MEET ATOM ELEMENT GRAPH OF A LATTICE

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Communicated by Ioan Tomescu

Let $\mathcal{L}$ be a bounded lattice. The meet atom element graph $\mathbb{AG}(\mathcal{L})$ of $\mathcal{L}$ is a simple undirected graph whose vertices are all nontrivial elements of $\mathcal{L}$ and any two distinct vertices a and b are adjacent if and only if $a \wedge b$ is an atom element of $\mathcal{L}$. The basic properties and possible structures of the graph $\mathbb{AG}(\mathcal{L})$ are investigated. The connectedness, clique number, domination number and independence number of $\mathbb{AG}(\mathcal{L})$ and their relations to algebraic properties of $\mathcal{L}$ are explored.

AMS 2020 Subject Classification: 06B15, 06B20, 05C25, 05C40.

Key words: lattice, atom, meet atom element graph.

1. INTRODUCTION

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words, they are bounded. The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in last decade. Associating a graph with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa (see, for example, [1–8, 11, 13–18, 20]).

Beck [4], Anderson and Naseer [2], and Anderson and Livingston [1] et. al. have studied graphs on commutative rings. One of the most important graphs which have been studied is the intersection graph. Bosak [5] defined the intersection graph of semigroups. Csàkàny and Pollàk [8] studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [7]. The intersection minimal ideal graph of a ring, i.e., a simple graph whose vertices are nontrivial ideals of a ring R and two vertices I, J are adjacent if the intersection of corresponding ideals is a minimal ideal, was investigated by Barman and Rajkhowa in [6]. The intersection graph of ideals of rings, submodules of modules and lattices has been investigated by several authors (see, for example, [3, 16–18, 20]).

MATH. REPORTS **25(75)** (2025), *1-2*, 89–102
doi: 10.59277/mrar.2025.27.77.1.2.89

Let $\mathcal{L}$ be a bounded distributive lattice. The purpose of this paper is to investigate a graph associated to a lattice $\mathcal{L}$ called the meet atom element graph of $\mathcal{L}$. This results in the characterization of lattices in terms of some specific properties of those graphs. The meet atom element graph of $\mathcal{L}$ is a simple graph $\mathbb{AG}(\mathcal{L})$ whose vertices are all nontrivial elements and two distinct vertices are adjacent if and only if the meet of the corresponding elements is an atom element of $\mathcal{L}$. Here is a brief outline of the article. Among many results in this paper, Section 2 contains elementary observations needed later on. In Section 3 and Section 4, we characterize the lattices for which the meet atom element graphs are connected, complete bipartite, star. The concepts of planarity, clique number, domination number and split character are also investigated. *Most of the results in this article are observed in Artinian lattices.*

2. PRELIMINARIES

Let G be a simple graph with vertex set $\mathcal{V}\ (G)$ and edge set $\mathcal{E}\ (G)$. The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n stands for a complete graph with n vertices. If the vertices of G can be partitioned into two disjoint sets V_1 and V_2 with every vertex of V_1 adjacent to any vertex of V_2 and no two vertices belonging to the same set are adjacent, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. If graph G does not have K_5 or $K_{3,3}$ as its subgraph, then G is planar.

Let u and v be elements of $\mathcal{V}\ (G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is

$$\operatorname{diam}(G) = \sup\big\{d(a, b) : a, b \in \mathcal{V}(G)\big\}.$$

The girth of a graph G, denoted by $\operatorname{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\operatorname{gr}(G) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. A subset S of $\mathcal{V}\ (G)$ is said to be an independent set if no two vertices of S are adjacent. If $\mathcal{V}\ (G)$ can be partitioned in an independent set and a clique then G is said to be split. A set

$D \subseteq \mathcal{V}(G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one of the members of D. The cardinality of smallest dominating set is the domination number of the graph G and is denoted by $\gamma(G)$. Note that a graph whose vertices set is empty is a null graph and a graph whose edge set is empty is an empty graph. For a connected graph G, x is a cut vertex of G if $G \setminus \{x\}$ is not connected [19].

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$). We say that an element x in a lattice $\mathcal{L}$ is an atom (resp. coatom) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). The set of all coatom (resp. atom) elements of $\mathcal{L}$ is denoted by $\mathcal{CA}(\mathcal{L})$ (resp. $\mathcal{A}(\mathcal{L})$). If $\mathcal{L}$ is a complete lattice, then the join of all the atoms of $\mathcal{L}$, denoted by $\mathrm{Soc}(\mathcal{L})$, is called the socle of the lattice $\mathcal{L}$ (i.e., $\mathrm{Soc}(\mathcal{L}) = \vee_{a \in \mathcal{A}(\mathcal{L})} a$). An element x of a lattice $\mathcal{L}$ is nontrivial (resp. proper) if $x \neq 0, 1$ (resp. $x \neq 1$). An element x of a lattice $\mathcal{L}$ is called essential (written $x \trianglelefteq \mathcal{L}$), if there is no nonzero $y \in \mathcal{L}$ such that $x \wedge y = 0$. In a lattice $\mathcal{L}$ with 0, an element c is called a complement of b in $\mathcal{L}$ if it is maximal relative to the property $b \wedge c = 0$ [9].

