EXTREMAL MULTIPLICATIVE SOMBOR INDEX OF GRAPHS WITH A GIVEN CHROMATIC NUMBER

DA-YEON HUH

Communicated by Ioan Tomescu

The multiplicative Sombor index of a graph $\mathbb{G} = (\mathbf{V}(\mathbb{G}), \mathbf{E}(\mathbb{G}))$ introduced in 2021 is defined as $\Pi so(\mathbb{G}) = \prod_{uv\in \mathbf{E}(\mathbb{G})} \sqrt{d_u^2 + d_v^2}$, where d_u indicates the degree of $u \in \mathbf{V}(\mathbb{G})$. In this paper, some bounds of the multiplicative Sombor index are presented using graph parameters. In particular, both the maximal and minimal graphs are determined when a chromatic number is given among graphs of fixed order. Moreover, the bounds of the index for the cartesian, tensor, and strong product of two graphs are represented with the index of each graph.

AMS 2020 Subject Classification: 05C09, 05C15, 05C76.

Key words: multiplicative Sombor index, chromatic number, Turán graph, product of graphs.

1. INTRODUCTION

A graph $\mathbb{G}$ considered throughout this paper is a simple graph with vertex set $\mathbf{V}(\mathbb{G})$ and edge set $\mathbf{E}(\mathbb{G})$. We write $\mathbf{N}_{\mathbb{G}}(v)$ for the set of neighbors of $v \in \mathbf{V}(\mathbb{G})$. $|\mathbf{N}_{\mathbb{G}}(v)|$ is denoted by $d_{v,\mathbb{G}}$ or simply d_v. By Δ and δ, we represent the maximum and minimum degree of vertices in $\mathbb{G}$, respectively. A r-coloring of $\mathbb{G}$ is a function $\kappa : \mathbf{V}(\mathbb{G}) \to \{1, 2, \ldots, r\}$ that satisfies $\kappa(u) \neq \kappa(v)$ whenever $uv \in \mathbf{E}(\mathbb{G})$. If $\mathbb{G}$ has an r-coloring, we call $\mathbb{G}$ r-colorable. The smallest r for which $\mathbb{G}$ is r-colorable is its chromatic number $\chi(\mathbb{G})$. $\mathcal{G}(n, r)$ is the set of connected graphs with n vertices and chromatic number r. We define the kite graph $Ki_{n,r}$ as a graph consisting of n vertices obtained by connecting a vertex of the complete graph K_r and an end vertex of the path graph P_{n-r} with an edge. Similarly, $C_{n,r}$ is a graph of order n obtained by connecting a vertex of cycle C_r and an end vertex of P_{n-r} with an edge.

A map from a set of graphs to real numbers is called a graph invariant if it remains unchanged under graph isomorphism [6]. Topological indices as graph invariants have been developed to describe molecules in modeling chemical compounds and to predict their physical or biological properties [12]. After the Wiener index [15], hundreds of topological indices in various forms were

MATH. REPORTS **27(77)** (2025), *1-2*, 103–120
doi: 10.59277/mrar.2025.27.77.1.2.103

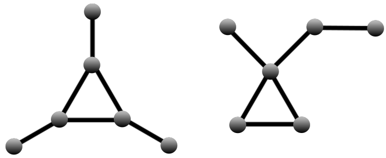

Figure 1 – $SO(\mathbb{G}) < SO(\mathbb{H})$ but $\Pi so(\mathbb{G}) > \Pi so(\mathbb{H})$.

proposed, combined, and transformed in the literature of mathematics and chemistry. Notably, dozens of vertex-degree-based indices were studied. One is the Sombor index $SO(\mathbb{G})$ introduced in 2021 by Gutman [7], defined as

$$SO(\mathbb{G}) = \sum_{uv \in \mathbf{E}(\mathbb{G})} \sqrt{d_u^2 + d_v^2}.$$

The Sombor index has inspired many researchers and has been extensively studied in a short period of time. Soon after, its variation, the multiplicative Sombor index, was put forward by Kulli [9] with the following definition,

$$\Pi so(\mathbb{G}) = \prod_{uv \in \mathbf{E}(\mathbb{G})} \sqrt{d_u^2 + d_v^2}.$$

We append the following definition to apply it to the product of graphs:

$$\Pi so(\mathbb{G}) = 1 \quad \text{if } \mathbf{E}(\mathbb{G}) = \phi.$$

Figure 1 shows one of the numerous examples where $SO(\mathbb{G}) < SO(\mathbb{H})$ but $\Pi so(\mathbb{G}) > \Pi so(\mathbb{H})$.

Nevertheless, research showed that the extremal graphs for both indices are the same in various classes of graphs (Table 1). It was proven in [7] and [10] that P_n and K_n are the minimal and maximal graphs of order n for both $SO(\mathbb{G})$ and $\Pi so(\mathbb{G})$, respectively. S_n was proven to be the maximal tree for the two indices in the same papers. Besides that, proofs for extremal graphs were made in classes such as bipartite graph, unicyclic graph etc., to give the same results in both [1–5, 16]. For given order n and chromatic number r, the maximal and minimal graphs were proven only for the Sombor index [5, 17].

In this paper, we evaluate some bounds of the multiplicative Sombor index for given parameters and identify the corresponding extremal graphs. Particularly, we prove that the Turán graph and kite graph are the maximal and minimal graph, respectively, for the multiplicative Sombor index among the graphs with n vertices and the chromatic number r, and conclude that the maximal and minimal graphs are the same for both the Sombor and the multiplicative Sombor index in $\mathcal{G}(n, r)$. We also represent the bounds of the

Table 1 – The minimal or maximal graphs for the Somber and the multiplicative Somber index are identical in certain classes

Classes	$\mathcal{G}(n)$		$\mathcal{T}(n)$		$\mathcal{U}(n)$		$\mathcal{B}(n)$	$\mathcal{P}(n,p)$	$\mathcal{W}(n,w)$	$\mathcal{F}(n,g)$
	min	max	min	max	min	max	max	max	min	min
Graph for $So(G)$ and $\Pi So(G)$	P_n [7,10]	K_n [7,10]	P_n [7,10]	S_n [7,10]	C_n [2,10]	C_n^3 [2,10]	$K_{\lceil \frac{n}{2}\rceil,\lfloor \frac{n}{2}\rfloor}$ [3,16]	K_n^{n-p**} [5,16]	$Ki_{n,n-w}$ [5,16]	$C_{n,g}$ [10,17]

n-indicates the order of graphs in each class.
$\mathcal{G}(n)$-the class of connected graph; $\mathcal{T}(n)$-the class of trees; $\mathcal{U}(n)$-the class of unicyclic graphs; $\mathcal{B}(n)$-the class of bipartite graphs; $\mathcal{P}(n,p)$-the class of graphs with p pendent vertices; $\mathcal{W}(n,w)$-the class of graphs with clique number w; $\mathcal{F}(n,g)$-the class of graphs with girth g.
C_n^3-a graph obtained by hanging $n-3$ pendent vertices with edges on the same vertex of C_3; K_n^{n-p}-a graph obtained by hanging p pendent vertices with edges on the same vertex of K_{n-p}.

index for three kinds of graph products using the indices of each graph in Section 3.

2. BOUNDS AND EXTREMAL GRAPHS

In this section, we focus exclusively on connected graphs.

Lemma 2.1 ([10]). *Πso is increasing, i.e., $\Pi so(\mathbb{G}+uv) > \Pi so(G)$ whenever $uv \notin \boldsymbol{E}(\mathbb{G})$.*

Proposition 2.2. *Let $\mathbb{G}$ be a graph having $n \geq 2$ vertices, Δ, and δ. Then*

$$\left(\sqrt{2}\ \delta\right)^{\frac{n\delta}{2}} \leq \Pi so(\mathbb{G}) \leq \left(\sqrt{2}\ \Delta\right)^{\frac{n\Delta}{2}}.$$

Equality holds if and only if $\mathbb{G}$ is a regular graph.

