdots < t_{n+1}^* = 1$ such that

$$\sum_{i=0}^{n+1} d\left(\beta'(t_i), \beta'(t_{i+1})\right) = \sum_{i=0}^{n+1} d\left(\beta(t_i^*), \beta(t_{i+1}^*)\right).$$

Taking supremum over all partitions $0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} = 1$, we arrive at $\ell(\beta') \leq \ell(\beta)$. □

LEMMA 3.4. *The image of the path β' is a subset of the image of the path γ, that is,* $\mathrm{Im}(\beta') \subset \mathrm{Im}(\gamma)$.

Proof. The geodesic η is a path from $\psi(0,0)$ to $\psi(1,0)$, as is the map $\psi|_{I\times\{0\}}$. Since η is a geodesic in the metric tree T, the image of the geodesic is a subset of the image of any path with the same initial and terminal points. Thus, $\mathrm{Im}(\eta) \subset \mathrm{Im}(\psi|_{I\times\{0\}})$. Therefore,

$$\mathrm{Im}(\beta') = \mathrm{Im}(\varphi \circ \eta) \subset \mathrm{Im}(\varphi \circ \psi|_{I\times\{0\}}) = \mathrm{Im}(\gamma). \qquad \square$$

LEMMA 3.5. *The Lipschitz constant of the path β' is bounded above by the Lipschitz constant of the path γ, that is,* $\mathrm{Lip}(\beta') \leq \mathrm{Lip}(\gamma)$.

Proof. Let $t, t' \in I$. Using that $\mathrm{Lip}(\varphi) = 1$ as well as (1) and Lemma 3.2, we have the following inequalities:

$$\begin{aligned} d\big(\beta'(t), \beta'(t')\big) &= d\big(\varphi(\eta(t)), \varphi(\eta(t'))\big) \\ &\leq d_T\big(\eta(t), \eta(t')\big) \\ &= d_T\big(\psi(0,0), \psi(1,0)\big)|t-t'| \\ &\leq \mathrm{diam}(T')|t-t'| \\ &\leq \mathrm{Lip}(\gamma)|t-t'|. \quad \square \end{aligned}$$

We now construct a homotopy H' from the initial path γ to the new path β' which has Lipschitz constant $\mathrm{Lip}(H') = \mathrm{Lip}(\gamma)$.

Let $t \in I$. There is a geodesic $g_t : I \to T$ from $g_t(0) = \psi(t,0)$ to $g_t(1) = \eta(t)$ where, for all $s, s' \in I$,

$$d_T\big(g_t(s), g_t(s')\big) = d_T\big(\psi(t,0), \eta(t)\big)|s-s'|. \tag{2}$$

Since points $\psi(0,0) = \eta(0)$ are equal, the geodesic $g_0(s) = \psi(0,0)$ is constant. Similarly, the geodesic $g_1(s) = \psi(1,0)$ is constant. Thus, the function $g : I \times I \to T$ given by $g(t,s) := g_t(s)$ is a homotopy from path $\psi|_{I\times\{0\}}$ to geodesic η provided g is Lipschitz. We show that g is a Lipschitz map with Lipschitz constant bounded by $\mathrm{Lip}(\gamma)$ and that the image of g is a subset of the image $\mathrm{Im}(\psi|_{I\times\{0\}})$.

Lemma 3.6. *The function g is Lipschitz with Lipschitz constant* $\mathrm{Lip}(g) \leq \mathrm{Lip}(\gamma)$.

Proof. Let $(t,s), (t',s') \in I \times I$. First, consider $d_T(g_t(s), g_{t'}(s))$. Fix t and t'. As $s \in I$ varies,

$$D(s) := d_T\big(g_t(s), g_{t'}(s)\big)$$

is a function from I to $\mathbb{R}$. By properties of metric trees, there exists $s_0 \in I$ such that the restriction $D|_{[0,s_0]}$ is decreasing and the restriction $D|_{[s_0,1]}$ is increasing. Thus, the maximum of the function D occurs when $s = 0$ or $s = 1$. Now, by Lemma 3.1,

$$\begin{aligned} D(0) = d_T\big(g_t(0), g_{t'}(0)\big) &= d_T\big(\psi(t,0), \psi(t',0)\big) \\ &\leq \mathrm{Lip}(\gamma)|t-t'|. \end{aligned}$$