A nonzero element x of a lattice $\mathcal{L}$ is called semisimple, if for each element y of $\mathcal{L}$ with $y < x$, there exists an element z of $\mathcal{L}$ such that $x = y \vee z$ and $y \wedge z = 0$. In this case, we say that y is a direct join of x, and we write $x = y \oplus z$. A lattice $\mathcal{L}$ is called semisimple if 1 is semisimple in $\mathcal{L}$ [15]. A lattice $\mathcal{L}$ is Artinian (satisfies DCC) if there is no infinite strictly descending chain $a_0 > a_1 > \cdots$ in $\mathcal{L}$.

Lemma 2.1 ([15, Lemma 2.3 and Theorem 3.9]). *If $\mathcal{L}$ is a complete distributive lattice, then the following hold:*

(1) *Every element of $\mathcal{L}$ has a complement in $\mathcal{L}$. Moreover, if s is a complement of $x \neq 0$, then $x \vee s \trianglelefteq \mathcal{L}$.*

(2) *If s is a nonzero element of $\mathcal{L}$, then s is semisimple if and only if $s = \vee_{i \in \Lambda} a_i$, where $\{a_i\}_{i \in \Lambda}$ is the set of all atoms a_i of L with $a_i \leq s$. In particular, if s is semisimple, then $s \leq \mathrm{Soc}(\mathcal{L})$.*

The undefined terms related to lattice theory are taken from [9, 10] and terms related to graph theory are taken from [19].

3. BASIC PROPERTIES OF $\mathbb{AG}(\mathcal{L})$

Throughout this paper, we assume, unless otherwise stated, that $\mathcal{L}$ is a *bounded distributive lattice.* In this section, we collect some basic properties concerning the meet atom element graph $\mathbb{AG}(\mathcal{L})$ of $\mathcal{L}$. We remind the reader the following definition.

Definition 3.1. The meet atom element graph $\mathbb{AG}(\mathcal{L})$ of $\mathcal{L}$ is simple undirected graph whose vertices are all nontrivial elements of $\mathcal{L}$ and any two distinct vertices a and b are adjacent if and only if $a \wedge b$ is an atom element of $\mathcal{L}$.

Lemma 3.2. *The following hold in* $\mathbb{AG}(\mathcal{L})$*:*

(1) *Every non-atom element of $\mathcal{L}$ is adjacent to at least one of the atom elements of $\mathcal{L}$.*

(2) *If $\mathcal{L}$ is complete and* $\mathrm{Soc}(\mathcal{L}) \neq 1$*, then every element of $\mathcal{A}(\mathcal{L})$ is adjacent to* $\mathrm{Soc}(\mathcal{L})$*.*

Proof. (1) Let b be a non-atom element of $\mathcal{L}$. Then there exists an atom element a of $\mathcal{L}$ such that $a \lneqq b$. Hence, $a \wedge b = a$ is an atom element, as needed.

(2) Let $a \in \mathcal{A}(\mathcal{L})$. Then $a \leq \mathrm{Soc}(\mathcal{L})$ gives $a \wedge \mathrm{Soc}(\mathcal{L}) = a \in \mathcal{A}(\mathcal{L})$ and so a adjacent to $\mathrm{Soc}(\mathcal{L})$. $\square$

Proposition 3.3. *The following hold in* $\mathbb{AG}(\mathcal{L})$*:*

(1) *The subgraph induced by the atom elements of $\mathcal{L}$ is empty.*

(2) *The subgraph induced by the non-atom elements of $\mathcal{L}$ is connected graph of diameter not bigger than* 4*.*

Proof. (1) Let $a, b \in \mathcal{A}(\mathcal{L})$ with $a \neq b$. Then $a \wedge b \leq a, b$ gives $a \wedge b = 0$ which implies that a is not adjacent to b, as required.

(2) Suppose that x and y are distinct non-atom vertices of the graph $\mathbb{AG}(\mathcal{L})$. If x adjacent to y, then $x \backsim y$ is a path. So we may assume that $x \wedge y$ is not atom. By the hypothesis, there exist $a, b \in \mathcal{A}(\mathcal{L})$ such that $a \lneqq x$ and $b \lneqq y$. If $a = b$, then $x \backsim a \backsim y$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. If $a \neq b$, then $x \backsim a \backsim a \vee b \backsim b \backsim y$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 4$, i.e., (2) holds. $\square$

In the following theorem, we give a condition under which $\mathbb{AG}(\mathcal{L})$ is an empty graph.

THEOREM 3.4. *The following hold in $\mathbb{AG}(\mathcal{L})$:*

(1) *Every nontrivial element of a lattice of $\mathcal{L}$ is atom if and only if $\mathbb{AG}(\mathcal{L})$ is an empty graph.*

(2) *If $a_1 \oplus a_2 = 1$ for some atom elements a_1 and a_2 of $\mathcal{L}$, then $\mathbb{AG}(\mathcal{L})$ is an empty graph.*

Proof. (1) If every nontrivial element of $\mathcal{L}$ is atom, then $\mathbb{AG}(\mathcal{L})$ is an empty graph by Proposition 3.3 (1). Conversely, assume that $\mathbb{AG}(\mathcal{L}$ is empty and let b be any vertex of the graph of $\mathbb{AG}(\mathcal{L}$ such that $b \notin \mathcal{A}(\mathcal{L})$. Then by Lemma 3.2 (1), there exists an atom element a of $\mathcal{L}$ such that a adjacent to b which is impossible, as needed.