Proof. This is obvious from the inequalities $\sqrt{2}\delta \leq \sqrt{d_u^2+d_v^2} \leq \sqrt{2}\Delta$ and $\frac{n\delta}{2} \leq |\mathbf{E}(\mathbb{G})| \leq \frac{n\Delta}{2}$. Four equalities hold simultaneously if and only if $d_u = d_v = \delta = \Delta$ for every $uv \in \mathbf{E}(\mathbb{G})$, which precisely matches the condition for regular graphs. □

Theorem 2.3. *Let $\mathbb{G}$ be a graph with $n \geq 2$ vertices. Then*

$$\Pi so(\mathbb{G}) \geq \left(\sqrt{2}^{\sum\limits_{u\in \boldsymbol{V}(\mathbb{G})} d_u} \cdot e^{\sum\limits_{u\in \boldsymbol{V}(\mathbb{G})} d_u \log d_u}\right)^{\frac{1}{2}}$$

or

$$\log \Pi so(\mathbb{G}) \geq \frac{1}{2}\left(m \log 2 + \sum_{u\in \boldsymbol{V}(\mathbb{G})} d_u \log d_u\right),$$

where $m = |\boldsymbol{E}(\mathbb{G})|$. Equality holds if and only if $\mathbb{G}$ is a regular graph.

Proof. From the AM-GM inequality, we have

$$\prod_{uv\in\mathbf{E}(\mathbb{G})}\sqrt{d_u^2+d_v^2}\geq\prod_{uv\in\mathbf{E}(\mathbb{G})}\sqrt{2d_ud_v}=\prod_{uv\in\mathbf{E}(\mathbb{G})}(\sqrt{2}d_u)^{\frac{1}{2}}(\sqrt{2}d_v)^{\frac{1}{2}}.$$

Since each vertex contributes $(\sqrt{2}d_u)^{\frac{1}{2}}$ to each of its incident edges,

$$\begin{aligned}\prod_{uv\in\mathbf{E}(\mathbb{G})}(\sqrt{2}d_u)^{\frac{1}{2}}(\sqrt{2}d_v)^{\frac{1}{2}}&=\prod_{u\in\mathbf{V}(\mathbb{G})}(\sqrt{2}d_u)^{\frac{1}{2}d_u}=\Big(\prod_{u\in\mathbf{V}(\mathbb{G})}(\sqrt{2}d_u)^{d_u}\Big)^{\frac{1}{2}}\\&=\Big(\sqrt{2}^{\sum\limits_{u\in\mathbf{V}(\mathbb{G})}d_u}\prod_{u\in\mathbf{V}(\mathbb{G})}e^{\,d_u\log d_u}\Big)^{\frac{1}{2}}\\&=\Big(\sqrt{2}^{\sum\limits_{u\in\mathbf{V}(\mathbb{G})}d_u}\cdot e^{\sum\limits_{u\in\mathbf{V}(\mathbb{G})}d_u\log d_u}\Big)^{\frac{1}{2}}.\end{aligned}$$

Equality holds if and only if $d_u=d_v$ for all $uv\in\mathbf{E}(\mathbb{G})$. □

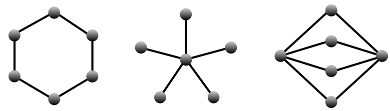

Figure 2 – Graphs having the same $d_u^2+d_v^2$ for all $uv\in\mathbf{E}(\mathbb{G})$ in Theorem 2.4.

THEOREM 2.4. *Let $\mathbb{G}$ be a graph with $m>0$ edges. Then*

$$\Pi so(\mathbb{G})\leq\Big(\frac{SO(\mathbb{G})}{m}\Big)^m$$

or

$$\log\Pi so(\mathbb{G})\leq m(\log SO(\mathbb{G})-\log m).$$

Equality holds if and only if $d_u^2+d_v^2$ is the same for all $uv\in\boldsymbol{E}(\mathbb{G})$.

Proof. Since $\log(x)$ is a concave function, it follows, by Jensen's inequality, that

$$\frac{\log\Pi so(\mathbb{G})}{m}=\frac{\sum_{uv\in\mathbf{E}(\mathbb{G})}\log\sqrt{d_u^2+d_v^2}}{m}\leq\log\Big(\frac{SO(\mathbb{G})}{m}\Big).$$

Since $\log(x)$ is not affine, equality in Jensen's inequality is true if and only if $\sqrt{d_u^2+d_v^2}$ is the same for all edges $uv\in\mathbf{E}(\mathbb{G})$ as shown in Figure 2. □

For the next theorem, assume that $\mathbb{G}\in\mathcal{G}(n,r)$. Let $\{\mathbf{V}_1,\mathbf{V}_2,\cdots,\mathbf{V}_r\}$ denote a partition of $\mathbf{V}(\mathbb{G})$ such that $\mathbf{V}_i$ is a set of independent vertices of the same color. The Turán graph $T_{n,r}$ [11] is known as the complete r-partite

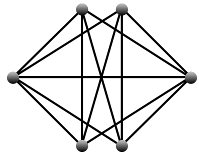

Figure 3 – Turán graph $T_{6,4}$ in $\mathcal{G}(6,4)$.

graph with n vertices satisfying $||\mathbf{V}_i| - |\mathbf{V}_j|| \leq 1$ for all $1 \leq i, j \leq r$ (Figure 3).

It was verified by Das et al. [5] that $T_{n,r}$ is the maximal graph in $\mathcal{G}(n,r)$ for the Sombor index. We prove here that the same holds for the multiplicative Sombor index.

THEOREM 2.5. *Let* $\mathbb{G} \in \mathcal{G}(n,r)$. *Then*

$$\Pi so(\mathbb{G}) \leq \Pi so(T_{n,r}).$$

Equality holds if and only if $\mathbb{G} = T_{n,r}$.

Proof. To begin with, the condition of $r = 2$ makes $\mathbb{G}$ a bipartite graph. Chunlei et al. [16] proved

$$\Pi so\big(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\big) \geq \Pi so(\mathbb{G})$$

for all bipartite graphs of order n. So, we only consider cases when $r \geq 3$. Also, it is trivial by Lemma 2.1 that $\mathbb{G} \in \mathcal{G}(4,3)$ implies $\Pi so(\mathbb{G}) \leq T_{4,3}$, and $\mathbb{G} \in \mathcal{G}(4,4)$ implies $\Pi so(\mathbb{G}) \leq T_{4,4} = K_4$. Therefore, we only investigate the case $n \geq 5$. To construct a proof by contradiction, suppose there is a graph $\mathbb{G} \in \mathcal{G}(n,r)$ that has the largest value of $\Pi so(\mathbb{G})$, but $\mathbb{G} \neq T_{n,r}$. It is obvious that $\mathbb{G}$ is complete r-partite because adding an edge increases $\Pi so(\mathbb{G})$.

Let $\{\mathbf{V}_i : 1 \leq i \leq r\}$ be a partition of $\mathbf{V}(\mathbb{G})$ according to r-coloring of $\mathbb{G}$. By assumption, there exist two partite sets, say $\mathbf{V}_{r-1}$ and $\mathbf{V}_r$, such that $|\mathbf{V}_{r-1}| - |\mathbf{V}_r| \geq 2$. And we let $|\mathbf{V}_i| = a_i$, $1 \leq i \leq r$. Note that $a_{r-1} - a_r \geq 2$. Now, we build another graph $\mathbb{H}$ from $\mathbb{G}$ by moving one vertex of $\mathbf{V}_{r-1}$ to $\mathbf{V}_r$, deleting and adding edges to form a new complete r-partite graph $\mathbb{H}$. We show that $\Pi so(\mathbb{G}) < \Pi so(\mathbb{H})$ to derive a contradiction.

