# EXISTENCE OF ALMOST SPLIT SEQUENCES IN FINITELY PRESENTED FUNCTOR CATEGORIES

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We study the existence of almost split sequences in the category (mod-$R$)-mod of finitely presented functors over an artin algebra $R$. Our results extend the existence theorem for almost split sequences in the category mod-$R$ of finitely presented $R$-modules.

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## 1. INTRODUCTION

Given a ring $R$, denote by mod-$R$ the category of finitely presented right $R$-modules and Ab the category of abelian groups. The categories (mod-$R$)-Mod of covariant additive functors and Mod-(mod-$R$) of contravariant additive functors from mod-$R$ to Ab have received widespread attention since the 1960s. Auslander proposed the study of the representation theory of $R$ in terms of the category Mod-(mod-$R$). He pointed out in [3, Proposition 2.1] that the full subcategory mod-(mod-$R$) consisting of covariant finitely presented functors is an abelian category if mod-$R$ is abelian and showed in [4, §III.2] that for any ring $R$, the category (mod-$R$)-mod is an abelian category. They play initial roles in the study of the model theory of modules and the representation theory of artin algebras, see [3–5, 7].

The theory of almost split sequences (or Auslander–Reiten sequences) in mod($\Lambda$) was introduced by Auslander and Reiten [7] for an artin algebra $\Lambda$, and it also has been playing an important role in the investigation of representation theory of artin algebras. The existence theorem for almost split sequences in different categories is still an interesting topic and extensively

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studied by many authors. For example, when $\Lambda$ is an artin algebra, Auslander and Smalø [9] developed a theory for the existence of almost split sequences in subcategories of $\mathrm{mod}(\Lambda)$. Years later, Ng [18] further discussed the existence theorem in subcategories of $\mathrm{mod}(\Lambda)$ and provided a necessary and sufficient condition for which Salarian and Vahed [19] proved the existence theorem in the category $\mathrm{C}^b(\mathrm{mod}(\Lambda))$ of bounded complexes of finitely generated modules. As an analogues of almost split sequences, almost split triangles were introduced by Happel in [12, 13], which are widely studied by Jorgensen and Krause (see [15, 16]). For more general context, such as the exact $R$-categories and quasi-abelian categories, the existence theorems for almost split sequences were investigated separately by Liu et al. [17] and Shah [20]. See [14, 21] for more contexts where the almost split sequences exist.

We note that for any $R$-module $M$, there is a fully faithful embedding from mod-$R$ to (mod-$R$)-mod via $M \mapsto \mathrm{Hom}_R(M, -)$, which motivates us to generalise the classical existence theorem for almost split sequence in mod-$R$ to more general category (mod-$R$)-mod over artin algebras and we obtain the following main result.

Theorem 1.1. *Let $R$ be an artin algebra, $\tau$ and $\tau^{-1}$ be the Auslander–Reiten translations. Then the following statements hold:*

(1) *For any indecomposable non-projective object $F$ in* (mod-$R$)-mod*, there exists an almost split sequence* $0 \to \tau F \to G \to F \to 0$ *in* (mod-$R$)-mod.

(2) *For any indecomposable non-injective object $H$ in* (mod-$R$)-mod*, there exists an almost split sequence* $0 \to H \to T \to \tau^{-1} H \to 0$ *in* (mod-$R$)-mod.

## 2. PRELIMINARIES

In this section, we recall some necessary definitions and notations. Let $\mathcal{A}$ be an abelian category. A functor $F : \mathcal{A} \to \mathrm{Ab}$ is called representable if it is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(X, -)$ for some $X \in \mathcal{A}$. We abbreviate the representable functors by $(X, -)$. A functor $F : \mathcal{A} \to \mathrm{Ab}$ is called finitely presented if there exists a sequence of natural transformations $(Y, -) \to (X, -) \to F \to 0$ such that for any $A \in \mathcal{A}$, the sequence of abelian groups $(Y, A) \to (X, A) \to F(A) \to 0$ is exact. The category consisting of all finitely presented functors together with the natural transformations is denoted by $\mathcal{A}$-mod and was studied extensively by Auslander in [3]. In particular, Auslander showed that $\mathcal{A}$-mod is abelian, and the projectives in $\mathcal{A}$-mod are exactly the representable functors. Given a ring $R$, for the category mod-$R$ of finitely presented right

$R$-modules, the injectives in (mod-$R$)-mod have been characterized by Gruson and Jensen [10, Proposition 5.5] as those functors isomorphic to $-\otimes M$ for $M \in$ mod-$R$. The exactness in $\mathcal{A}$-mod is characterized by the exactness in Ab through evaluating. That is, a sequence $F \to G \to H$ of functors in $\mathcal{A}$-mod is exact if and only if for each object $X \in \mathcal{A}$, the corresponding sequence of abelian groups $F(X) \to G(X) \to H(X)$ is exact.