Also, by (1) and Lemma 3.2,

$$\begin{aligned} D(1) = d_T\big(g_t(1), g_{t'}(1)\big) &= d_T\big(\eta(t), \eta(t')\big) \\ &= d_T\big(\eta(0), \eta(1)\big)|t-t'| \\ &\leq \mathrm{diam}(T')|t-t'| \end{aligned}$$

$$\leq \operatorname{Lip}(\gamma)|t-t'|.$$

Therefore, for any $s \in I$,

$$d_T\big(g_t(s), g_{t'}(s)\big) = D(s) \leq \operatorname{Lip}(\gamma)|t-t'|.$$

Now, consider the value $d_T(g_t(s), g_t(s'))$. Since η is a geodesic from $\psi(0,0)$ to $\psi(1,0)$ and $T' = \operatorname{Im}(\psi|_{I\times\{0\}}) \subset T$ is a subtree containing these points, $\eta(t) \in T'$. Thus, by (2) and Lemma 3.2,

$$\begin{aligned} d_T\big(g_t(s), g_t(s')\big) &= d_T\big(\psi(t,0), \eta(t)\big)|s-s'| \\ &\leq \operatorname{diam}(T')|s-s'| \\ &\leq \operatorname{Lip}(\gamma)|s-s'|. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} d_T\big(g_t(s), g_{t'}(s')\big) &\leq d_T\big(g_t(s), g_{t'}(s)\big) + d_T\big(g_{t'}(s), g_{t'}(s')\big) \\ &\leq \operatorname{Lip}(\gamma)|t-t'| + \operatorname{Lip}(\gamma)|s-s'| \\ &= \operatorname{Lip}(\gamma)\ d^1\big((t,s),(t',s')\big). \quad \square \end{aligned}$$

Lemma 3.7. *The image of the function g is a subset of the image of the restricted map $\psi|_{I\times\{0\}}$, that is,* $\operatorname{Im}(g) \subset \operatorname{Im}(\psi|_{I\times\{0\}})$.

Proof. Let $(t,s) \in I \times I$. The path g_t is a geodesic from $\psi(t,0)$ to $\eta(t)$. Since η is a geodesic from $\psi(0,0)$ to $\psi(1,0)$ and $T' = \operatorname{Im}(\psi|_{I\times\{0\}})$ is a subtree containing these points, $\eta(t) \in T'$. Thus, since g_t is a geodesic between points $\psi(t,0), \eta(t) \in T'$ and T' is a subtree, $g_t(s) \in T'$ for all $s \in I$. Thus,

$$\operatorname{Im}(g) \subset \operatorname{Im}(\psi|_{I\times\{0\}}). \qquad \square$$

We are now ready to define the new homotopy $H' : I \times I \to M$ by $H'(t,s) := \varphi \circ g(t,s)$. The function H' is indeed a homotopy from γ to β' as

$$\begin{aligned} H'(t,0) &= \varphi\big(g_t(0)\big) = \varphi\big(\psi(t,0)\big) = \gamma(t), \\ H'(t,1) &= \varphi\big(g_t(1)\big) = \varphi\big(\eta(t)\big) = \beta'(t), \\ H'(0,s) &= \varphi\big(g_0(s)\big) = \varphi\big(\psi(0,0)\big) = \gamma(0), \\ H'(1,s) &= \varphi\big(g_1(s)\big) = \varphi\big(\psi(1,0)\big) = \gamma(1). \end{aligned}$$

Moreover, since $\operatorname{Lip}(\varphi) = 1$ and, by Lemma 3.6, for $(s,t), (s',t') \in I \times I$,

$$\begin{aligned} d\big(H'(t,s), H'(t',s')\big) &= d\big(\varphi\big(g(t,s)\big), \varphi\big(g(t',s')\big)\big) \\ &\leq d_T\big(g(t,s), g(t',s')\big) \\ &\leq \operatorname{Lip}(\gamma)\ d^1\big((t,s),(t',s')\big). \end{aligned}$$

Therefore, $\operatorname{Lip}(H') \leq \operatorname{Lip}(\gamma)$. In fact, since $\operatorname{Lip}(H'|_{I\times\{0\}}) = \operatorname{Lip}(\gamma)$, we have that $\operatorname{Lip}(H') = \operatorname{Lip}(\gamma)$. Also, an immediate consequence of Lemma 3.7 is that $\operatorname{Im}(H') \subset \operatorname{Im}(\gamma)$.