In a complete r-partite graph $\mathbb{G}$, all the vertices in $\mathbf{V}_i$ have the same degree $n - a_i$. We choose a vertex in each $\mathbf{V}_i$ and consider it as a representative vertex v_i of $\mathbf{V}_i$, and define

$$\phi_{\mathbb{G}}(v_i) = \prod_{j=i+1}^{r} \left(\sqrt{(n-a_i)^2 + (n-a_j)^2}\right)^{a_j}.$$

Then we have

$$\begin{aligned}\Pi so(\mathbb{G}) &= \prod_{uv\in\mathbf{E}(\mathbb{G})} \sqrt{d_u^2+d_v^2} = \prod_{i=1}^{r-1}\prod_{j=i+1}^{r}\left(\sqrt{(n-a_i)^2+(n-a_j)^2}\right)^{a_ia_j} \\ &= \prod_{i=1}^{r-1}\left(\prod_{j=i+1}^{r}\left(\sqrt{(n-a_i)^2+(n-a_j)^2}\right)^{a_j}\right)^{a_i} \qquad (1) \\ &= \prod_{i=1}^{r-1}\left(\phi_{\mathbb{G}}(v_i)\right)^{a_i}.\end{aligned}$$

First, we compare $\phi_{\mathbb{G}}(v_i)$ and $\phi_{\mathbb{H}}(v_i)$ for $1 \le i \le r-2$. Since $\mathbb{H}$ has $|\mathbf{V}_{r-1}| = a_{(r-1)} - 1$ and $|\mathbf{V}_r| = a_r + 1$,

(2)

$$\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)} = \left(\frac{((n-a_i)^2+(n-a_{(r-1)}+1)^2)^{a_{(r-1)}-1}((n-a_i)^2+(n-a_r-1)^2)^{a_r+1}}{((n-a_i)^2+(n-a_{(r-1)})^2)^{a_{(r-1)}}((n-a_i)^2+(n-a_r)^2)^{a_r}}\right)^{\frac{1}{2}},$$

where all terms other than the two terms are reduced. Let $a_r + a_{r-1} = k$, and $a_r = x$ so that $a_{r-1} = k - x$. Then, from $a_{r-1} - a_r \ge 2$, we obtain

$$1 \le x \le \frac{k}{2} - 1.$$

Thus, equation (2) becomes

$$\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)} = \left(\frac{(M^2+(n-(k-x)+1)^2)^{k-x-1}(M^2+(n-x-1)^2)^{x+1}}{(M^2+(n-(k-x))^2)^{k-x}(M^2+(n-x)^2)^{x}}\right)^{\frac{1}{2}}, \qquad (3)$$

where $M = n - a_i$. Taking the logarithm of equation (3) yields

$$\begin{aligned}\log\left(\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)}\right) = \frac{1}{2}\big[&(k-x-1)\log\big(M^2+(n-k+x+1)^2\big) \\ &+(x+1)\log\big(M^2+(n-x-1)^2\big) \qquad (4)\\ &-(k-x)\log\big(M^2+(n-k+x)^2\big) \\ &-x\log\big(M^2+(n-x)^2\big)\big].\end{aligned}$$

Let

$$\theta(x) = (k-x)\log\big(M^2+\big(n-(k-x)\big)^2\big) + x\log\big(M^2+(n-x)^2\big)$$

to simplify (4) as

$$\log\left(\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)}\right) = \frac{1}{2}\big[\theta(x+1)-\theta(x)\big].$$

We claim $\theta(x)$ is a strictly increasing function on $1 \le x < \frac{k}{2}$. The derivative of $\theta(x)$ with respect to x is

$$\frac{d}{dx}\theta(x) = \log\big(M^2+(n-x)^2\big) - \frac{2x(n-x)}{M^2+(n-x)^2}$$

$$-\log\big(M^2+(n-k+x)^2\big)+\frac{2(k-x)(n-k+x)}{M^2+(n-k+x)^2}.$$

Let $y=k-x$. Since $y>x$ and $x+y=k<n$, we observe that

$$(n-x)^2-(n-k+x)^2=(n-x)^2-(n-y)^2>0, \tag{5}$$

$$(k-x)(n-k+x)-x(n-x)=(y-x)\big(n-(x+y)\big)>0, \tag{6}$$

$$\big(M^2+(n-x)^2\big)-\big(M^2+(n-k+x)^2\big)=(n-x)^2-(n-y)^2>0. \tag{7}$$

From (5),(6), and (7),

$$\begin{aligned}\frac{d}{dx}\theta(x)=&\big[\log\big(M^2+(n-x)^2\big)-\log\big(M^2+(n-k+x)^2\big)\big]\\&+2\left[\frac{(k-x)(n-k+x)}{M^2+(n-k+x)^2}-\frac{x(n-x)}{M^2+(n-x)^2}\right]>0.\end{aligned}$$

This proves that $\theta(x)$ is strictly increasing on $1\le x<\frac{k}{2}$. As a consequence,

$$\log\left(\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)}\right)=\frac{1}{2}\big[\theta(x+1)-\theta(x)\big]>0,$$

for $1\le x\le\frac{k}{2}-1$, or

$$\frac{\phi_{\mathbb{H}}(v_i)}{\phi_{\mathbb{G}}(v_i)}>1 \qquad \text{on } 1\le i\le r-2. \tag{8}$$

Finally, we compare $\phi_{\mathbb{G}}(v_{r-1})^{a_{(r-1)}}$ and $\phi_{\mathbb{H}}(v_{r-1})^{a_{(r-1)}-1}$, as follows:

$$\frac{\phi_{\mathbb{H}}(v_{r-1})^{a_{(r-1)}-1}}{\phi_{\mathbb{G}}(v_{r-1})^{a_{(r-1)}}}=\left(\frac{((n-a_{(r-1)}+1)^2+(n-a_r-1)^2)^{(a_{(r-1)}-1)(a_r+1)}}{((n-a_{(r-1)})^2+(n-a_r)^2)^{a_{(r-1)}a_r}}\right)^{\frac{1}{2}}.$$

In a similar way to the previous method, let $x=a_r$ and $a_{r-1}=k-x$ so that $1\le x\le\frac{k}{2}-1$. After taking the logarithm, we have

$$\begin{aligned}&\log\left(\frac{\phi_{\mathbb{H}}(v_{r-1})^{a_{(r-1)}-1}}{\phi_{\mathbb{G}}(v_{r-1})^{a_{(r-1)}}}\right)\\&=\frac{1}{2}\log\left(\frac{((n-k+x+1)^2+(n-x-1)^2)^{(k-x-1)(x+1)}}{((n-k+x)^2+(n-x)^2)^{(k-x)x}}\right)\\&=\frac{1}{2}\big[\eta(x+1)-\eta(x)\big],\end{aligned}$$

where $\eta(x)=x(k-x)\log((n-x)^2+(n-(k-x))^2)$. Again, we claim that $\eta(x)$ is a strictly increasing function on $1\le x<\frac{k}{2}$. The derivative of $\eta(x)$ with respect to x is

$$\frac{d}{dx}\eta(x)=(k-2x)\left[\log\big((n-k+x)^2+(n-x)^2\big)-\frac{2x(k-x)}{(n-k+x)^2+(n-x)^2}\right].$$

Since

$2x(k-x) \le x^2+(k-x)^2$ and $(n-k+x)^2+(n-x)^2 = 2n(n-k)+x^2+(k-x)^2$,

we observe

$$\frac{2x(k-x)}{(n-k+x)^2+(n-x)^2} \le \frac{x^2+(k-x)^2}{2n(n-k)+x^2+(k-x)^2} < 1.$$

Hence, it follows from $n-x \ge 2$ and $k-2x \ge 2$ that

$$\frac{d}{dx}\eta(x) > (k-2x)\big[\log\big((n-k+x)^2+(n-x)^2\big)-1\big] \ge 0.$$

Therefore, we conclude

$$\log\left(\frac{\phi_{\mathbb{H}}(v_{r-1})^{a_{(r-1)}-1}}{\phi_{\mathbb{G}}(v_{r-1})^{a_{(r-1)}}}\right) = \frac{1}{2}\big[\eta(x+1)-\eta(x)\big] > 0,$$

on $1 \le x \le \frac{k}{2}-1$ or

$$\frac{\phi_{\mathbb{H}}(v_{r-1})^{a_{(r-1)}-1}}{\phi_{\mathbb{G}}(v_{r-1})^{a_{(r-1)}}} > 1. \tag{9}$$

Finally, from (1), (8) and (9),

$$\frac{\Pi so(\mathbb{H})}{\Pi so(\mathbb{G})} = \frac{(\phi_{\mathbb{H}}(v_1))^{a_1}\cdots(\phi_{\mathbb{H}}(v_{r-2}))^{a_{(r-2)}}(\phi_{\mathbb{H}}(v_{r-1}))^{a_{(r-1)}-1}}{(\phi_{\mathbb{G}}(v_1))^{a_1}\cdots(\phi_{\mathbb{G}}(v_{r-2}))^{a_{(r-2)}}(\phi_{\mathbb{G}}(v_{r-1}))^{a_{(r-1)}}} > 1,$$

a contradiction. □

For the next theorem, we call $\mathbb{G} \in \mathcal{G}(n,r)$ is *r-critical* if $\chi(\mathbb{H}) < \chi(\mathbb{G}) = r$ for every proper subgraph $\mathbb{H}$ of $\mathbb{G}$.