For two objects $F, G \in \mathcal{A}$-mod, we use the abbreviation $(F, G)$ to denote the group $\mathrm{Hom}_{\mathcal{A}}(F, G)$ of all natural transformations from $F$ to $G$, and $\mathcal{P}(F, G)$ the subgroup of $(F, G)$ consisting of all natural transformations that factor through a projective object in $\mathcal{A}$-mod. The quotient category of $\mathcal{A}$-mod, denoted by $\mathcal{A}$-$\underline{\mathrm{mod}}$, is called the projectively stable category of $\mathcal{A}$-mod. Its objects are the same as those of $\mathcal{A}$-mod, the morphism set $(\underline{F}, \underline{G})$ is defined as

$$(\underline{F}, \underline{G}) = (F, G)/\mathcal{P}(F, G).$$

For any abelian category $\mathcal{A}$, an object $X \in \mathcal{A}$ is called indecomposable if $X$ is non-zero and $X$ has no direct sum decomposition $X \cong X_1 \oplus X_1$, where $X_1$ and $X_2$ are non-zero objects. A morphism $\alpha : X \to Y$ in $\mathcal{A}$ is said to be right minimal if each morphism $\gamma : X \to X$ satisfying $\alpha\gamma = \alpha$ is an automorphism. Left minimal morphism in $\mathcal{A}$ is defined dually.

Based on the Jacobson radical of a ring, the following definition, radical of an additive category, can be found in [2].

*Definition* 2.1. The Jacobson radical $\mathrm{rad}_{\mathcal{C}}(X, Y)$ of an additive category $\mathcal{C}$ is a two-sided ideal $\mathrm{rad}_{\mathcal{C}}$ in $\mathcal{C}$ defined by

$$\{\alpha : X \to Y | 1_X - \beta\alpha \text{ is invertible for any morphism } \beta : Y \to X\}$$

for all objects $X, Y \in \mathcal{C}$.

For the finitely presented functor category (mod-$R$)-mod, in the presented paper, we denote by $\mathrm{rad}(F, G)$ the Jacobson radical of (mod-$R$)-mod for any functors $F, G \in$ (mod-$R$)-mod.

The next lemma is useful in the later sections.

Lemma 2.2 ([2], Proposition 3.5). *Let $\mathcal{C}$ be an additive category. Then the following statements hold:*

(1) *For any object $X \in \mathcal{C}$, $\mathrm{rad}_{\mathcal{C}}(X, X)$ is the Jacobson radical of the endomorphism ring $\mathrm{End}_{\mathcal{C}}X$.*

(2) *Assume that $X$ and $Y$ are objects in $\mathcal{C}$ such that the endomorphism rings $\mathrm{End}_{\mathcal{C}}X$ and $\mathrm{End}_{\mathcal{C}}Y$ are local. Then $\mathrm{rad}_{\mathcal{C}}(X, Y)$ is the abelian group of all non-isomorphisms from $X$ to $Y$ in $\mathcal{C}$.*

Next, we recall from [8] some definitions for the Auslander–Reiten theory.

*Definition* 2.3. Let $\alpha : G \to H$ be a morphism in (mod-$R$)-mod.

(1) $\alpha$ is called right almost split if $\alpha$ is not a split epimorphism and for each object $T \in$ (mod-$R$)-mod, each morphism $\varphi : T \to H$ in (mod-$R$)-mod that is not a split epimorphism factors through $\alpha$, i.e., there is a morphism $\psi : T \to G$ in (mod-$R$)-mod such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dl, dotted, "\psi"'] \arrow[d, "\varphi"] \\ G \arrow[r, "\alpha"] & H . \end{tikzcd}$$

The notion of a left almost split morphism in (mod-$R$)-mod is defined dually.

(2) $\alpha$ is called minimal right (resp., left) almost split if it is right (resp., left) minimal and right (resp., left) almost split in (mod-$R$)-mod.

(3) $\alpha$ is called irreducible if $\alpha$ is neither a split monomorphism nor a split epimorphism and if $\alpha = \nu\mu$ for some morphisms $\mu : G \to W$ and $\nu : W \to H$ in (mod-$R$)-mod, then $\mu$ is a split monomorphism or $\nu$ is a split epimorphism.

We collect some basic properties of (minimal) almost split morphisms in (mod-$R$)-mod.

LEMMA 2.4. *Let $\alpha : F \to G$ be a morphism in* (mod-$R$)-mod.

(1) *If $\alpha$ is right almost split in* (mod-$R$)-mod, *then the functor $G$ is indecomposable.*

(2) *If $\alpha$ is left almost split in* (mod-$R$)-mod, *then the functor $F$ is indecomposable.*

*Proof.* We just prove (1) since the proof of (2) is similar.

Suppose that $G = G_1 \oplus G_2$ with $G_1, G_2$ non-zero. Let $\epsilon_i : G_i \to G$ be the canonical injections for $i = 1, 2$. Then $\epsilon_i$ is not a split epimorphism. Since $\alpha$ is a right almost split morphism in (mod-$R$)-mod, there exists a morphism $\varphi_i : G_i \to F$ such that $\alpha\varphi_i = \epsilon_i$. So one gets a morphism

$$\varphi = (\varphi_1, \varphi_2) : G_1 \oplus G_2 \to F$$

such that $\alpha\varphi = 1$, which implies that $\alpha$ is a split epimorphism, a contradiction. □

Lemma 2.5. *The following statements hold:*

(1) *If* $\alpha : G \to H$ *and* $\alpha' : G' \to H$ *are minimal right almost split morphisms in* (mod-$R$)-mod, *then there exists an isomorphism* $\lambda : G \to G'$ *in* (mod-$R$)-mod *such that* $\alpha = \alpha'\lambda$.

(2) *If* $\beta : F \to G$ *and* $\beta' : F \to G'$ *are minimal left almost split morphisms in* (mod-$R$)-mod, *then there exists an isomorphism* $\tau : G \to G'$ *in* (mod-$R$)-mod *such that* $\beta' = \tau\beta$.