(2) Let x be a nontrivial element of $\mathcal{L}$. Then $x = x \wedge 1 = (x \wedge a_1) \vee (x \wedge a_2)$. If $x \wedge a_1 = 0 = x \wedge a_2$, then $x = 0$, a contradiction. If $x \wedge a_1 \neq 0$ and $x \wedge a_2 \neq 0$, then a_1, a_2 are atoms gives $x = a_1 \vee a_2 = 1$ which is impossible. Without loss of generality, let $x \wedge a_1 \neq 0$ (so $a_1 \leq x$) and $x \wedge a_2 = 0$ which implies that $x = a_1$. Therefore, every nontrivial element of $\mathcal{L}$ is an atom element. Now, the assertion follows from (1). $\square$

PROPOSITION 3.5. *The following hold in $\mathbb{AG}(\mathcal{L})$:*

(1) *If $\mathbb{AG}(\mathcal{L})$ has a non-atom universal vertex c, then c is coatom.*

(2) *If $\mathbb{A}(\mathcal{L}) = \{a\}$, then a is a universal vertex.*

(3) *If $\mathbb{AG}(\mathcal{L})$ has an atom universal vertex a, then $\mathbb{A}(\mathcal{L}) = \{a\}$.*

Proof. (1) By [14, Lemma 2.1], there exists a coatom element c' of $\mathcal{L}$ such that $c \leq c'$. If $c \neq c'$, then c is a universal vertex gives $c = c \wedge c'$ is an atom element which is impossible. Thus, $c = c'$.

(2) If x is a nontrivial element of $\mathcal{L}$, then $a \leq x$ gives $x \backsim a$, as needed.

(3) Apply Proposition 3.3 (1). $\square$

PROPOSITION 3.6. *If $\mathbb{AG}(\mathcal{L})$ is a complete graph, then $|\mathcal{A}(\mathcal{L})| = 1$.*

Proof. If $a, a' \in \mathcal{A}(\mathcal{L})$ with $a \neq a'$, then a and a' are not adjacent in $\mathbb{AG}(\mathcal{L})$ by Proposition 3.3 (1) which is impossible, as $\mathbb{AG}(\mathcal{L})$ is complete. Hence, $|\mathcal{A}(\mathcal{L})| = 1$. $\square$

The following example shows that, in general, the converse of Proposition 3.6 is not true.

Example 3.7. Assume that $\mathcal{L} = \{0, a, b, c, d, e, 1\}$ is a lattice with the relations $0 \leq e \leq a \leq b \leq c \leq 1$, $0 \leq e \leq a \leq d \leq c \leq 1$, $d \vee b = c$ and $d \wedge b = a$. Clearly, $\mathcal{A}(\mathcal{L}) = \{e\}$. Then, by stating that $b \wedge d = a$ is not an atom element, gives $\mathbb{AG}(\mathcal{L})$ is not a complete graph.

In the following theorem, we give a condition under which the graph $\mathbb{AG}(\mathcal{L})$ is complete.

THEOREM 3.8. *Let a be an element $\mathcal{V}$ $(\mathbb{AG}(\mathcal{L}))$ with degree* 1. *If a is atom in $\mathcal{L}$ which is not a coatom element, then $\mathbb{AG}(\mathcal{L}) \cong K_2$ the complete graph with two vertices.*

Proof. By [14, Lemma 2.1], we have that $a \lneqq c$ for some coatom element c of $\mathcal{L}$; so $a \backsim c$. If c' is an element of $\mathcal{L}$ such that $a \lneqq c' \lneqq c$, then $a \backsim c'$ gives $\deg_{\mathbb{AG}(\mathcal{L})}(a) \geq 2$ which is a contradiction. Thus, $\mathcal{V}$ $(\mathbb{SG}(\mathcal{L})) = \{a, c\}$, as required. $\square$

PROPOSITION 3.9. *The following hold in $\mathbb{AG}(\mathcal{L})$:*

(1) *If $\mathcal{L}$ is complete, $a \in \mathcal{A}(\mathcal{L})$ and $\mathrm{Soc}(\mathcal{L}) \lneqq x$, then $a \backsim x$.*

(2) *If $x \notin \mathcal{A}(\mathcal{L})$ and $x \lneqq y$, then x is not adjacent to y.*

Proof. (1) Since $a \leq \mathrm{Soc}(\mathcal{L}) \lneqq x$, we conclude that $x \wedge a = a$ and so $a \backsim x$.

(2) If $x \lneqq y$, then $x \wedge y = x$. Since $x \notin \mathcal{A}(\mathcal{L})$, we infer that x is not adjacent to y. $\square$

The next theorem gives a more explicit description of the diameter of $\mathbb{AG}(\mathcal{L})$.

THEOREM 3.10. *The graph $\mathbb{AG}(\mathcal{L})$ is connected with* $\mathrm{diam}(\mathbb{AG}(\mathcal{L})) \leq 4$ *if and only if the join of any two distinct atom elements of $\mathcal{L}$ is not* 1 *or* $|\mathcal{A}(\mathcal{L})| = 1$.