PROPOSITION 2.6. *There is no r-critical graph in $\mathcal{G}(r+1,r)$.*

Proof. Assume $\mathbb{G} \in \mathcal{G}(r+1,r)$ is a r-critical graph. Since the number of vertices is only one more than the chromatic number, only two vertices of $\mathbb{G}$, say v and w, share the same color. Also, since $\delta \ge r-1$, v and w are adjacent to the remaining $r-1$ vertices, which means $d_v = d_w = r-1$. By the assumption, $\mathbb{G}-v$ (deleting v from $\mathbb{G}$) is $(r-1)$-colorable. But since v and w have the same neighbors in $\mathbb{G}$, assigning v the same color as w when adding it back to $\mathbb{G}-v$ shows that G is $(r-1)$-colorable, a contradiction. □

A pendant path $P = a_1a_2\cdots a_k$ of $\mathbb{G}$ is a subgraph of $\mathbb{G}$ which satisfies $d_{a_1,\mathbb{G}} \ge 3$ for one end vertex a_1, $d_{a_k,\mathbb{G}} = 1$ for the other end vertex a_k, and $d_{a_i,\mathbb{G}} = 2$ for the rest vertices of $P = a_1a_2\cdots a_k$. Both $Ki_{n,r}$ and $C_{n,r}$ have a pendent path P_{n-r+1} (Figure 4).

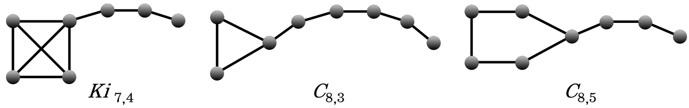

Figure 4 – $Ki_{7,4}$ is minimal in $\mathcal{G}(7,4)$. $C_{8,3}$ and $C_{8,5}$ are minimal in $\mathcal{G}(8,3)$.

LEMMA 2.7 ([10]). *Suppose a graph* $\mathbb{G}$ *has two pendant paths* $P = a_1a_2\cdots a_k$ *and* $P' = b_1b_2\cdots b_l$ *with* $d_{a_k} = d_{b_l} = 1$*, and possibly* $a_1 = b_1$*. Then*

$$\Pi so(\mathbb{G} - a_1a_2 + a_2b_l) < \Pi so(\mathbb{G}).$$

LEMMA 2.8 ([10]). *Suppose* $\mathbb{G}$ *is a unicyclic graph with* n *vertices. Then*

$$\Pi so(C_n) \leq \Pi so(\mathbb{G}),$$

with equality if and only if $\mathbb{G} = C_n$.

LEMMA 2.9 ([10]). *Suppose* $\mathbb{G}$ *is a unicyclic graph with* n *vertices and girth* g*. Then*

$$\Pi so(C_{n,g}) \leq \Pi so(\mathbb{G}),$$

with equality if and only if $\mathbb{G} = C_{n,g}$.

Now, we are ready to characterize the minimal graph having n vertices and chromatic number r. It was shown by Zhang et al. [17] that $Ki_{n,r}$ is minimal for the Sombor index in $\mathcal{G}(n,r)$.

THEOREM 2.10. *Let* $\mathbb{G} \in \mathcal{G}(n,r)$*,* $n \geq r \geq 2$*. When* $r \neq 3$*,*

$$\Pi so(\mathbb{G}) \geq \Pi so(Ki_{n,r}) \tag{10}$$

for all $n \geq r$*. Equality holds if and only if* $\mathbb{G} = Ki_{n,r}$*. When* $r = 3$*,*

$$\Pi so(\mathbb{G}) \geq \begin{cases} \Pi so(C_n) & \text{for all odd } n \geq 3, \\ \Pi so(C_{n,3}) & \text{for } n = 4 \text{ or } 6, \\ \Pi so(C_{n,2k+1}), 1 \leq k \leq \frac{n}{2} - 2 & \text{for all even } n \geq 8. \end{cases}$$

Proof. We consider three cases.

Case 1: $r = 2$. It is trivial that $\Pi so(P_n) \leq \Pi so(\mathbb{G})$ for all connected graph $\mathbb{G}$ of order n from Lemmas 2.1 and 2.7. Since $P_n = Ki_{n,2}$ belongs to bipartite graphs, the proof is done for the case of $r = 2$.

Case 2: $r \geq 4$. When $n = r$, the only graph in $\mathcal{G}(r,r)$ is K_r. So, there is nothing to prove. For the remaining part of $n \geq r+1$, the proof is by induction on n. When $n = r + 1$, by Proposition 2.6, there exists a vertex v such that

$\chi(\mathbb{G}-v) = r$ for every $\mathbb{G} \in \mathcal{G}(r+1,r)$, where $\mathbb{G}-v$ is K_r. Therefore, by Lemma 2.1, we may remove edges of v until $d_v = 1$ in $\mathbb{G}$ to conclude that

$$\Pi so(Ki_{r+1,r}) \leq \Pi so(\mathbb{G}) \tag{11}$$

for all $\mathbb{G} \in \mathcal{G}(r+1,r)$. Note that equality holds if and only if $\mathbb{G} = Ki_{r+1,r}$.

Assume the result (10) holds for n, and let $\mathbb{G} \in \mathcal{G}(n+1,r)$. We consider $h(x,y) = \sqrt{x^2+y^2}$.

(i) If $\mathbb{G}$ is r-critical, from $\delta \geq r-1$ and Proposition 2.2, we have

$$\begin{aligned}
\frac{\Pi so(\mathbb{G})}{\Pi so(Ki_{n+1,r})} &\geq \frac{(\sqrt{2}(r-1))^{\frac{1}{2}(n+1)(r-1)}}{h(r-1,r-1)^{\binom{r-1}{2}} h(r,r-1)^{r-1} h(r,2)\, h(2,2)^{n-r-1}\, h(2,1)} \\
&= \frac{(\sqrt{2}(r-1))^{\frac{1}{2}(r-1)(n-r+3)}}{\sqrt{r^2+(r-1)^2}^{\,r-1}\sqrt{8}^{\,n-r-1}\sqrt{5(r^2+4)}} \\
&= \frac{(r-1)^{\frac{3}{2}(r-1)}}{\sqrt{2r^2-2r+1}^{\,r-1}} \frac{(\sqrt{2}(r-1))^{\frac{1}{2}(r-1)(n-r-1)}}{\sqrt{8}^{\,n-r-1}} \frac{(4(r-1))^{\frac{1}{2}(r-1)}}{\sqrt{5(r^2+4)}} \\
&> \left(\frac{(r-1)^3}{2r^2-2r+1}\right)^{\frac{1}{2}(r-1)} \left(\frac{3\sqrt{2}}{\sqrt{8}}\right)^{n-r-1} \left(\frac{64(r-1)^3}{5(r^2+4)}\right)^{\frac{1}{2}} \\
&> 1
\end{aligned}$$

as $r \geq 4$. Thus, $\Pi so(\mathbb{G}) > \Pi so(Ki_{n+1,r})$ holds.

(ii) If $\mathbb{G}$ is not r-critical and can be made r-critical in $\mathcal{G}(n+1,r)$ by removing edges, $\Pi so(\mathbb{G}) > \Pi so(Ki_{n+1,r})$ is established according to the result of (i) and by Lemma 2.1 again.