*Proof.* Again, we only prove (1) since the proof of (2) is similar.

Since $\alpha$ and $\alpha'$ are right almost split morphisms in (mod-$R$)-mod, there exist morphisms $\lambda : G \to G'$ and $\lambda' : G' \to G$ in (mod-$R$)-mod such that $\alpha = \alpha'\lambda$ and $\alpha' = \alpha\lambda'$. Then one gets that $\alpha = \alpha'\lambda = \alpha\lambda'\lambda$ and $\alpha' = \alpha\lambda' = \alpha'\lambda\lambda'$. Moreover, $\alpha$ and $\alpha'$ are right minimal yields that $\lambda'\lambda$ and $\lambda\lambda'$ are automorphisms. Hence, $\lambda$ is an isomorphism. $\square$

*Definition* 2.6. A short exact sequence of functors

$$0 \to F \xrightarrow{\beta} G \xrightarrow{\alpha} H \to 0$$

in (mod-$R$)-mod is called an almost split sequence if $\beta$ is minimal left almost split and $\alpha$ is minimal right almost split in (mod-$R$)-mod.

*Remark* 2.7. (1) It follows from the above definition that an almost split sequence is not split and hence $F$ is not an injective object and $H$ is not a projective object in (mod-$R$)-mod.

(2) Lemma 2.5 implies that an almost split sequence is unique up to isomorphism for its end terms.

## 3. MINIMAL RIGHT (RESPECTIVELY, LEFT) ALMOST SPLIT MORPHISMS AND IRREDUCIBLE MORPHISMS

In this section, we mainly consider minimal right (resp., left) almost split morphisms and irreducible morphisms in (mod-$R$)-mod. The ideas we deal with come from [2]. We first begin with some characterizations for irreducible monomorphisms (resp., epimorphisms).

Proposition 3.1. *Let* $0 \to F \xrightarrow{\beta} G \xrightarrow{\alpha} H \to 0$ *be a non-split short exact sequence in* (mod-$R$)-mod. *The following statements hold:*

(1) *The morphism* $\alpha : G \to H$ *is irreducible in* (mod-$R$)-mod *if and only if for any morphism* $\varphi : F \to S$ *in* (mod-$R$)-mod, *there exists either a morphism* $\mu_1 : G \to S$ *such that* $\varphi = \mu_1\beta$ *or* $\mu_2 : S \to G$ *such that* $\beta = \mu_2\varphi$.

(2) *The morphism* $\beta : F \to G$ *is irreducible in* (mod-$R$)-mod *if and only if for any morphism* $\psi : T \to H$ *in* (mod-$R$)-mod, *there exists either a morphism* $\nu_1 : T \to G$ *such that* $\psi = \alpha\nu_1$ *or* $\nu_2 : G \to T$ *such that* $\alpha = \psi\nu_2$.

*Proof.* We only prove (1); the statement (2) is proved similarly.

Suppose that $\alpha : G \to H$ is an irreducible morphism in (mod-$R$)-mod and let $\varphi : F \to S$ be any morphism in (mod-$R$)-mod. Then one gets the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\varphi"'] & G \arrow[r, "\alpha"] \arrow[d, "\delta"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & S \arrow[r, "\beta'"] & W \arrow[r, "\alpha'"] & H \arrow[r] & 0 \end{tikzcd}$$

where the left square is a pushout square. Since $\alpha = \alpha'\delta$ is irreducible, $\delta$ is a split monomorphism or $\alpha'$ is a split epimorphism in (mod-$R$)-mod. On one hand, if $\alpha'$ is a split epimorphism, then $\beta'$ is a split monomorphism. Thus there exists a morphism $\lambda : W \to S$ such that $\lambda\beta' = 1_S$. Set $\mu_1 = \lambda\delta : G \to S$. Then one has $\mu_1\beta = \lambda\delta\beta = \lambda\beta'\varphi = \varphi$. On the other hand, if $\delta$ is a split monomorphism, then there exists a morphism $\delta' : W \to G$ such that $\delta'\delta = 1_G$. Set $\mu_2 = \delta'\beta' : S \to G$. Then one gets $\mu_2\varphi = \delta'\beta'\varphi = \delta'\delta\beta = \beta$.

Conversely, since the short exact sequence $0 \to F \xrightarrow{\beta} G \xrightarrow{\alpha} H \to 0$ in (mod-$R$)-mod is not split, $\alpha$ is neither a split monomorphism nor a split epimorphism. Assume that $\alpha = \phi_2\phi_1$, where $\phi_1 : G \to T$ and $\phi_2 : T \to H$ for some object $T \in$ (mod-$R$)-mod. Clearly, $\phi_2$ is an epimorphism in (mod-$R$)-mod. So one gets the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\lambda"'] & G \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r, "\epsilon"] & T \arrow[r, "\phi_2"] & H \arrow[r] & 0 \end{tikzcd}$$

where $K = \mathrm{Ker}\phi_2$. According to [2, Proposition 5.3, Appendix A], the left square is a pushout square. On one hand, if there is a morphism $\mu_1 : G \to K$ in (mod-$R$)-mod such that $\lambda = \mu_1\beta$, then there exists a morphism $\pi : T \to K$ in (mod-$R$)-mod such that $\pi\epsilon = 1_K$ by the universal property of pushout. It follows that $\epsilon$ is a split monomorphism, and thus $\phi_2$ is a split epimorphism.