We have thus defined a Lipschitz homotopy with all of the desired properties, which are collected in the following lemma.

Lemma 3.8. *Let M be a purely 2-unrectifiable space. Given Lipschitz paths $\gamma : I \to M$ and $\beta : I \to M$ that are homotopic rel endpoints, there exists a Lipschitz map $H' : I \times I \to M$ and a Lipschitz path $\beta' : I \to M$ such that*

- *the map H' is a homotopy from γ to β',*
- $\mathrm{Lip}(H') = \mathrm{Lip}(\gamma)$,
- $\mathrm{Im}(H') \subset \mathrm{Im}(\gamma)$,
- $\mathrm{Lip}(\beta') \leq \mathrm{Lip}(\gamma)$,
- $\mathrm{Im}(\beta') \subset \mathrm{Im}(\gamma)$, *and*
- $\ell(\beta') \leq \ell(\beta)$.

3.2. **Finding the length minimizer of a homotopy class**

We now prove the primary result of the paper: the existence of a length minimizer in any homotopy class of paths in a purely 2-unrectifiable metric space. We use Lemma 3.8 to fashion a sequence of Lipschitz paths in a given homotopy class, as well as associated homotopies, that have a uniform bound on their Lipschitz constants and then apply Arzelà–Ascoli theorem to find the length minimizer.

Theorem 3.9. *Let M be a purely 2-unrectifiable metric space. For any homotopy class $[\gamma]$ of Lipschitz paths in M, there exists a length minimizing Lipschitz path $\gamma_\infty \in [\gamma]$ where*

$$\ell(\gamma_\infty) = \inf\big\{\ell(\gamma) \mid \gamma \in [\gamma]\big\}.$$

Proof. Let M be a purely 2-unrectifiable metric space. Let $[\gamma]$ be a homotopy class of Lipschitz paths and define $\ell_{\min} := \inf\{\ell(\gamma) \mid \gamma \in [\gamma]\}$ to be the infimum of all lengths of paths in $[\gamma]$.

For each natural number n, let $\gamma_n \in [\gamma]$ be a Lipschitz path such that $\ell(\gamma_n) \leq \ell_{\min} + \frac{1}{n}$. Furthermore, since γ_1 is homotopic rel endpoints to γ_n, via Lemma 3.8, we can assume that $\mathrm{Lip}(\gamma_n) \leq \mathrm{Lip}(\gamma_1)$ and $\mathrm{Im}(\gamma_n) \subset \mathrm{Im}(\gamma_1)$. Additionally, there is a homotopy $H_n : I \times I \to M$ from γ_1 to γ_n such that $\mathrm{Lip}(H_n) = \mathrm{Lip}(\gamma_1)$ and $\mathrm{Im}(H_n) \subset \mathrm{Im}(\gamma_1)$.

Now, for any $n \in \mathbb{N}$, $\mathrm{Lip}(\gamma_n) \leq \mathrm{Lip}(\gamma_1)$. Since the images of the paths in the sequence (γ_n) are subsets of the compact set $\mathrm{Im}(\gamma_1)$, by Arzelà–Ascoli theorem, there exists a subsequence (γ_{n_k}) that uniformly converges to a Lipschitz path γ_∞. Now, utilizing lower semi-continuity of the length measure, $\ell(\gamma_\infty) \le \liminf_k \ell(\gamma_{n_k})$. In fact, due to how the sequence (γ_n) was selected, $\ell(\gamma_\infty) \le \ell_{\min}$.