(iii) If $\mathbb{G}$ is not r-critical and there is a vertex for which $\mathbb{G}-v \in \mathbb{G}(n,r)$, define $\mathbb{H}$ as $\mathbb{H} = \mathbb{G}-v$. We may reduce $\Pi so(\mathbb{G})$ by erasing the edges of v one by one until it reaches $\mathbb{H}+wv$, where w is a neighbor of v in $\mathbb{H}$ and $d_{v,\mathbb{H}+wv} = 1$. Since $\mathbb{H} \in \mathcal{G}(n,r)$, $\Pi so(\mathbb{H}) \geq \Pi so(Ki_{n,r})$ is satisfied by induction hypothesis. Therefore,

$$\begin{aligned}
\Pi so(\mathbb{G}) &\geq \Pi so(\mathbb{H}+wv) \\
&= \Pi so(\mathbb{H}) \cdot \frac{\prod_{v_i \in \mathbf{N}_{\mathbb{H}}(w)} h(d_{v_i}, d_{w,\mathbb{H}}+1)}{\prod_{v_i \in \mathbf{N}_{\mathbb{H}}(w)} h(d_{v_i}, d_{w,\mathbb{H}})} \cdot h(d_{w,\mathbb{H}}+1, 1),
\end{aligned}$$

where $\mathbf{N}_{\mathbb{H}}(w)$ is the set of neighbors of w in $\mathbb{H}$. Since $\frac{h(d_{v_i}, d_{w,\mathbb{H}}+1)}{h(d_{v_i}, d_{w,\mathbb{H}})} > 1$, for a $v_k \in \mathbf{N}_{\mathbb{H}}(w)$,

$$\frac{\prod_{v_i \in \mathbf{N}_{\mathbb{H}}(w)} h(d_{v_i}, d_{w,\mathbb{H}}+1)}{\prod_{v_i \in \mathbf{N}_{\mathbb{H}}(w)} h(d_{v_i}, d_{w,\mathbb{H}})} \geq \frac{h(d_{v_k}, d_{w,\mathbb{H}}+1)}{h(d_{v_k}, d_{w,\mathbb{H}})}.$$

Put a function $f(x)$ as

$$f(x) = \left(\frac{h(k,x+1)}{h(k,x)} \cdot h(x+1,1)\right)^2 = \frac{((x+1)^2+k^2)((x+1)^2+1)}{x^2+k^2}$$

on $x \geq 1$ and $k \geq 1$. Then $f(x)$ is increasing because

$$\frac{d}{dx} f(x) = \frac{2x^5+4x^4+4k^2x^3+(10k^2-16)x^2+(2k^4+10k^2-4)x+2k^2(k^2+3)}{(x^2+k^2)^2} > 0.$$

As a result,

$$\Pi so(\mathbb{G}) \geq \Pi so(\mathbb{H}) \cdot \frac{h(d_{v_k},2)}{h(d_{v_k},1)} \cdot h(2,1) \tag{12}$$

$$\geq \Pi so(Ki_{n,r}) \cdot \frac{h(d_{v_k},2)}{h(d_{v_k},1)} \cdot h(2,1) \tag{13}$$

$$= \Pi so(Ki_{n+1,r}). \tag{14}$$

The equality of (12) is satisfied when $d_{w,\mathbb{H}} = 1$ and the equality of (13) holds if and only if $\mathbb{H} = Ki_{n,r}$ by assumption. There is only one vertex of degree 1 in $Ki_{n,r}$ and this makes the equality of (14) true. Hence, the proof for $r \geq 4$ is done.

Case 3: $r = 3$. Since 3-critical graphs always contain an odd cycle, every graph in $\mathcal{G}(n,3)$ can be transformed into an odd-unicyclic graph–a unicyclic graph whose cycle is odd–by removing edges to make it contain a minimal number of edges. Here, an odd-unicyclic graph means a unicyclic graph whose cycle is odd. So, we only need to compare the value $\Pi so(\mathbb{G})$ of odd-unicyclic graphs. When n is odd, by Lemma 2.8, $\Pi so(C_n) \leq \Pi so(\mathbb{G})$, done. When $n = 4$, $\mathbb{G} \in \mathcal{G}(4,3)$ and by (11), we have $\Pi so(C_{4,3}) = \Pi so(Ki_{4,3}) \leq \Pi so(\mathbb{G})$, done. When $n = 6$, $\mathbb{G} \in \mathcal{G}(6,3)$ can contain only two odd cycles C_3, and C_5. By Lemma 2.9, $C_{6,3}$ and $C_{6,5}$ are minimal graphs having girth 3 and 5, respectively. But we have

$$\frac{\Pi so(C_{6,3})}{\Pi so(C_{6,5})} = \frac{h(2,2)^2 \cdot h(2,3)^3 \cdot h(2,1)}{h(2,2)^3 \cdot h(2,3)^2 \cdot h(3,1)} = \frac{\sqrt{65}}{\sqrt{80}} < 1.$$

Hence, $\Pi so(C_{6,3}) \leq \Pi so(\mathbb{G})$, done. A similar method applies when $n \geq 8$ and even. But simple computation gives

$$\Pi so(C_{n,3}) = \Pi so(C_{n,4}) = \cdots = \Pi so(C_{n,n-2}) < \Pi so(C_{n,n-1}).$$

Therefore, multiple odd-unicyclic graphs have the same minimum value, i.e.,

$$\Pi so(C_{n,2k+1}) \leq \Pi so(\mathbb{G}),$$

for $2k+1 \leq n-2$ or $k \leq \frac{n-3}{2}$. More exactly, $k \leq \frac{n-4}{2}$ for n is even. $\square$

Remark 2.11. A similar method to the above proof can be applied to the Sombor index, so we modify the result of Theorem 2 in [17] by adding the missing graph in the "equality condition" as in the theorem above when $r = 3$.

3. BOUNDS FOR PRODUCTS OF GRAPHS

Let $\mathbb{G} = (\mathbf{V}(\mathbb{G}), \mathbf{E}(\mathbb{G}))$ and $\mathbb{H} = (\mathbf{V}(\mathbb{H}), \mathbf{E}(\mathbb{H}))$ be two graphs. We define $\mathbb{G} \cup \mathbb{H} := \mathbb{G}(\mathbf{V}(\mathbb{G}) \cup \mathbf{V}(\mathbb{H}), \mathbf{E}(\mathbb{G}) \cup \mathbf{E}(\mathbb{H}))$. $\mathbb{G}$ and $\mathbb{H}$ are disjoint if $\mathbf{V}(\mathbb{G}) \cap \mathbf{V}(\mathbb{H}) = \phi$, and edge-disjoint if $\mathbf{E}(\mathbb{G}) \cap \mathbf{E}(\mathbb{H}) = \phi$. $\overline{\mathbb{G}}$ is the complement of $\mathbb{G}$.

LEMMA 3.1. *Let two graphs* $\mathbb{G}$ *and* $\mathbb{H}$ *be edge-disjoint on the same vertex set* $\mathbf{V} = \mathbf{V}(\mathbb{G}) = \mathbf{V}(\mathbb{H})$ *with* $|\mathbf{V}| = n$. *Then*

$$\Pi so(\mathbb{G})\, \Pi so(\mathbb{H}) \leq \Pi so(\mathbb{G} \cup \mathbb{H}).$$

Equality holds if and only if $d_{v,\mathbb{G}} = 0$ *or* $d_{v,\mathbb{H}} = 0$ *for every* $v \in \mathbf{V}$. *Especially,*

$$\Pi so(\mathbb{G})\, \Pi so(\overline{\mathbb{G}}) \leq \Pi so(K_n).$$

Equality holds if and only if $\mathbb{G} = K_n$ *or* $\mathbb{G} = \overline{K_n}$.