On the other hand, if there is a morphism $\mu_2 : K \to G$ such that $\mu_2\lambda = \beta$, then, similarly, one has that $\phi_1$ is a split monomorphism, as desired. $\square$

COROLLARY 3.2. *Let $\alpha : F \to G$ be a morphism in* (mod-$R$)-mod.

(1) *If $\alpha$ is an irreducible monomorphism in* (mod-$R$)-mod, *then* Coker$\alpha$ *is an indecomposable object in* (mod-$R$)-mod.

(2) *If $\alpha$ is an irreducible epimorphism in* (mod-$R$)-mod, *then* Ker$\alpha$ *is an indecomposable object in* (mod-$R$)-mod.

*Proof.* We just prove the second statement since the first follows by duality.

Assume that $\mathrm{Ker}\alpha = K_1 \oplus K_2$ with $K_i \neq 0$ for $i = 1, 2$. Next, we consider $\rho_i : \mathrm{Ker}\alpha \to K_i$ the canonical projections. If there exists a morphism $\mu_i : K_i \to F$ such that $\mu_i\rho_i = \gamma$ for the morphism $\gamma : \mathrm{Ker}\alpha \to F$, then $\rho_i$ is a monomorphism in (mod-$R$)-mod. This implies that $\rho_i$ is an isomorphism, which is contrary to the assumption that $K_i \neq 0$ for $i = 1, 2$. So by Proposition 3.1, there exists a morphism $\nu_i : F \to K_i$ in (mod-$R$)-mod such that $\rho_i = \nu_i\gamma$. Thus, one gets a morphism $\nu = \binom{\nu_1}{\nu_2} : F \to \mathrm{Ker}\alpha$ such that $\nu\gamma = \binom{\nu_1}{\nu_2}\gamma = 1_{\mathrm{Ker}\alpha}$, which yields that $\gamma$ is a split monomorphism, and further, $\alpha$ is a split epimorphism, this is contrary to the fact that $\alpha$ is irreducible. $\square$

According to [6, Proposition 2.5, Proposition 2.6], (mod-$R$)-mod is a dualizing variety. Since the endomorphism ring of each object in (mod-$R$)-mod is an artin algebra, it follows that (mod-$R$)-mod is a Krull–Schmidt category. Hence, the endomorphism ring of any indecomposable object in (mod-$R$)-mod is local.

Let $F$ be a functor in the category (mod-$R$)-mod. We denote by End$F$ the endomorphism ring $\mathrm{End}_{(\text{mod-}R)\text{-mod}}F$ of functor $F$, and radEnd$F$ the Jacobson radical of the endomorphism ring End$F$. In what follows, we consider the connections between irreducible morphisms and minimal right (resp., left) almost split morphisms, and before which, the next lemma is needed.

LEMMA 3.3. *The following statements hold*:

(1) *Let $\alpha : G \to H$ be a non-zero morphism in* (mod-$R$)-mod *with $H$ indecomposable. Then $\alpha$ is not a split epimorphism if and only if* $\mathrm{Im}(H, \alpha) \subseteq$ radEnd$H$.

(2) *Let $\beta : F \to G$ be a non-zero morphism in* (mod-$R$)-mod *with $F$ indecomposable. Then $\beta$ is not a split monomorphism if and only if* $\mathrm{Im}(\beta, F) \subseteq$ radEnd$F$.

*Proof.* We only prove (1); the proof of (2) is similar.

$\Rightarrow$) Assume that $\alpha$ is not a split epimorphism in (mod-$R$)-mod. For each $\delta \in \mathrm{Im}(H,\alpha) \subseteq \mathrm{End}H$, there exists a morphism $\alpha' : H \to G$ such that $\delta = (H,\alpha)(\alpha') = \alpha\alpha'$. If $\mathrm{Im}(H,\alpha) \nsubseteq \mathrm{radEnd}H$, then $\delta$ is an isomorphism by Lemma 2.2. So there exists a morphism $\delta' : H \to H$ such that $1_H = \delta\delta' = \alpha\alpha'\delta'$, which yields that $\alpha$ is a split epimorphism, a contradiction.

$\Leftarrow$) Assume that $\mathrm{Im}(H,\alpha) \subseteq \mathrm{radEnd}H$. If $\alpha$ is a split epimorphism, then $(H,\alpha)$ is an epimorphism and hence $\mathrm{Im}(H,\alpha) = \mathrm{End}H$, which is contrary to the assumption. $\square$

PROPOSITION 3.4. *The following statements hold:*

(1) *Let $\alpha : G \to H$ be a minimal right almost split morphism in* (mod-$R$)-mod. *Then $\alpha$ is an irreducible morphism in* (mod-$R$)-mod. *Moreover, a morphism $\alpha' : G' \to H$ is irreducible in* (mod-$R$)-mod *if and only if $G' \neq 0$ and there exists a direct decomposition $G \cong G' \oplus G''$ and a morphism $\alpha'' : G'' \to H$ such that $(\alpha', \alpha'') : G' \oplus G'' \to H$ is minimal right almost split in* (mod-$R$)-mod.

