ENTIRE FUNCTIONS WHOSE DERIVATIVES AND SHIFTS SHARE TWO PAIRS OF SMALL FUNCTIONS

PULAK SAHOO and SONIYA SULTANA

Communicated by Lucian Beznea

In this paper, we study the relationship between the k-th derivative of an entire function and its shift sharing two distinct pairs of small functions. For such a function f, a nonzero complex value c and a positive integer k, we establish a linear relationship between $f(z+c)$ and $f^{(k)}(z)$ which generalizes a result due to Huang, Deng and Fang [Open Mathematics, 19 (2021), 144–156]. The examples are also given in support of our result.

AMS 2020 Subject Classification: 30D35.

Key words: entire functions, derivatives, shift, small functions, sharing.

1. INTRODUCTION, DEFINITIONS AND RESULTS

Let f be a meromorphic function in the complex plane $\mathbb{C}$. Throughout this paper, we adopt the standard notations of Nevanlinna value distribution theory as described in [3,6,13]. We use $S(r,f)$ to denote any quantity satisfying $S(r,f) = o\{T(r,f)\}$ for all r outside a possible exceptional set E of finite linear measure. A meromorphic function $a(z)$ is said to be a small function of f provided that $T(r,a) = S(r,f)$. Let f and g be any two nonconstant functions and let a and b be any two small functions of f and g. The functions f and g are said to share a pair (a,b) CM if $f-a$ and $g-b$ have the same zeros counting multiplicities; f and g are said to share a pair (a,b) IM if $f-a$ and $g-b$ have the same zeros ignoring multiplicities. We see that f and g share (a,a) CM (resp., IM) if and only if f and g share a CM (resp., IM). The same argument is also applicable when a and b are two values in $\mathbb{C}\cup\{\infty\}$. The order $\rho(f)$ and hyper-order $\rho_2(f)$ of a meromorphic function f is defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 1977, Rubel and Yang [12] proved the following result.

THEOREM 1.1. *If a nonconstant entire function f and its derivative f' share two distinct finite complex values a, b CM, then $f \equiv f'$.*

doi: 10.59277/mrar.2025.27.77.3.4.121

In 1979, replacing CM sharing as IM sharing, Mues and Steinmetz [8] proved the following theorem.

Theorem 1.2. *Let f be a nonconstant entire function, and let a, b be two distinct finite complex numbers. If f and f' share a, b IM, then $f \equiv f'$.*

In case of small functions sharing, Zheng and Wang [14] proved the following result.

Theorem 1.3. *Let f be a nonconstant entire function, and let $a(\not\equiv \infty)$, $b(\not\equiv \infty)$ be two distinct small functions of f. If f and f' share a, b CM, then $f \equiv f'$.*

In 2000, Qiu [10] proved the following result for IM sharing.

Theorem 1.4. *Let f be a nonconstant entire function, and let $a(\not\equiv \infty)$, $b(\not\equiv \infty)$ be two distinct small functions of f. If f and f' share a, b IM, then $f \equiv f'$.*

Considering k-th derivative instead of first derivative, Li and Yang [7] proved the result as follows.

Theorem 1.5. *Let f be a nonconstant entire function, and let a, b be two distinct small functions of f. If f and $f^{(k)}$ share a CM and b IM, then $f = f^{(k)}$.*

In 2020, Qi and Yang [9] considered the first derivative of f and its shift and proved the following result.

Theorem 1.6. *Let f be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then we have $f'(z) \equiv f(z+c)$.*

In 2024, Huang [4] extended the result of Qi and Yang [9], and the result is as follows.

Theorem 1.7. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$, let c be a nonzero finite value, k be a positive integer, and let $a(z) \not\equiv \infty$, $b(z) \not\equiv \infty$ be two distinct small functions. If $f^{(k)}(z)$ and $f(z+c)$ share $a(z)$, ∞ CM and share $b(z)$ IM, then $f^{(k)}(z) \equiv f(z+c)$.*

In 2021, Huang, Deng and Fang [5] considered the sharing of two pairs of small functions and obtained the following theorem.

Theorem 1.8. *Let f be a transcendental entire function, and let a_1, a_2, b_1 and b_2 be four small functions of f such that $a_1 \not\equiv b_1$ and $a_2 \not\equiv b_2$, and none of them is identically equal to ∞. If f and $f^{(k)}$ share (a_1, a_2) CM and (b_1, b_2) IM, then $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} \equiv a_2 b_1 - a_1 b_2$.*

Now the following question is inevitable.