Proof. Since $\mathbb{G}$ and $\mathbb{H}$ are edge-disjoint, $d_{v,\mathbb{G}\cup\mathbb{H}} = d_{v,\mathbb{G}} + d_{v,\mathbb{H}}$ holds, implying that $d_{u,\mathbb{G}} \leq d_{u,\mathbb{G}\cup\mathbb{H}}$ and $d_{u,\mathbb{H}} \leq d_{u,\mathbb{G}\cup\mathbb{H}}$. Thus,

$$\begin{aligned}
\Pi so(K_n) &\geq \Pi so(\mathbb{G} \cup \mathbb{H}) \qquad (15)\\
&= \prod_{uv \in \mathbf{E}(\mathbb{G}\cup\mathbb{H})} \sqrt{d^2_{u,\mathbb{G}\cup\mathbb{H}} + d^2_{v,\mathbb{G}\cup\mathbb{H},}}\\
&= \prod_{uv \in \mathbf{E}(\mathbb{G}\cup\mathbb{H})} \sqrt{(d_{u,\mathbb{G}} + d_{u,\mathbb{H}})^2 + (d_{v,\mathbb{G}} + d_{v,\mathbb{H}})^2}\\
&\geq \prod_{uv \in \mathbf{E}(\mathbb{G})} \sqrt{d^2_{u,\mathbb{G}} + d^2_{v,\mathbb{G}}} \prod_{uv \in \mathbf{E}(\mathbb{H})} \sqrt{d^2_{u,\mathbb{H}} + d^2_{v,\mathbb{H}}} \qquad (16)\\
&= \Pi so(\mathbb{G})\, \Pi so(\mathbb{H}).
\end{aligned}$$

The equality in (15) holds if and only if $K_n = \mathbb{G} \cup \mathbb{H}$. Another equality in (16) is true if and only if $d_{v,\mathbb{G}} = d_{v,\mathbb{G}\cup\mathbb{H}}$ or $d_{v,\mathbb{H}} = d_{v,\mathbb{G}\cup\mathbb{H}}$ for all $v \in \mathbf{V}$. This condition occurs if and only if $d_{v,\mathbb{G}} = 0$ or $d_{v,\mathbb{H}} = 0$ for all $v \in \mathbf{V}$. Both equals are valid if and only if $\mathbb{G} = K_n$ or $\mathbb{H} = K_n$. □

The cartesian product $\mathbb{G} \,\Box\, \mathbb{H}$ of graphs $\mathbb{G}$ and $\mathbb{H}$ has a vertex set $\mathbf{V}(\mathbb{G}) \times \mathbf{V}(\mathbb{H})$ and an edge set satisfying the following condition: two vertices (u, v), $(u,' v') \in \mathbf{V}(\mathbb{G}) \times \mathbf{V}(\mathbb{H})$ are adjacent if and only if (1) $u = u'$ and $vv' \in \mathbf{E}(\mathbb{H})$, or (2) $v = v'$ and $uu' \in \mathbf{E}(\mathbb{G})$ [13].

Lemma 3.2. *Let* $\mathbb{G}$ *be a graph of order* n. *Then*

$$\Pi so(\mathbb{G} \,\square\, \overline{K_n}) = \Pi so(\mathbb{G})^{n}.$$

Proof. By definition of the cartesian product of graph,

$$\mathbb{G} \,\square\, \overline{K_n} = \bigcup_{k=1}^{n} \mathbb{G}_k,$$

where $\mathbb{G}_k$, $1 \leq k \leq n$, are disjoint but each $\mathbb{G}_k$ is isomorphic to $\mathbb{G}$. Therefore,

$$\Pi so(\mathbb{G} \,\square\, \overline{K_n}) = \Pi so\Big(\bigcup_{k=1}^{n} \mathbb{G}_k\Big) = \prod_{k=1}^{n} \Pi so(\mathbb{G}_k) = \Pi so(\mathbb{G})^{n}. \qquad \square$$

Theorem 3.3. *Let* $\mathbb{G}$ *and* $\mathbb{H}$ *be two disjoint graphs with* $|\mathbf{V}(\mathbb{G})| = n$ *and* $|\mathbf{V}(\mathbb{H})| = p$. *Then*

$$\Pi so(\mathbb{G} \,\square\, \mathbb{H}) \geq \Pi so(\mathbb{G})^{|\mathbf{V}(\mathbb{H})|} \Pi so(\mathbb{H})^{|\mathbf{V}(\mathbb{G})|}.$$

Equality holds if and only if $\mathbb{G} = \overline{K_n}$ *or* $\mathbb{H} = \overline{K_p}$.

Proof. Let $\mathbf{V}(\mathbb{G}) = \mathbf{V}(\overline{K_n})$, $\mathbf{V}(\mathbb{H}) = \mathbf{V}(\overline{K_p})$. Then $\mathbb{G} \,\square\, \mathbb{H}$ is represented as

$$\mathbb{G} \,\square\, \mathbb{H} = (\mathbb{G} \,\square\, \overline{K_p}) \cup (\overline{K_n} \,\square\, \mathbb{H}).$$

Since $\mathbb{G} \,\square\, \overline{K_p}$ and $\overline{K_n} \,\square\, \mathbb{H}$ are edge-disjoint on the same vertex set, by Lemmas 3.1 and 3.2,

$$\begin{aligned} \Pi so(\mathbb{G} \,\square\, \mathbb{H}) &= \Pi so\big((\mathbb{G} \,\square\, \overline{K_p}) \cup (\overline{K_n} \,\square\, \mathbb{H})\big) \\ &\geq \Pi so(\mathbb{G} \,\square\, \overline{K_p})\, \Pi so(\overline{K_n} \,\square\, \mathbb{H}) \\ &= \Pi so(\mathbb{G})^{p}\, \Pi so(\mathbb{H})^{n}. \end{aligned} \tag{17}$$

Let $(u, v) \in \mathbf{V}(\mathbb{G} \,\square\, \mathbb{H})$. By Lemma 3.2, equality in (17) holds true if and only if $d_{(u,v),\, \mathbb{G}\square \overline{K_p}} = 0$ or $d_{(u,v),\, \overline{K_n}\square \mathbb{H}} = 0$ for all $(u, v) \in \mathbf{V}(\mathbb{G} \square \mathbb{H})$. Since $d_{(u,v),\mathbb{G}\square\mathbb{H}} = d_{u,\mathbb{G}} + d_{v,\mathbb{H}}$,

$$d_{(u,v),\, \mathbb{G}\square \overline{K_p}} = d_{u,\mathbb{G}} + d_{v,\overline{K_p}} = d_{u,\mathbb{G}}$$

and

$$d_{(u,v),\, \overline{K_n}\square \mathbb{H}} = d_{u,\overline{K_n}} + d_{v,\mathbb{H}} = d_{v,\mathbb{H}}.$$

Claim. $d_{u,\mathbb{G}} = 0$ or $d_{v,\mathbb{H}} = 0$ for all $(u, v) \in \mathbf{V}(\mathbb{G} \square \mathbb{H})$ if and only if $\mathbb{G} = \overline{K_n}$ or $\mathbb{H} = \overline{K_p}$.

Proof of Claim. If $\mathbb{G} = \overline{K_n}$ or $\mathbb{H} = \overline{K_p}$, it is clear that $d_{u,\mathbb{G}} = 0$ or $d_{v,\mathbb{H}} = 0$. Conversely, suppose $\mathbb{G} \neq \overline{K_n}$ and $\mathbb{H} \neq \overline{K_p}$. Then there exist vertices $u \in \mathbf{V}(\mathbb{G})$ and $v \in \mathbf{V}(\mathbb{H})$ such that $d_{u,\mathbb{G}} > 0$ and $d_{v,\mathbb{H}} > 0$, contradiction. This completes the proof. $\square$

The tensor product $\mathbb{G}\times\mathbb{H}$ of graphs $\mathbb{G}$ and $\mathbb{H}$ is a graph whose vertex set is $\mathbf{V}(\mathbb{G})\times\mathbf{V}(\mathbb{H})$ and an edge set satisfies the following condition: two vertices (u,v), $(u',v')\in\mathbf{V}(\mathbb{G})\times\mathbf{V}(\mathbb{H})$ are adjacent if and only if $uu'\in\mathbf{E}(\mathbb{G})$ and $vv'\in\mathbf{E}(\mathbb{G})$ [14].