Proof. Suppose that $\mathcal{A}(\mathcal{L}) = \{a\}$ and let x and y be distinct non-atom vertices of the graph $\mathbb{AG}(\mathcal{L})$. Then $x \backsim a \backsim y$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. So suppose that $|\mathcal{A}(\mathcal{L})| \neq 1$ and the join of any two distinct atom elements of $\mathcal{L}$ is not 1. Consider two distinct vertices b and c of $\mathbb{AG}(\mathcal{L})$. If b adjacent to c, then $b \backsim c$ is a path. So we may assume that $b \wedge c$ is not an atom element of $\mathcal{L}$. Then either $b \wedge c = 0$ or $a \lneqq b \wedge c$ for some $a \in \mathcal{A}(\mathcal{L})$. If $a \lneqq b \wedge c$ (so $a \lneqq b$ and $a \lneqq c$), then $b \backsim a \backsim c$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. If $b \wedge c = 0$, we split the proof into three cases:

Case 1: $b, c \in \mathcal{A}(\mathcal{L})$. Then $b \backsim b \vee c \backsim c$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 2$.

Case 2: If exactly one of b and c is atom, then without loss of generality, assume that $b \in \mathcal{A}(\mathcal{L})$ and $c \notin \mathcal{A}(\mathcal{L})$. By Lemma 3.2 (1), there is an atom a of $\mathcal{L}$ such that $a \lneqq c$ which implies that $b \backsim a \vee b \backsim a \backsim c$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 4$.

Case 3: $b, c \notin \mathcal{A}(\mathcal{L})$. Then there exist $a, a' \in \mathcal{A}(\mathcal{L})$ such that $a \lneqq b$ and $a' \lneqq c$. If $a = a'$, then $b \backsim a \backsim c$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 2$. If $a \neq a'$, then $b \backsim a \backsim a \vee a' \backsim a' \backsim c$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 4$. Hence, we infer that $\mathbb{AG}(\mathcal{L})$ is connected with $\mathrm{diam}(\mathbb{AG}(\mathcal{L})) \leq 4$.

Conversely, assume that $\mathbb{AG}(\mathcal{L})$ is connected. If $|\mathcal{A}(\mathcal{L})| = 1$, then we are done. So we may assume that $|\mathcal{A}(\mathcal{L})| \neq 1$. On the contrary, assume that there are two atom elements a_1 and a_2 of $\mathcal{L}$ such that $a_1 \vee a_2 = 1$. We claim that a_2 is a coatom element of $\mathcal{L}$. Assume to the contrary, $a_2 \lneqq x \lneqq 1$ for some $x \in \mathcal{L}$. Then $1 = a_1 \vee a_2 \leq x \vee a_1$ gives $x \vee a_1 = 1$. If $x \wedge a_1 \neq 0$, then $a_1 \leq x$ implies that $1 = x \vee a_1 = x$ which is impossible. Thus, $x \wedge a_1 = 0$. Now, we have $a_2 = a_2 \vee 0 = a_2 \vee (x \wedge a_1) = (a_2 \vee x) \wedge (a_2 \vee a_1) = x$, a contradiction. So a_2 is a coatom. similarly, a_1 is a coatom. By the hypothesis, $a_1 \backsim u$ for some vertex u of $\mathbb{AG}(\mathcal{L})$. Then $a_1 \wedge u = a_1$ since $a_1 \wedge u$ is an atom (so $a_1 \lneqq u$), a contradiction since a_1 is a coatom, as required. $\square$

THEOREM 3.11. *Assume that $\mathcal{L}$ is a complete lattice with* $\mathrm{Soc}(\mathcal{L}) \neq 1$ *and let $\mathbb{AG}(\mathcal{L})$ be a graph which contains a cycle. Then* $\mathrm{gr}(\mathbb{AG}(\mathcal{L})) = 3, 4$.

Proof. Assume that $\mathrm{Soc}(\mathcal{L}) \neq 1$ and let $b \backsim c$. By Proposition 3.3 (1), either $b \notin \mathcal{A}(\mathcal{L})$ or $c \notin \mathcal{A}(\mathcal{L})$. If $b, c \notin \mathcal{A}(\mathcal{L})$, then $b \backsim b \wedge c \backsim c \backsim b$ is a cycle (so $\mathrm{gr}(\mathbb{AG}(\mathcal{L})) = 3$). Suppose that one of b or c is an atom element. We can assume that $b \in \mathcal{A}(\mathcal{L})$ and $c \notin \mathcal{A}(\mathcal{L})$. By Lemma 3.2 (1), there is an atom element a of $\mathcal{A}(\mathcal{L})$ such that $a \lneqq c$. Hence, we obtain the cycle $b \backsim c \backsim a \backsim \mathrm{Soc}(\mathcal{L}) \backsim b$ which implies that $\mathrm{gr}(\mathbb{AG}(\mathcal{L})) = 4$, as needed. $\square$

Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2), \ldots, (\mathcal{L}_n, \leq_n)$ are lattices $(n \geq 2)$ and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \ldots, n\}$. The following notation below is used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

The proof of the following lemma can be found in [12, Proposition 2.3] for $n = 2$, but we give details for convenience.

LEMMA 3.12. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice. If b_i is an atom element of $\mathcal{L}_i$ for $i \in \{1, \ldots, n\}$, then $a_1 = (b_1, 0, \ldots, 0)$, $a_2 = (0, b_2, 0, \ldots, 0), \ldots,$ and $a_n = (0, 0, \ldots, b_n)$ are atom elements of $\mathcal{L}$.*

Proof. On the contrary, assume that

$$(0, 0, \ldots, 0) < (x_1, x_2, \ldots, x_n) = x < (b_1, 0, \ldots, 0) = a_1$$

for some element x of $\mathcal{L}$ (so $x_i = 0$ for $i \in \{2, 3, \ldots, n\}$). It follows that $0 < x_1 < b_1$ which is impossible, as b_1 is an atom element. Thus a_1 is an atom element of $\mathcal{L}$. Similarly, $a_2, \ldots, a_n$ are atom elements of $\mathcal{L}$. $\square$

THEOREM 3.13. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice such that $\mathcal{A}(\mathcal{L}_i) = \{b_i\}$ for $i \in \{1, 2, \ldots, n\}$. Then* $\operatorname{diam}(\mathbb{AG}(\mathcal{L})) \leq 2$.