(2) *Let $\beta : F \to G$ be a minimal left almost split morphism in* (mod-$R$)-mod. *Then $\beta$ is an irreducible morphism in* (mod-$R$)-mod. *Moreover, a morphism $\beta' : F \to G'$ is irreducible in* (mod-$R$)-mod *if and only if $G' \neq 0$ and there exists a direct decomposition $G \cong G' \oplus G''$ and a morphism $\beta'' : F \to G''$ such that $\binom{\beta'}{\beta''} : F \to G' \oplus G''$ is minimal left almost split in* (mod-$R$)-mod.

*Proof.* Again, we just prove the first statement since the second is proved similarly.

Let $\alpha : G \to H$ be a minimal right almost split morphism in (mod-$R$)-mod. Then $\alpha$ is not a split epimorphism and $H$ is an indecomposable object in (mod-$R$)-mod by Lemma 2.4. So $\alpha$ is not an isomorphism, and moreover, $\alpha$ is not a split monomorphism since $H$ is indecomposable. Suppose that $\alpha = \psi_2\psi_1$, where $\psi_1 : G \to T$ and $\psi_2 : T \to H$ for some object $T \in$ (mod-$R$)-mod. Assume that $\psi_2$ is not a split epimorphism in (mod-$R$)-mod. One needs to prove that $\psi_1$ is a split monomorphism in (mod-$R$)-mod. Since $\alpha$ is right almost split, there exists a morphism $\lambda : T \to G$ such that $\alpha\lambda = \psi_2$. Thus $\alpha = \psi_2\psi_1 = \alpha\lambda\psi_1$. If the morphism $\alpha$ is right minimal, this yields that $\lambda\psi_1 = 1_G$. Hence $\psi_1$ is a split monomorphism.

Now let $\alpha' : G' \to H$ be an irreducible morphism in (mod-$R$)-mod. Clearly, $G' \neq 0$. And it is obvious that there exists a morphism $\varphi : G' \to G$ such that $\alpha\varphi = \alpha'$. Since $\alpha$ is not a split epimorphism, $\varphi$ is a split monomorphism. Let $G'' = \mathrm{Coker}\varphi$. Hence there exists a morphism $\psi : G'' \to G$ such that

$(\varphi, \psi) : G' \oplus G'' \to G$ is an isomorphism, and $\alpha(\varphi, \psi) = (\alpha', \alpha\psi) : G' \oplus G'' \to H$ is a minimal right almost split morphism in (mod-$R$)-mod, where $\alpha'' = \alpha\psi$.

Conversely, assume that $\alpha' : G' \to H$ satisfies the stated conditions, one needs to verify that $\alpha'$ is an irreducible morphism in (mod-$R$)-mod. If $\alpha'$ is an isomorphism, then there exists a morphism $\beta' : H \to G'$ such that $\alpha'\beta' = 1_H$. So one has $(\alpha', \alpha'')\binom{\beta'}{0} = 1_H$, which yields that $(\alpha', \alpha'')$ is a split epimorphism in (mod-$R$)-mod, contrary to $(\alpha', \alpha'') : G' \oplus G'' \to H$ is right almost split. Thus, $\alpha'$ is not an isomorphism, and moreover, $\alpha'$ is not a split monomorphism, either. Similar as above, one gets that $\alpha'$ is not a split epimorphism. Now assume that $\alpha' = \psi_2'\psi_1'$, where $\psi_1' : G' \to S$ and $\psi_2' : S \to H$ for some object $S \in$ (mod-$R$)-mod. Suppose that $\psi_2'$ is not a split epimorphism, one needs to prove that $\psi_1'$ is a split monomorphism. Note that

$$(\alpha', \alpha'') = (\psi_2', \alpha'') \begin{pmatrix} \psi_1' & 0 \\ 0 & 1 \end{pmatrix},$$

where $\left(\begin{smallmatrix} \psi_1' & 0 \\ 0 & 1 \end{smallmatrix}\right) : G' \oplus G'' \to S \oplus G''$ and $(\psi_2', \alpha'') : S \oplus G'' \to H$. Since $\psi_2'$ is not a split epimorphism, it follows from Lemma 3.3 that $\mathrm{Im}(H, \psi_2') \subseteq \mathrm{radEnd}H$. Similarly, one has $\mathrm{Im}(H, \alpha'') \subseteq \mathrm{radEnd}H$. Hence, one gets

$$\mathrm{Im}\big(H, (\psi_2', \alpha'')\big) \subseteq \mathrm{radEnd}H.$$

Again by Lemma 3.3, one has that $(\psi_2', \alpha'')$ is not a split epimorphism in (mod-$R$)-mod. Since $(\alpha', \alpha'')$ is minimal right almost split, $(\alpha', \alpha'')$ is an irreducible morphism. Hence, $\left(\begin{smallmatrix} \psi_1' & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a split monomorphism, which implies that $\psi_1'$ is a split monomorphism, as desired. $\square$

## 4. EXISTENCE OF ALMOST SPLIT SEQUENCES IN (MOD-$R$)-MOD

In this section, we prove that the existence theorem for almost split sequences holds in the category (mod-$R$)-mod by using (minimal) right (resp., left) almost split morphisms and irreducible morphisms. We begin with the following lemma.