THEOREM 3.4. *Let $\mathbb{G}$ be a graph of order n. Then*

$$\Pi so(\mathbb{G}\times K_2)=\Pi so(\mathbb{G})^2.$$

Proof. Let

$$\mathbf{V}(\mathbb{G}\times K_2)=\{(v_i,u_j)|\ 1\le i\le n, j=1,2\}$$

for $\mathbf{V}(\mathbb{G})=\{v_i|\ 1\le i\le n\}$, $\mathbf{V}(K_2)=\{u_1,u_2\}$. By the definition of tensor product of graphs, the edge set of $\mathbb{G}\times K_2$ becomes

$$\mathbf{E}(\mathbb{G}\times K_2)=\{((v_i,u_1)(v_j,u_2))|\ v_iv_j\in\mathbf{E}(\mathbb{G})\}.$$

This means that, for each $v_iv_j\in\mathbf{E}(\mathbb{G})$ and $u_1u_2\in\mathbf{E}(K_2)$, there are two different edges $((v_i,u_1)(v_j,u_2))$ and $((v_j,u_1)(v_i,u_2))$ in $\mathbb{G}\times K_2$. In addition, since $\mathbf{N}_{\mathbb{G}\times K_2}((v_i,u_1))=\{(v,u_2)|\ v\in\mathbf{N}_{\mathbb{G}}(v_i)\}$, we have $d_{(v_i,u_j),\mathbb{G}\times K_2}=d_{v_i,\mathbb{G}}$. Therefore,

$$\begin{aligned}\Pi so(\mathbb{G}\times K_2)&=\prod_{\substack{((v_i,u_1)(v_j,u_2))\\ \in\mathbf{E}(\mathbb{G}\times K_2)}}\sqrt{d^2_{(v_i,u_1),\mathbb{G}\times K_2}+d^2_{(v_j,u_2),\mathbb{G}\times K_2}}\\ &=\prod_{(v_i,v_j)\in\mathbf{E}(\mathbb{G})}\left(\sqrt{d^2_{v_i,\mathbb{G}}+d^2_{v_j,\mathbb{G}}}\sqrt{d^2_{v_j,\mathbb{G}}+d^2_{v_i,\mathbb{G}}}\right)\\ &=\left(\prod_{(v_i,v_j)\in\mathbf{E}(\mathbb{G})}\sqrt{d^2_{v_i,\mathbb{G}}+d^2_{v_j,\mathbb{G}}}\right)^2\\ &=\Pi so(\mathbb{G})^2.\end{aligned}$$

□

THEOREM 3.5. *Let $\mathbb{G}$ and $\mathbb{H}$ be two disjoint graphs with $|\mathbf{V}(\mathbb{G})|=n$ and $|\mathbf{V}(\mathbb{H})|=p$. Then*

$$\Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|}\le\Pi so(\mathbb{G}\times\mathbb{H})\le\left(\Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|}\right)^2.$$

The first equality holds if and only if

$$\mathbb{G}=(\cup_{i=1}^{n'}K_2)\cup\overline{K_{n-2n'}}\quad\textit{and}\quad\mathbb{H}=(\cup_{j=1}^{p'}K_2)\cup\overline{K_{p-2p'}}$$

for some $1\le n'\le\frac{n}{2}, 1\le p'\le\frac{p}{2}$, or

$$\mathbb{G}=\overline{K_n}\ \textit{or}\ \mathbb{H}=\overline{K_p}.$$

The second equality holds if and only if $\mathbb{G}=\overline{K_n}$ or $\mathbb{H}=\overline{K_p}$.

Proof. Since the edge set of $\mathbb{G}\times\mathbb{H}$ is defined as

$$\mathbf{E}(\mathbb{G}\times\mathbb{H})=\Big\{\big((u_i,v_k)(u_j,v_l)\big)\mid u_iu_j\in\mathbf{E}(\mathbb{G}) \text{ and } v_ku_l\in\mathbf{E}(\mathbb{H})\Big\},$$

the degree of a vertex $(u_i,v_k)\in\mathbf{V}(\mathbb{G}\times\mathbb{H})$ is the product of $|\mathbf{N}_{\mathbb{G}}(u_i)|$ and $|\mathbf{N}_{\mathbb{H}}(v_k)|$, or

$$d_{(u_i,v_k),\mathbb{G}\times\mathbb{H}}=d_{u_i,\mathbb{G}}\,d_{v_k,\mathbb{H}}.$$

Suppose $\mathbb{G}\neq\overline{K_n}$ and $\mathbb{H}\neq\overline{K_p}$. Then by Theorem 3.4,

$$\begin{aligned}
\Pi so(\mathbb{G}\times\mathbb{H}) &= \prod_{\substack{((u_i,v_k)(u_j,v_l))\\ \in\mathbf{E}(\mathbb{G}\times\mathbb{H})}}\sqrt{d^2_{(u_i,v_k),\mathbb{G}\times\mathbb{H}}+d^2_{(u_j,v_l),\mathbb{G}\times\mathbb{H}}}\\
&= \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\prod_{\substack{(u_i,u_j)\\ \in\mathbf{E}(\mathbb{G})}}\sqrt{(d_{u_i,\mathbb{G}}d_{v_k,\mathbb{H}})^2+(d_{u_j,\mathbb{G}}d_{v_l,\mathbb{H}})^2}\sqrt{(d_{u_i,\mathbb{G}}d_{v_l,\mathbb{H}})^2+(d_{u_j,\mathbb{G}}d_{v_k,\mathbb{H}})^2}\\
&< \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\prod_{\substack{(u_i,u_j)\\ \in\mathbf{E}(\mathbb{G})}}\sqrt{(d^2_{u_i,\mathbb{G}}+d^2_{u_j,\mathbb{G}})(d^2_{v_k,\mathbb{H}}+d^2_{v_l,\mathbb{H}})}\sqrt{(d^2_{u_i,\mathbb{G}}+d^2_{u_j,\mathbb{G}})(d^2_{v_l,\mathbb{H}}+d^2_{v_k,\mathbb{H}})}\\
&= \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\prod_{\substack{(u_i,u_j)\\ \in\mathbf{E}(\mathbb{G})}}\left(\sqrt{d^2_{u_i,\mathbb{G}}+d^2_{u_j,\mathbb{G}}}\right)^2\left(\sqrt{d^2_{v_k,\mathbb{H}}+d^2_{v_l,\mathbb{H}}}\right)^2\\
&= \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\Pi so(\mathbb{G})^2\left(\sqrt{d^2_{v_k,\mathbb{H}}+d^2_{v_l,\mathbb{H}}}\right)^{2|\mathbf{E}(\mathbb{G})|}\\
&= \Pi so(\mathbb{G})^{2|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{2|\mathbf{E}(\mathbb{G})|}.
\end{aligned}$$

Furthermore, when $\mathbb{G}\neq\overline{K_n}$ and $\mathbb{H}\neq\overline{K_p}$,

(18)

$$\begin{aligned}
&\Pi so(\mathbb{G}\times\mathbb{H})\\
&= \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\prod_{\substack{(u_i,u_j)\\ \in\mathbf{E}(\mathbb{G})}}\sqrt{(d_{u_i,\mathbb{G}}d_{v_k,\mathbb{H}})^2+(d_{u_j,\mathbb{G}}d_{v_l,\mathbb{H}})^2}\sqrt{(d_{u_i,\mathbb{G}}d_{v_l,\mathbb{H}})^2+(d_{u_j,\mathbb{G}}d_{v_k,\mathbb{H}})^2}
\end{aligned}$$

(19)

$$\begin{aligned}
&\geq \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\prod_{\substack{(u_i,u_j)\\ \in\mathbf{E}(\mathbb{G})}}\sqrt{(d^2_{u_i,\mathbb{G}}+d^2_{u_j,\mathbb{G}})}\sqrt{(d^2_{v_l,\mathbb{H}}+d^2_{v_k,\mathbb{H}})}\\
&= \prod_{\substack{(v_k,v_l)\\ \in\mathbf{E}(\mathbb{H})}}\Pi so(\mathbb{G})\left(\sqrt{d^2_{v_k,\mathbb{H}}+d^2_{v_l,\mathbb{H}}}\right)^{|\mathbf{E}(\mathbb{G})|}
\end{aligned}$$

$$= \Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|}.$$

Equality (19) is satisfied if and only if $d_{u_i,\mathbb{G}} = d_{u_j,\mathbb{G}} = d_{v_k,\mathbb{H}} = d_{v_l,\mathbb{H}} = 1$ for all $u_iu_j \in \mathbf{E}(\mathbb{G})$ and $v_kv_l \in \mathbf{E}(\mathbb{H})$ if and only if $\mathbb{G}$ and $\mathbb{H}$ are the form of

$$\mathbb{G} = (\cup_{i=1}^{n'} K_2) \cup \overline{K_{n-2n'}} \text{ and } \mathbb{H} = (\cup_{j=1}^{p'} K_2) \cup \overline{K_{p-2p'}}.$$