Proof. By Lemma 3.12, $\mathcal{A}(\mathcal{L}) = \{a_1, \ldots, a_n\}$, where we have that $a_k = (0, 0, \ldots, 0, b_k, 0, \ldots, 0)$ for $k \in \{1, 2, \ldots, n\}$. Let x and y be two vertices of $\mathbb{AG}(\mathcal{L})$ such that they are not adjacent. If $a_i \leq x$ and $a_i \leq y$ for some a_i, then $\mathrm{d}(x, y) = 2$. Otherwise, there are $a_i, a_j \in \mathcal{A}(\mathcal{L})$ such that $a_i \leq x$, $a_i \nleq y$, $a_j \leq y$ and $a_j \nleq x$. We may assume that $i < j$. Consider the element

$$z = (0, 0, \ldots, 0, b_i, 0, \ldots, 0, b_j, 0, \ldots, 0).$$

Since $a_j \nleq x$ and $a_i \nleq y$, we conclude that $z \wedge x = a_i$ and $z \wedge y = a_j$. This shows that $x \backsim z \backsim y$ is a path in $\mathbb{AG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. Thus, $\operatorname{diam}(\mathbb{AG}(\mathcal{L})) \leq 2$. $\square$

THEOREM 3.14. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice such that $\mathcal{A}(\mathcal{L}_i) = \{b_i\}$ for $i \in \{1, 2, \ldots, n\}$. Then* $\operatorname{gr}(\mathbb{AG}(\mathcal{L})) = 3$.

Proof. By Lemma 3.12, $\mathcal{A}(\mathcal{L}) = \{a_1, \ldots, a_n\}$, where we have that $a_k = (0, 0, \ldots, 0, b_k, 0, \ldots, 0)$ for $k \in \{1, 2, \ldots, n\}$. Now, we consider the elements $x = (b_1, b_2, 0, \ldots, 0)$, $y = (b_1, 0, b_3, 0, \ldots, 0)$ and $z = (0, b_2, b_3, 0, \ldots, 0)$. Since $x \wedge y = a_1$, $x \wedge z = a_2$ and $y \wedge z = a_3$, we get the cycle $x \backsim y \backsim z \backsim x$. This shows that $\operatorname{gr}(\mathbb{AG}(\mathcal{L})) = 3$. $\square$

PROPOSITION 3.15. *If $\mathcal{V}$ $(\mathbb{AG}(\mathcal{L}))$ is a totally ordered set, then $\mathbb{AG}(\mathcal{L})$ is a star graph.*

Proof. By assumption, there is an element a of $\mathcal{A}(\mathcal{L})$ such that $a \backsim x$ for every $x \in \mathcal{V}$ $(\mathbb{AG}(\mathcal{L}))$. If $x \neq a$ and $y \neq a$ are two distinct vertices of $\mathbb{AG}(\mathcal{L})$, then either $x \leq y$ or $y \leq x$. For both cases, x and y are not adjacent vertices. Thus $\mathbb{AG}(\mathcal{L})$ is a star graph with center a. $\square$

LEMMA 3.16. *The following hold in a complete lattice $\mathcal{L}$:*

(1) $\mathrm{Soc}(\mathcal{L}) = \bigwedge_{e \trianglelefteq \mathcal{L}} e$.

(2) *If* $x \in \mathcal{L}$*, then* $x \trianglelefteq \mathcal{L}$ *if and only if* $\mathrm{Soc}(\mathcal{L}) \leq x$.

(3) *If* x *is a nontrivial element of* $\mathcal{L}$ *and* $x \lneqq \mathrm{Soc}(\mathcal{L})$*, then* x *is not essential in* $\mathcal{L}$.

Proof. (1) Suppose that $\mathrm{Soc}(\mathcal{L}) = \vee_{a \in \mathcal{A}(\mathcal{L})} a$ and let $c = \wedge_{e \trianglelefteq \mathcal{L}} e$. Let a be an atom element of $\mathcal{L}$. If $e \trianglelefteq \mathcal{L}$, then $a \wedge e \neq 0$ gives $a \leq e$ which implies that $\mathrm{Soc}(\mathcal{L}) \leq c$. It suffices to show that $c \leq \mathrm{Soc}(\mathcal{L})$. At first, we claim that c is semisimple. Let b be an element of $\mathcal{L}$ such that $b \lneqq c$. If $b \trianglelefteq \mathcal{L}$, then $c \leq b$, a contradiction. So we may assume that b is not essential in $\mathcal{L}$. Let b' be a complement of b in $\mathcal{L}$; so $b \vee b' \trianglelefteq \mathcal{L}$ by Lemma 2.1 (1). It follows that $b \leq c \leq b \vee b'$ and so $c = c \wedge (b \vee b') = (c \wedge b) \vee (c \wedge b') = b \vee (c \wedge b')$ with $b \wedge (c \wedge b') = 0$ which implies that c is semisimple; thus $c \leq \mathrm{Soc}(L)$ by Lemma 2.1 (2) and so we have equality.