LEMMA 4.1. *Let*

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\mu"'] & G \arrow[r, "\alpha"] \arrow[d, "\nu"'] & H \arrow[r] \arrow[d, "\omega"'] & 0 \\ 0 \arrow[r] & F \arrow[r, "\beta"] & G \arrow[r, "\alpha"] & H \arrow[r] & 0 \end{tikzcd}$$

*be a commutative diagram in* (mod-$R$)-mod *with non-split exact rows.*

(1) *If $F$ is an indecomposable object and $\omega$ is an automorphism in* (mod-$R$)-mod*, then $\mu$ and hence $\nu$ are automorphisms in* (mod-$R$)-mod*.*

(2) *If $H$ is an indecomposable object and $\mu$ is an automorphism in* (mod-$R$)-mod*, then $\omega$ and hence $\nu$ are automorphisms in* (mod-$R$)-mod*.*

*Proof.* We just prove the statement (2) since the other is similar.

Suppose that $\mu = 1_F$. If $\omega$ is not an isomorphism, then $F$ is indecomposable yields that $\omega$ is nilpotent. So we have $m$ such that $\omega^m = 0$. According to the commutativity of the above diagram, one has $\alpha\nu^m = \omega^m\alpha = 0$, then there exists a morphism $\delta : G \to F$ such that $\beta\delta = \nu^m$ by the universal property of kernels. Thus $\beta = \nu^m\beta = \beta\delta\beta$. Since $\beta$ is a monomorphism, $1_F = \delta\beta$. It follows that $\beta$ is a split monomorphism, a contradiction. $\square$

Now, we give some equivalent characterizations for almost split sequences in (mod-$R$)-mod. See [2, IV, Theorem 1.13] for the case of modules.

Proposition 4.2. *Let $0 \to F \xrightarrow{\beta} G \xrightarrow{\alpha} H \to 0$ be a short exact sequence in* (mod-$R$)-mod*. Then the following statements are equivalent:*

(1) *The above sequence is an almost split sequence.*

(2) *$F$ is an indecomposable object and $\alpha$ is a right almost split morphism.*

(3) *$H$ is an indecomposable object and $\beta$ is a left almost split morphism.*

(4) *$\alpha$ is a minimal right almost split morphism.*

(5) *$\beta$ is a minimal left almost split morphism.*

(6) *$F$ and $H$ are indecomposable objects, $\alpha$ and $\beta$ are irreducible morphisms.*

*Proof.* The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Lemma 2.4.

The implications (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are clear.

The implication (1) $\Rightarrow$ (6) follows from Lemma 2.4 and Proposition 3.4.

(4) $\Rightarrow$ (2) Since $\alpha$ is a minimal right almost split morphism in (mod-$R$)-mod, $\alpha$ is an irreducible morphism in (mod-$R$)-mod by Proposition 3.4. Hence $F$ is an indecomposable object in (mod-$R$)-mod by Corollary 3.2.

The implication (5) $\Rightarrow$ (3) is proved similarly.

(2) $\Rightarrow$ (3) Since $\alpha$ is a right almost split morphism in (mod-$R$)-mod, it follows from Lemma 2.4 that $H$ is an indecomposable object in (mod-$R$)-mod. Next, we show that $\beta$ is a left almost split morphism in (mod-$R$)-mod. Firstly, $\beta$ is not a split monomorphism since $\alpha$ is not a split epimorphism. Let $\varphi : F \to T$ be a morphism in (mod-$R$)-mod such that $\varphi$ cannot factor

through $\beta$, where $T$ is an object in (mod-$R$)-mod. We prove that $\varphi$ is a split monomorphism. By [2, Proposition 5.3, Appendix A], there is a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\varphi"'] & G \arrow[r, "\alpha"] \arrow[d, "\omega"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r, "\delta"] & W \arrow[r, "\sigma"] & H \arrow[r] & 0 \end{tikzcd}$$

where the left square is a pushout square. Note that $\sigma$ is not a split epimorphism, so there exists a morphism $\omega' : W \to G$ in (mod-$R$)-mod such that $\alpha\omega' = \sigma$. Then one gets the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "\delta"] \arrow[d, "\varphi'"'] & W \arrow[r, "\sigma"] \arrow[d, "\omega'"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & F \arrow[r, "\beta"] & G \arrow[r, "\alpha"] & H \arrow[r] & 0 \end{tikzcd}$$

where $\varphi' : T \to F$ is obtained by the universal property of kernels. Combining the above two commutative diagrams, one gets the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\varphi'\varphi"'] & G \arrow[r, "\alpha"] \arrow[d, "\omega'\omega"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & F \arrow[r, "\beta"] & G \arrow[r, "\alpha"] & H \arrow[r] & 0\,. \end{tikzcd}$$

By Lemma 4.1, one gets $\varphi'\varphi = 1_F$. Hence $\varphi$ is a split monomorphism in (mod-$R$)-mod.

The implication (3) $\Rightarrow$ (2) is proved similarly.

(2) $\Rightarrow$ (1) We first prove that $\alpha$ is a right minimal morphism in (mod-$R$)-mod. Let $\gamma : G \to G$ be a morphism such that $\alpha\gamma = \alpha$. Then one gets the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r, "\beta"] \arrow[d, "\lambda"'] & G \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & H \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & F \arrow[r, "\beta"] & G \arrow[r, "\alpha"] & H \arrow[r] & 0 \end{tikzcd}$$

where $\lambda : F \to F$ is obtained by the universal property of kernels. So Lemma 4.1 yields that $\gamma$ is an automorphism. Thus $\alpha$ is a minimal right almost split morphism in (mod-$R$)-mod. Since the statements (2), (3) are equivalent, similarly, one gets $\beta$ is a minimal left almost split morphism in (mod-$R$)-mod.