Finally, suppose $\mathbb{G} = \overline{K_n}$ or $\mathbb{H} = \overline{K_p}$. Then $\mathbf{E}(\mathbb{G} \times \mathbb{H}) = \phi$, so $\Pi so(\mathbb{G} \times \mathbb{H}) = 1$. Hence, bounds on both sides satisfy

$$\Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|} = \left(\Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|}\right)^2 = 1$$

and these make both equalities valid. □

The strong product $\mathbb{G} \boxtimes \mathbb{H}$ of graphs $\mathbb{G}$ and $\mathbb{H}$ is a graph on the vertex set $\mathbf{V}(\mathbb{G}) \times \mathbf{V}(\mathbb{H})$, whose edge set satisfies the following condition: two vertices (u, v), $(u', v') \in \mathbf{V}(\mathbb{G}) \times \mathbf{V}(\mathbb{H})$ are adjacent if and only if (1) $u = u'$ and $vv' \in \mathbf{E}(\mathbb{H})$, or (2) $v = v'$ and $uu' \in \mathbf{E}(\mathbb{G})$, or (3) $uu' \in \mathbf{E}(\mathbb{G})$ and $vv' \in \mathbf{E}(\mathbb{G})$ [8].

Theorem 3.6. *Let $\mathbb{G}$ and $\mathbb{H}$ be two disjoint graphs. Then*

$$\Pi so(\mathbb{G} \boxtimes \mathbb{H}) \geq \Pi so(\mathbb{G})^{|\mathbf{V}(\mathbb{H})|+|\mathbf{E}(\mathbb{H})|}\Pi so(\mathbb{H})^{|\mathbf{V}(\mathbb{G})|+|\mathbf{E}(\mathbb{G})|}. \tag{20}$$

The equality holds if and only if $\mathbb{G} = \overline{K_n}$ or $\mathbb{H} = \overline{K_p}$.

Proof. Suppose $\mathbb{G} \neq \overline{K_n}$ and $\mathbb{G} \neq \overline{K_p}$. $\mathbb{G} \boxtimes \mathbb{H}$ can be expressed as

$$\mathbb{G} \boxtimes \mathbb{H} = (\mathbb{G} \,\square\, \mathbb{H}) \cup (\mathbb{G} \times \mathbb{H}).$$

Since $\mathbb{G} \,\square\, \mathbb{H}$ and $\mathbb{G} \times \mathbb{H}$ are edge-disjoint on the same set of vertices, it follows from Proposition 3.1, Theorem 3.3, and Theorem 3.5 that

$$\begin{aligned}\Pi so(\mathbb{G} \boxtimes \mathbb{H}) &= \Pi so\big((\mathbb{G} \,\square\, \mathbb{H}) \cup (\mathbb{G} \times \mathbb{H})\big)\\ &> \Pi so(\mathbb{G} \,\square\, \mathbb{H})\, \Pi so(\mathbb{G} \times \mathbb{H}) \qquad (21)\\ &> \Pi so(\mathbb{G})^{|\mathbf{V}(\mathbb{H})|}\, \Pi so(\mathbb{H})^{|\mathbf{V}(\mathbb{G})|}\, \Pi so(\mathbb{G})^{|\mathbf{E}(\mathbb{H})|}\, \Pi so(\mathbb{H})^{|\mathbf{E}(\mathbb{G})|}.\end{aligned}$$

Note that the equality does not hold in (21) because there exist $u \in \mathbf{V}(\mathbb{G})$, $v \in \mathbf{V}(\mathbb{H})$ such that $d_{u,\mathbb{G}} > 0$, $d_{v,\mathbb{H}} > 0$, which explains $d_{(u,v),\mathbb{G}\square\mathbb{H}} = d_{u,\mathbb{G}} + d_{v,\mathbb{H}} > 0$ and $d_{(u,v),\mathbb{G}\times\mathbb{H}} = d_{u,\mathbb{G}}d_{v,\mathbb{H}} > 0$.

Suppose $\mathbb{G} = \overline{K_n}$. Then we obtain $\mathbb{G} \boxtimes \mathbb{H} = \mathbb{G} \,\square\, \mathbb{H}$ from $\mathbf{E}(\mathbb{G} \times \mathbb{H}) = \phi$. Hence, the left side of equation (20) is equal to $\Pi so(\mathbb{G} \,\square\, \mathbb{H}) = \Pi so(\mathbb{H})^{|\mathbf{V}(\mathbb{G})|}$ from Lemma 3.2. The right side of (20) also gives the same value and proves the equality. □

4. DISCUSSION

To conclude, it is still uncertain whether there are infinitely many graphs where $SO(\mathbb{G}) < SO(\mathbb{H})$ but $\Pi so(\mathbb{G}) > \Pi so(\mathbb{H})$. However, despite the existence of many such graphs, it has been established that the extremal graphs in certain classes considered are equal for both indices. In this article, we prove that this result also applies to the class $\mathcal{G}(n, r)$. We have confirmed that the methods used in the proof can similarly be applied to the Sombor index, thereby addressing the missing part of the “equality condition” in Theorem 2 from Zhang et al.’s paper [17]. The theorems presented in Section 3 illustrate how graph indices can be utilized in various graph operations. These approaches can offer a way to predict bounds when applying graph indices to large networks.

REFERENCES

[1] A. Aashtab, S. Akbari, S. Madadinia, M. Noei, and F. Salehi, *On the graphs with minimum Sombor index.* MATCH Commun. Math. Comput. Chem. **88** (2022), *3*, 553–559.

[2] R. Cruz and J. Rada, *Extremal values of the Sombor index in unicyclic and bicyclic graphs.* J. Math. Chem. **59** (2021), *4*, 1098–1116.

[3] K.C. Das, A.S. Çevik, I.N. Cangul, and Y. Shang, *On Sombor index.* Symmetry **13** (2021), *1*, article no. 140.

[4] K.C. Das and I. Gutman, *On Sombor index of trees.* Appl. Math. Comput. **412** (2022), article no. 126575.

[5] K.C. Das and Y. Shang, *Some extremal graphs with respect to Sombor index.* Mathematics **9** (2021), *11*, article no. 1202.

[6] R. Diestel, *Graph Theory.* Grad. Texts in Math. 173, Springer, Heidelberg, 2005.

[7] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices.* MATCH Commun. Math. Comput. Chem. **86** (2021), *1*, 11–16.

[8] K. Kozawa, Y. Otachi, and K. Yamazaki, *Lower bounds for treewidth of product graphs.* Discrete Appl. Math. **162** (2014), 251–258.

[9] V. Kulli, *Multiplicative Sombor indices of certain nanotubes.* Int. J. Math. Arch. **12** (2021), 1–5.

[10] H. Liu, *Multiplicative Sombor index of graphs.* Discrete Math. Lett. **9** (2022), 80–85.

[11] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie.* Mat. Fiz. Lapok. **48** (1941), 436–452.

[12] K. Varmuza, M. Dehmer, and D. Bonchev, *Statistical modelling of molecular descriptors in QSAR/QSPR.* Wiley Online Library, 2012.

[13] V.G. Vizing, *The Cartesian product of graphs.* Vycisl. Sistemy. **9** (1963), 30–43.

[14] P.M. Weichsel, *The Kronecker product of graphs.* Proc. Amer. Math. Soc. **13** (1962), 47–52.

[15] H. Wiener, *Structural determination of paraffin boiling points.* J. Amer. Chem. Soc. **69** (1947), 17–20.

[16] C. Xu, B. Horoldagva, L. Buyantogtokh, K.C. Das, and T.A. Selenge, *Multiplicative Sombor index of graphs.*

[17] W. Zhang, J. Meng, and N. Wang, *Extremal graphs for Sombor index with given parameters.* Axioms **12** (2023), *2*, article no. 203.

Received 13 March 2025

Sungkyunkwan University
Department of Mathematics
Suwon, Republic of Korea
`dayeonhuh7@gmail.com`