(2) One side is clear by (1). To prove the other side, assume to the contrary, that x is not essential in $\mathcal{L}$. Then there exists a nontrivial element b of $\mathcal{L}$ such that $x \wedge b = 0$. Therefore, there is an atom element a of $\mathcal{L}$ such that $a \leq b$ and $a \leq \mathrm{Soc}(\mathcal{L}) \leq x$. So we have $a = a \wedge x \leq x \wedge b = 0$ which is a contradiction. Thus $x \trianglelefteq \mathcal{L}$.

(3) It is a direct consequence of (2). □

The following theorem shows when the meet atom element graph is a complete bipartite graph.

THEOREM 3.17. *If* $\mathcal{L}$ *is a complete lattice with* $\mathrm{Soc}(\mathcal{L}) \neq 1$*, then the following assertions are equivalent:*

(1) *Every vertex of* $\mathbb{AG}(\mathcal{L})$ *is either atom or essential.*

(2) $\mathbb{AG}(\mathcal{L})$ *is a complete bipartite graph.*

Proof. (1) $\Rightarrow$ (2) Let $V_1 = \mathcal{A}(\mathcal{L})$ and V_2 be the set of all essential elements of $\mathcal{L}$. If $a, a' \in V_1$, then a and a' are not adjacent by Proposition 3.3 (1). If $x, y \in V_2$, then $x \wedge y$ is essential in $\mathcal{L}$ by [14, Lemma 2.3 (2)]. So $\mathrm{Soc}(\mathcal{L}) \lneqq x \wedge y$ by Lemma 3.16 (2) which implies that any two vertices of V_2 are not adjacent. Moreover, every vertex in V_1 is adjacent to each vertex V_2 by Proposition 3.9. Therefore, $\mathbb{AG}(\mathcal{L})$ is a complete bipartite graph.

(2) $\Rightarrow$ (1) Suppose that V_1 and V_2 are parts of $\mathbb{AG}(\mathcal{L})$ and let a be an atom element of $\mathcal{L}$. Without loss of generality, let $a \in V_1$. If $a \neq a' \in \mathcal{A}(\mathcal{L})$ with $a' \notin V_1$, then $a' \in V_2$ gives $a \wedge a'$ is atom which is impossible. Thus

$\mathcal{A}(\mathcal{L}) \subseteq V_1$. If $b \in V_1$ with $b \notin \mathcal{A}(\mathcal{L})$, then there exists an atom element a of $\mathcal{L}$ (so $a \in V_1$) such that $b \backsim a$ by Lemma 3.2 (1), a contradiction. Therefore, $V_1 = \mathcal{A}(\mathcal{L})$. Suppose that $\mathrm{Soc}(\mathcal{L}) \leq c$ for some vertex c and let $d \neq 0$ be any element of $\mathcal{L}$. If d is atom, then $d \wedge c = d \neq 0$. If d is not atom, then there exists an atom element T of $\mathcal{L}$ such that $T \leq d \wedge c$ which implies that c is an essential element of $\mathcal{L}$; so $\mathcal{K} = \{e \in \mathcal{V}\ (\mathbb{AG}(\mathcal{L}) : \mathrm{Soc}(\mathcal{L}) \leq e\}$ is the set of all essential elements of $\mathcal{L}$. An easy inspection shows that $V_2 = \mathcal{K}$, as required. $\square$

THEOREM 3.18. *Suppose that the join of any two distinct atom elements of $\mathcal{L}$ is not 1 and let c be a cut vertex of $\mathbb{AG}(\mathcal{L})$. Then there exist $a, a' \in \mathbb{A}(\mathcal{L})$ such that $c = a \vee a'$.*

Proof. If $c \in \mathbb{A}(\mathcal{L})$, then $c = c \vee c$. So we may assume that $c \notin \mathbb{A}(\mathcal{L})$. Let f and g be two vertices of $\mathbb{AG}(\mathcal{L})$ such that $f \in V_1$ and $g \in V_2$, where V_1 and V_2 are the distinct components $\mathbb{AG}(\mathcal{L}) \setminus \{c\}$. We split the proof into four cases.

Case 1: $f, g \in \mathbb{A}(\mathcal{L})$. Since $f \backsim f \vee g \backsim g$ is a path in $\mathbb{AG}(\mathcal{L})$ and c is a cut vertex, we conclude that $c = f \vee g$.

Case 2: $f \in \mathbb{A}(\mathcal{L})$ and $g \notin \mathbb{A}(\mathcal{L})$. By the hypothesis, $a \leq g$ for some atom element a of $\mathcal{L}$ by Lemma 3.2 (1); so $a \in V_2$. Since $f \backsim f \vee a \backsim a$ is a path in $\mathbb{AG}(\mathcal{L})$, $f \in V_1$ and c is a cut vertex, we conclude that $c = f \vee a$.

Case 3: $f \notin \mathbb{A}(\mathcal{L})$ and $g \in \mathbb{A}(\mathcal{L})$. By an argument like that in Case 2, $c = g \vee a$ for some atom element a of $\mathcal{L}$ with $a \leq f$.