We now want to show that $\gamma_\infty \in [\gamma]$. Associated to the subsequence (γ_{n_k}), there is a sequence of homotopies (H_{n_k}) such that $\mathrm{Lip}(H_{n_k}) = \mathrm{Lip}(\gamma_1)$ for each homotopy in the sequence. Since $\mathrm{Im}(H_{n_k}) \subset \mathrm{Im}(\gamma_1)$ for each n_k and $\mathrm{Im}(\gamma_1)$ is compact, by Arzelà–Ascoli theorem, there exists a subsequence $(H_{n_{k_j}})$ that converges uniformly to a Lipschitz map $H_\infty : I \times I \to M$.

Now, $H_\infty|_{I\times\{0\}} = \gamma_1$ since $H_{n_{k_j}}|_{I\times\{0\}} = \gamma_1$ for all n_{k_j}. Also, since the paths $H_{n_{k_j}}|_{I\times\{1\}} = \gamma_{n_{k_j}}$ converge uniformly to γ_∞, then $H_\infty|_{I\times\{1\}} = \gamma_\infty$. So, the map H_∞ is a homotopy from γ_1 to γ_∞. Therefore, $\gamma_\infty \in [\gamma]$ and thus $\ell(\gamma_\infty) = \ell_{\min}$. □

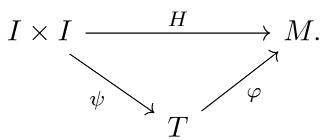
3.3. **Consequences of the existence of a length minimizer**

A length minimizer $\gamma_\infty \in [\gamma]$ can be thought of as the core of the homotopy class $[\gamma]$ where the extraneous branches of the paths in the class have been pruned in the sense that the image of γ_∞ is a subset of the image of any path contained in $[\gamma]$, as is now shown. A consequence of Theorem 3.10 is that a length minimizer for a homotopy class is unique up to reparametrization.

THEOREM 3.10. *Let M be a purely 2-unrectifiable metric space and let $[\gamma]$ be a homotopy class of Lipschitz paths in M with length minimizer $\gamma_\infty \in [\gamma]$. Additionally, assume that the length minimizer γ_∞ is arc length parametrized. Let $\gamma : I \to M$ be a Lipschitz path that is homotopic rel endpoints to γ_∞. Then the Lipschitz path γ'_∞ produced by Lemma* 3.8 *is equal to the length minimizer γ_∞. Furthermore, the image of a length minimizer γ_∞ is a subset of the image of γ, that is,*

$$\mathrm{Im}(\gamma_\infty) \subset \mathrm{Im}(\gamma).$$

Proof. Let $H : I \times I \to M$ be a Lipschitz homotopy from γ to γ_∞. By Theorem 2.5, the map H factors through a metric tree T.



We now show that the path $\psi|_{I\times\{1\}}$ in the metric tree T is the geodesic from $\psi(0,1)$ to $\psi(1,1)$. Let $t, t' \in I$ where $t < t'$ and let c be a Lipschitz path in $I \times I$ from $(t,1)$ to $(t',1)$. Since c is homotopic to the inclusion $(\mathbb{1}, 1) : [t, t'] \hookrightarrow$

$I \times I$, the paths $H \circ c$ and $H \circ (\mathbb{1}, 1) = \gamma_\infty|_{[t,t']}$ are homotopic. Since γ_∞ is the length minimizer for $[\gamma]$, the restriction $\gamma_\infty|_{[t,t']}$ is also a length minimizer in its homotopy class. Thus, $\ell(\gamma_\infty|_{[t,t']}) \leq \ell(H \circ c)$. Therefore, by the definition of the metric on T,

$$d_T\big(\psi(t,1), \psi(t',1)\big) = \ell(\gamma_\infty|_{[t,t']})$$

and in particular, $d_T(\psi(0,1), \psi(1,1)) = \ell(\gamma_\infty)$. Now, since γ_∞ is arc length parametrized,

$$\begin{aligned} d_T\big(\psi(t,1), \psi(t',1)\big) &= \ell(\gamma_\infty|_{[t,t']}) \\ &= \ell(\gamma_\infty)\ |t'-t| \\ &= d_T\big(\psi(0,1), \psi(1,1)\big)\ |t'-t|. \end{aligned}$$

Therefore, the path $\psi|_{I\times\{1\}}$ is indeed the geodesic from $\psi(0,1)$ to $\psi(1,1)$.