(6) $\Rightarrow$ (2) One needs to show that $\alpha$ is a right almost split morphism in (mod-$R$)-mod. Clearly, $\alpha$ is not a split epimorphism. Assume that $\mu : S \to H$

is not a split epimorphism for any object $S \in$ (mod-$R$)-mod. We may suppose that $S$ is an indecomposable object in (mod-$R$)-mod. Since $\beta$ is an irreducible morphism, there exists a morphism $\nu : S \to G$ such that $\alpha\nu = \mu$, or a morphism $\upsilon : G \to S$ such that $\mu\upsilon = \alpha$ by Proposition 3.1. The first case yields that $\alpha$ is a right almost split morphism in (mod-$R$)-mod. In the second case, since $\alpha$ is an irreducible morphism and $\mu$ is not a split epimorphism, $\upsilon$ is a split monomorphism. Then $S$ is indecomposable yields that $\upsilon$ is an isomorphism, so there exists a morphism $\upsilon' : S \to G$ such that $\upsilon\upsilon' = 1_S$, further, one gets $\alpha\upsilon' = \mu\upsilon\upsilon' = \mu$, as desired. $\square$

Let $R$ be an artin algebra. The classical existence theorem for almost split sequences in mod-$R$ is derived from the Auslander–Reiten formula in mod-$R$. Inspired by this, in what follows, we use the Auslander–Reiten formula in (mod-$R$)-mod studied in [11] to verify the existence of almost split sequences in (mod-$R$)-mod. Firstly, we recall some definitions and facts needed later.

*Definition* 4.3 ([11]). Let $R$ be an artin algebra. The Auslander–Reiten translations $\tau$ and $\tau^{-1}$ are defined as

$$\tau = D\mathrm{Tr} : (\mathrm{mod} - R) - \underline{\mathrm{mod}} \to (\mathrm{mod} - R) - \overline{\mathrm{mod}},$$
$$\tau^{-1} = \mathrm{Tr}D : (\mathrm{mod} - R) - \overline{\mathrm{mod}} \to (\mathrm{mod} - R) - \underline{\mathrm{mod}}.$$

In the above definition, the functors Tr : (mod-$R$)-$\underline{\mathrm{mod}}\longrightarrow$ $\underline{\mathrm{mod}}$-(mod-$R$) and $D$ : $\underline{\mathrm{mod}}$-(mod-$R$) $\to$(mod-$R$)-$\overline{\mathrm{mod}}$ are dualities between categories, where the duality $D$ is induced by the functor $D$ : (mod-$R$)-mod$\to$ mod-(mod-$R$). Here, we use the same symbol $D$; see [11, Lemma 3.6, Lemma 3.9]. And the category (mod-$R$)-$\overline{\mathrm{mod}}$ = (mod-$R$)-mod$/D\mathcal{P}$ is defined to be the injectively stable category of (mod-$R$)-mod. Its objects are the same as those of (mod-$R$)-mod, the morphism set from $F$ to $G$ is defined by

$$(\overline{F}, \overline{G}) = (F, G)/D\mathcal{P}(F, G),$$

where $D\mathcal{P}(F, G)$ denotes the subset of $(F, G)$ consisting of all natural transformations that factor through the dual of a projective object in mod-(mod-$R$). See [11] for more details.

Lemma 4.4 ([11]). *Let $R$ be an artin algebra and $\tau$ and $\tau^{-1}$ as in Definition* 4.3.

(1) $\tau = D\mathrm{Tr}$ : (mod-$R$)-$\underline{\mathrm{mod}} \to$ (mod-$R$)-$\overline{\mathrm{mod}}$, $\tau^{-1} = \mathrm{Tr}D$ : (mod-$R$)-$\overline{\mathrm{mod}}$ $\to$ (mod-$R$)-$\underline{\mathrm{mod}}$ *are mutually inverse equivalences.*

(2) *There exist functorial isomorphisms*

$$D(\underline{F}, \underline{G}) \xrightarrow{\cong} \mathrm{Ext}^1(G, \tau F), D(\overline{F}, \overline{G}) \xrightarrow{\cong} \mathrm{Ext}^1(\tau^{-1}G, F)$$

*for any functors $F, G \in$ (mod-$R$)-mod.*

*Remark* 4.5. From Lemma 4.4 (1), it is easy to verify that $\tau$ in Definition 4.3 gives a bijection from indecomposable non-projective objects in (mod-$R$)-mod to indecomposable non-injective objects in (mod-$R$)-mod, and the inverse is given by $\tau^{-1}$.

Finally, we give the proof of Theorem 1.1.

*Proof.* We only prove (2), the statement (1) is proved similarly.