Case 4: $f, g \notin \mathbb{A}(\mathcal{L})$. By assumption, $a \lneqq f$ and $a' \lneqq g$ for some atom elements a and a' of $\mathcal{L}$ by Lemma 3.2 (1); so $a \in V_1$ and $a' \in V_2$. Since $a \backsim a \vee a' \backsim a'$ is a path in $\mathbb{AG}(\mathcal{L})$ and c is a cut vertex, we infer that $c = a \vee a'$. $\square$

4. CLIQUE NUMBER, DOMINATION NUMBER AND PLANARITY OF $\mathbb{AG}(\mathcal{L})$

We continue this section with the investigation of the stability of meet atom element graphs in various lattice-theoretic constructions. Let us begin this section with the following proposition.

PROPOSITION 4.1. *Let $x, y \notin \mathcal{A}(\mathcal{L})$ such that they are adjacent. Then there is a unique atom element a of $\mathcal{L}$ such that $a \in N(x) \cap N(y)$.*

Proof. Since $x \wedge y \in \mathcal{A}(\mathcal{L})$, we conclude that $x \wedge y$ is adjacent to both x and y which implies that $x \wedge y \in N(x)$ and $x \wedge y \in N(y)$. Let $a \in \mathcal{A}(\mathcal{L})$ such that $a \in N(x) \cap N(y)$. It suffices to show that $a = x \wedge y$. On the contrary, assume that $a \neq x \wedge y$. At first, we show that $a \in N(x) \cap N(y)$ if and only if

$a \in N(x \wedge y)$. If $a \backsim x \wedge y$, then $a \wedge (x \wedge y)$ is an atom element of $\mathcal{L}$ which gives $a \wedge x \neq 0$ and $a \wedge y \neq 0$. Since $a \wedge x, a \wedge y \leq a$, we conclude that $a \wedge x = a = a \wedge y$ and so $a \backsim x$ and $a \backsim y$. Conversely, assume that $a \backsim x$ and $a \backsim y$. Then the fact that $a \wedge x$ and $a \wedge y$ are atom elements gives $a \wedge x = a = a \wedge y$ which implies that $(x \wedge y) \wedge a = a$; so $a \backsim x \wedge y$. Since $a \wedge (x \wedge y)$ is an atom element, we infer that $a \wedge (x \wedge y) \neq 0$ which implies that $a \leq x \wedge y$. Therefore, $a = x \wedge y$, as $x \wedge y$ is an atom element, a contradiction. Thus $a = x \wedge y$. $\square$

Proposition 4.2. *Let $\mathcal{C}$ be a clique in $\mathbb{AG}(\mathcal{L})$. Then $\mathcal{C}$ is contained in the subgraph induced by $\{x \in \mathcal{V}\ (\mathbb{AG}(\mathcal{L})) : a \leq x\}$ for some atom element a of $\mathcal{L}$.*

Proof. By Proposition 3.3 (1), clique $\mathcal{C}$ has at most one atom element. If $C \cap \mathcal{A}(\mathcal{L}) = \{a'\}$, then $a' \leq x$ for every $x \in \mathcal{C}$ and so $\mathcal{C}$ is contained in the subgraph induced by $\{x \in \mathcal{V}\ (\mathbb{AG}(\mathcal{L})) : a' \leq x\}$. So we may assume that $C \cap \mathcal{A}(\mathcal{L}) = \emptyset$. The adjacency of every two vertices of $\mathcal{C}$ and Proposition 4.1 shows that there exists a unique atom element a of $\mathcal{L}$ for which $\mathcal{C}$ is a subgraph of the graph induced by $\{x \in \mathcal{V}\ (\mathbb{AG}(\mathcal{L})) : a \leq x\}$. $\square$

In the following results, we show that domination numbers are really of interest in indecomposable lattices.

Theorem 4.3. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice such that $\mathbb{A}(\mathcal{L}_i) = \{a_i\}$ for $i \in \{1, 2\}$. Then $\gamma(\mathbb{AG}(\mathcal{L})) = 2$.*

Proof. By Lemma 3.12, $\mathcal{A}(\mathcal{L}) = \{a'_1, a'_2\}$, where $a'_1 = (a_1, 0)$ and $a'_2 = (0, a_2)$. Let $c = (c_1, c_2)$ be a nontrivial element of $\mathcal{L}$. Since $c \wedge_c a'_1 = (c_1 \wedge a_1, 0)$ and $c \wedge_c a'_2 = (0, c_2 \wedge a_2)$, we conclude that any vertex of the graph $\mathbb{AG}(\mathcal{L})$ is adjacent to at least one of the elements of the set $\{a'_1, a'_2\}$. This shows that $\gamma(\mathbb{AG}(\mathcal{L})) = 2$. $\square$

Theorem 4.4. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ $(n \geq 3)$ be a decomposable lattice such that $\mathbb{A}(\mathcal{L}_i) = \{a_i\}$ for $i \in \{1, 2, \ldots, n\}$. Then $\gamma(\mathbb{AG}(\mathcal{L})) \leq n$.*

Proof. By Lemma 3.12, $\mathcal{A}(\mathcal{L}) = \{a'_1, \ldots, a'_n\}$, where we have that $a'_k = (0, 0, \ldots, 0, a_k, 0, \ldots, 0)$ for $k \in \{1, 2, \ldots, n\}$. Then the set $\mathcal{A}(\mathcal{L})$ dominates all the vertices of the graph $\mathbb{AG}(\mathcal{L})$; hence $\gamma(\mathbb{AG}(\mathcal{L})) \leq n$. $\square$

The following example shows that, in general, Theorem 4.4 is not true in the case $\gamma(\mathbb{AG}(\mathcal{L})) = n$.