From the argument of Lemma 3.8, the path γ'_∞ is equal to the geodesic in T from $\psi(0,0) = \psi(0,1)$ to $\psi(1,0) = \psi(1,1)$ post-composed by φ. As the geodesic in discussion is $\psi|_{I\times\{1\}}$, we have that for all $t \in I$,

$$\gamma_\infty(t) = \varphi \circ \psi(t,1) = \gamma'_\infty(t).$$

Therefore, $\mathrm{Im}(\gamma_\infty) \subset \mathrm{Im}(\gamma)$ follows quickly from γ_∞ factoring through a geodesic segment. Indeed, for $t \in I$, the point $\psi(t,1) \in T$ is in the geodesic segment $\mathrm{Im}(\psi|_{I\times\{1\}})$ connecting $\psi(0,0) = \psi(0,1)$ to $\psi(1,0) = \psi(1,1)$. As the image $\mathrm{Im}(\psi|_{I\times\{0\}}) \subset T$ is a subtree containing these points, the geodesic segment $\mathrm{Im}(\psi|_{I\times\{1\}}) \subset \mathrm{Im}(\psi|_{I\times\{0\}})$ is a subset of the subtree. Hence, there exists $t' \in I$ such that $\psi(t,1) = \psi(t',0)$. Therefore,

$$\gamma_\infty(t) = \varphi \circ \psi(t,1) = \varphi \circ \psi(t',0) = \gamma(t').$$

Thus, $\mathrm{Im}(\gamma_\infty) \subset \mathrm{Im}(\gamma)$ as desired. □

Of note, in the proof of Theorem 3.10 we have shown that given an arc length parametrized length minimizer and any homotopy of the length minimizer, the length minimizer factors through a geodesic segment in the metric tree generated by the homotopy via Theorem 2.5.

An immediate consequence of Theorem 3.10 is that for every point in a purely 2-unrectifiable metric space, only the trivial class in the first Lipschitz homotopy group can be represented by a loop within each neighborhood of the point, as is now shown.

Corollary 3.11. *Let M be a purely 2-unrectifiable metric space and let $x \in M$. Let $[\alpha] \in \pi_1^{\mathrm{Lip}}(M,x)$ be a homotopy class of loops based at x such that for every open neighborhood $U \subset M$ of the point x, there exists a Lipschitz loop $\alpha_U \in [\alpha]$ based at x whose image is contained in U. Then $[\alpha]$ is the trivial homotopy class, $[\alpha] = [x]$.*

Proof. Let $[\alpha] \in \pi_1^{\mathrm{Lip}}(M, x)$ be a homotopy class of loops based at x such that for every open neighborhood $U \subset M$ of the point x, there exists a Lipschitz loop $\alpha_U \in [\alpha]$ based at x whose image is contained in U. Then, by Theorem 3.9, $[\alpha]$ has a length minimizer α_∞ and, by Theorem 3.10, the image of the length minimizer α_∞ is a subset of every neighborhood U of x. Therefore, α_∞ is the constant loop at x and thus $[\alpha] = [x]$. □

Using the wording of [15], Corollary 3.11 says that every point in a purely 2-unrectifiable supports only trivial local representation. In the language of [4], every point in a purely 2-unrectifiable metric space is non-singular. The harmonic archipelago is an instructive example of a space wherein not every point is non-singular. See Example 1.1 in [4].

As is detailed in [15], for a given metric space M, in order for the pseudometric $d_{\mathcal{P}}$ on the universal Lipschitz path space $\mathcal{P}_M$ to be a metric, it is sufficient that for every point in the underlying metric space M, only the trivial class in the first Lipschitz homotopy group can be represented by a loop within each neighborhood of the point. As such, Corollary 3.11 implies that the universal Lipschitz path space $\mathcal{P}_M$ over a purely 2-unrectifiable space M is a metric space.

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