Let $H$ be an indecomposable non-injective object in (mod-$R$)-mod and set $S(H,H) = (H,H)/\mathrm{rad}(H,H)$. Then Lemma 2.2 (1) yields that $S(H,H) = \mathrm{End}H/\mathrm{radEnd}H$. Note that by $D\mathcal{P}(H,H) \subseteq \mathrm{rad}(H,H)$, one gets the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D\mathcal{P}(H,H) \arrow[r] \arrow[d, hook] & (H,H) \arrow[r] \arrow[d, equal] & (\overline{H},\overline{H}) \arrow[r] \arrow[d, dotted, "\omega_{H,H}"'] & 0 \\
0 \arrow[r] & \mathrm{rad}(H,H) \arrow[r] & (H,H) \arrow[r] & S(H,H) \arrow[r] & 0
\end{tikzcd}$$

where $\omega_{H,H} : \overline{\mathrm{End}}H = (\overline{H},\overline{H}) \to S(H,H) = \mathrm{End}H/\mathrm{radEnd}H$ is obtained by the universal property of cokernels. Since $\mathrm{End}H$ is a local ring, $\overline{\mathrm{End}}H$ is a local ring. It follows from Lemma 2.2 (1) and [1, Proposition 15.15] that $\mathrm{radEnd}H$ is the unique maximal ideal of $\mathrm{End}H$. Hence $S(H,H) = \mathrm{End}H/\mathrm{radEnd}H$ is a simple $\mathrm{End}H$-module. Moreover, one has

$$\begin{aligned}
\mathrm{top}(\overline{\mathrm{End}}H) &= \overline{\mathrm{End}}H/\mathrm{rad}\overline{\mathrm{End}}H \\
&= \big((H,H)/D\mathcal{P}(H,H)\big)\big/\big(\mathrm{rad}(H,H)/D\mathcal{P}(H,H)\big) \\
&= S(H,H),
\end{aligned}$$

so $S(H,H)$ is the simple top of $\overline{\mathrm{End}}H$. Applying the functor

$$D : (\mathrm{mod}-R) - \overline{\mathrm{mod}} \to \underline{\mathrm{mod}} - (\mathrm{mod}-R)$$

to the above commutative diagram, one gets the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & DS(H,H) \arrow[r] \arrow[d, "D\omega_{H,H}"'] & D(H,H) \arrow[r] \arrow[d, equal] & D(\mathrm{rad}(H,H)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D(\overline{H},\overline{H}) \arrow[r] & D(H,H) \arrow[r] & D(D\mathcal{P}(H,H)) \arrow[r] & 0
\end{tikzcd}$$

where $D\omega_{H,H} : DS(H,H) \to D(\overline{H},\overline{H})$ is a monomorphism, and one gets that $\mathrm{Im}(D\omega_{H,H})$ is the simple socle of $D(\overline{H},\overline{H})$. By Lemma 4.4, there exists a functorial isomorphism $D(\overline{H},\overline{H}) \xrightarrow{\cong} \mathrm{Ext}^1(\tau^{-1}H,H)$.

Let $\xi'$ be a non-zero element in $DS(H,H)$ and $[\xi] \in \mathrm{Ext}^1(\tau^{-1}H,H)$ its image under $D\omega_{H,H}$. One needs to claim that if $[\xi]$ is represented by the following short exact sequence

$$[\xi]: \quad 0 \to H \xrightarrow{\beta} T \xrightarrow{\alpha} \tau^{-1}H \to 0,$$

then the sequence is an almost split sequence.

Since $\xi' \neq 0$ and $D\omega_{H,H}$ is a monomorphism, the above sequence is not split. By Remark 4.5, $\tau^{-1}H$ is an indecomposable object in (mod-$R$)-mod. Hence, it is sufficient to verify that $\beta: H \to T$ is a left almost split morphism in (mod-$R$)-mod by Proposition 4.2. Clearly, $\beta$ is not a split monomorphism. Let $G$ be any object in (mod-$R$)-mod and $\sigma: H \to G$ any morphism that is not split monic in (mod-$R$)-mod. We may assume that $G$ is an indecomposable object. Then $\sigma$ is not an isomorphism. Consider the following commutative diagram:

$$\begin{tikzcd}
DS(H,H) \arrow[r, "D\omega_{H,H}"] \arrow[d, "DS(\sigma{,}H)"] & D(\overline{H},\overline{H}) \arrow[r, "\cong"] \arrow[d, "D(\sigma{,}\overline{H})"] & \mathrm{Ext}^1(\tau^{-1}H,H) \arrow[d, "\mathrm{Ext}^1(\tau^{-1}H{,}\sigma)"] \\
DS(G,H) \arrow[r, "D\omega_{G,H}"] & D(\overline{G},\overline{H}) \arrow[r, "\cong"] & \mathrm{Ext}^1(\tau^{-1}H,G).
\end{tikzcd}$$

Since $\sigma \in \mathrm{rad}(H,G)$, one gets that $DS(\sigma,H)(\xi') = 0$. Then the above commutative diagram yields that the image $\mathrm{Ext}^1(\tau^{-1}H,\sigma)([\xi]) = 0$ in $\mathrm{Ext}^1(\tau^{-1}H,G)$, i.e., there exists a commutative diagram with exact rows:

$$\begin{tikzcd}
{[\xi]:} & 0 \arrow[r] & H \arrow[r, "\beta"] \arrow[d, "\sigma"] & T \arrow[r, "\alpha"] \arrow[d, "\delta"] & \tau^{-1}H \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{Ext}^1(\tau^{-1}H,\sigma)([\xi]): & 0 \arrow[r] & G \arrow[r, "\beta'"] & T' \arrow[r, "\alpha'"] & \tau^{-1}H \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is split. Thus there exists a morphism $\gamma: T' \to G$ such that $\gamma\beta' = 1_G$. Set $\varphi = \gamma\delta: T \to G$. Then one gets

$$\varphi\beta = \gamma\delta\beta = \gamma\beta'\sigma = \sigma,$$

which yields that $\beta$ is a left almost split morphism in (mod-$R$)-mod. This completes the proof. $\square$

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