Example 4.5. We suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \{0,1\}$ and we let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$ be a decomposable lattice. Then, we have

$$\mathcal{V}\big(\mathbb{AG}(\mathcal{L})\big) = \big\{(1,0,0),(1,1,0),(0,1,0),(0,1,1),(1,0,1),(0,0,1)\big\}.$$

If $D = \{(1,1,0),(0,1,1)\}$, then the set D dominates all the vertices of the graph $\mathbb{AG}(\mathcal{L})$; hence $\gamma(\mathbb{AG}(\mathcal{L})) = 2 \neq 3$.

In the following theorem, we give a condition under which we characterize the lattices for which the meet atom element graphs are split and planar. The concept of clique number is also investigated.

THEOREM 4.6. *Suppose that $\mathbb{AG}(\mathcal{L})$ is not empty and let*

$$\mathcal{V}(\mathbb{SG}(\mathcal{L})) = \mathbb{A}(\mathcal{L}) \cup \mathcal{CA}(\mathcal{L}).$$

Then the following hold:

(1) *The subgraph induced by the coatom elements of $\mathcal{L}$ is a complete graph.*

(2) $\mathbb{AG}(\mathcal{L})$ *is a split graph.*

(3) *If $|\mathcal{CA}(\mathcal{L})| \leq 3$, then $\mathbb{AG}(\mathcal{L})$ is a planar graph.*

(4) $|\mathcal{CA}(\mathcal{L})| \leq \omega(\mathbb{AG}(\mathcal{L}))$.

Proof. (1) It suffices to show if $c, d \in \mathcal{CA}(\mathcal{L})$ with $c \neq d$, then $c \backsim d$. At first, we claim that $c \wedge d \neq 0$. On the contrary, let $c \wedge d = 0$. Since $c \lneqq c \vee d \leq 1$, we infer that $c \vee d = 1$. Let $0 \neq e \leq c$ for some element e of $\mathcal{L}$. Then $e \wedge d = 0$ and $e \vee d = 1$ which implies that $c = c \wedge 1 = c \wedge (e \vee d) = (c \wedge e) \vee (c \wedge d) = c \wedge e = e$. This shows that c is atom. Similarly, d is atom. It follows that $\mathcal{V}$ $(\mathbb{AG}(\mathcal{L})) = \mathbb{A}(\mathcal{L})$ and so $\mathbb{AG}(\mathcal{L})$ is empty by Proposition 3.3 (1) which is impossible. Therefore, $c \wedge d \neq 0$. It is clear that $c \wedge d \notin \mathcal{CA}(\mathcal{L})$; so $c \wedge d \in \mathbb{A}(\mathcal{L})$, i.e., any two coatom elements are adjacent.

(2) Consider the subgraph induced by $\mathcal{CA}(\mathcal{L})$ of $\mathbb{AG}(\mathcal{L})$. Let $c, d \in \mathcal{CA}(\mathcal{L})$ with $c \neq d$. Then $c \wedge d \in \mathbb{A}(\mathcal{L})$ by (1) which implies that the subgraph induced by $\mathcal{CA}(\mathcal{L})$ is complete. Moreover, by Proposition 3.3 (1), the subgraph induced by $\mathbb{A}(\mathcal{L})$ is empty. Thus, $\mathbb{AG}(\mathcal{L})$ is a split graph.

(3) Recall that $\mathbb{AG}(\mathcal{L})$ is a split graph by (2). Since $|\mathcal{CA}(\mathcal{L})| \leq 3$, we conclude that any subgraph induced by five vertices is not complete; so K_5 is not a subgraph of $\mathbb{AG}(\mathcal{L})$. Now, we show that $K_{3,3}$ is not a subgraph of $\mathbb{AG}(\mathcal{L})$. On the contrary, assume that $K_{3,3}$ is a subgraph of $\mathbb{AG}(\mathcal{L})$ with partite sets $|V_1| = 3$ and $|V_2| = 3$. Clearly, either $V_1 \subseteq \mathbb{A}(\mathcal{L})$ or $V_2 \subseteq \mathbb{A}(\mathcal{L})$. If $V_1 \subseteq \mathbb{A}(\mathcal{L})$, then $V_2 \subseteq \mathcal{CA}(\mathcal{L})$ which is impossible by (1). Therefore, $\mathbb{AG}(\mathcal{L})$ is a planar graph.

(4) Since any two coatom distinct elements of $\mathcal{L}$ are adjacent by (1), we infer that the subgraph of $\mathbb{AG}(\mathcal{L})$ with the vertex set of $\mathcal{CA}(\mathcal{L})$ is a complete subgraph of $\mathbb{AG}(\mathcal{L})$. Therefore, $|\mathcal{CA}(\mathcal{L})| \leq \omega(\mathbb{AG}(\mathcal{L}))$. $\square$

The following example shows that the equality does not hold necessarily in Theorem 4.6 (4).

Example 4.7. Let $\mathcal{L} = \{0, a, b, c, 1\}$ be a lattice with the following relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. Clearly, the nontrivial elements of $\mathcal{L}$ are a, b and c. An inspection shows that $\mathcal{CA}(\mathcal{L}) = \{c\}$, $\mathcal{A}(\mathcal{L}) = \{a, b\}$ and $\mathrm{C} = \{a, c\}$ is a clique. Hence $|\mathcal{CA}(\mathcal{L})| = 1 < \omega(\mathbb{AG}(\mathcal{L})) = 2$.

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Received 28 January 2025

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