Question 1.9. Is it possible to establish a linear relationship between $f(z+c)$ and $f^{(k)}(z)$ whenever they share two pairs of small functions?

In this paper, we prove the following theorem which answers the above question.

THEOREM 1.10. *Let $k \geq 1$ be an integer and $c(\neq 0)$ be a finite complex value. Suppose that f is a transcendental entire function of finite order, and $a_1 \not\equiv \infty$, $a_2 \not\equiv \infty$, $b_1 \not\equiv \infty$, $b_2 \not\equiv \infty$ are four small functions of f such that $a_1 \not\equiv b_1$ and $a_2 \not\equiv b_2$, and a_1, b_1 are periodic functions with period c. If $f(z+c)$ and $f^{(k)}(z)$ share (a_1, a_2) CM and (b_1, b_2) IM, then*

$$(a_2 - b_2)f(z+c) - (a_1 - b_1)f^{(k)}(z) \equiv a_2b_1 - a_1b_2.$$

We provide the following example in support of Theorem 1.10.

Example 1.11. Let $f(z) = e^{2z}$, $a_1 = e^4$, $a_2 = 4$, $b_1 = 0$, $b_2 = 0$. Consider $c = 2$. Clearly, a_1, b_1 are periodic functions. Then we see that $f(z+2) - a_1 = e^4(e^{2z} - 1)$, $f''(z) - a_2 = 4(e^{2z} - 1)$, $f(z+2) - b_1 = e^4 e^{2z}$ and $f''(z) - b_2 = 4e^{2z}$. Thus $f(z+2)$ and $f''(z)$ share (a_1, a_2) CM and (b_1, b_2) IM. Clearly, we find that $(a_2 - b_2)f(z+2) - (a_1 - b_1)f''(z) \equiv a_2b_1 - a_1b_2$.

In the next two examples, we show that the conditions in Theorem 1.10 are sufficient.

Example 1.12. Let $f(z) = z + e^{2z}$, $a_1 = e^2 + z + 1$, $a_2 = 4$, $b_1 = z + 1$ and $b_2 = 0$. Consider $c = 1$. Then we see that $f(z+1) - a_1 = e^2(e^{2z} - 1)$, $f''(z) - a_2 = 4(e^{2z} - 1)$, $f(z+1) - b_1 = e^2 e^{2z}$ and $f''(z) - b_2 = 4e^{2z}$. Thus $f(z+1)$ and $f''(z)$ share (a_1, a_2) CM and (b_1, b_2) IM. Clearly, we find that $(a_2 - b_2)f(z+1) - (a_1 - b_1)f''(z) \equiv a_2b_1 - a_1b_2$.

Example 1.13. Let $f(z) = z^2 + e^{3z}$, $a_1 = e^3 + z^2 + 2z + 1$, $a_2 = 27$, $b_1 = z^2 + 2z + 1$ and $b_2 = 0$. Take $c = 1$. Then we see that $f(z+1) - a_1 = e^3(e^{3z} - 1)$, $f'''(z) - a_2 = 27(e^{3z} - 1)$, $f(z+1) - b_1 = e^3 e^{3z}$ and $f'''(z) - b_2 = 27e^{3z}$. Thus $f(z+1)$ and $f'''(z)$ share (a_1, a_2) CM and (b_1, b_2) IM. Clearly, we find that $(a_2 - b_2)f(z+1) - (a_1 - b_1)f'''(z) \equiv a_2b_1 - a_1b_2$.

The following example shows that the condition of sharing of two pairs of small functions in Theorem 1.10 is sharp.

Example 1.14. Let $f(z) = e^z$, $a_1 = e^c$, $a_2 = 1$, $b_1 = 2$ and $b_2 = 0$, where c is a nonzero finite complex value. Then $f(z+c) - a_1 = e^c(e^z - 1)$, $f''(z) - a_2 = e^z - 1$, $f(z+c) - b_1 = e^{z+c} - 2$ and $f''(z) - b_2 = e^z$. Therefore $f(z+c)$ and $f''(z)$ share (a_1, a_2) CM but do not share (b_1, b_2) IM. Now it is easy to verify that $(a_2 - b_2)f(z+c) - (a_1 - b_1)f''(z) \not\equiv a_2b_1 - a_1b_2$.

2. LEMMAS

In order to prove our results, we need the following lemmas.

LEMMA 2.1 ([3,11, Second fundamental theorem]). *Let $f(z)$ be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, a_2, \ldots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i's are small functions with respect to f for all $i = 1, \ldots, q$. Then the inequality*

$$(q-2)T(r,f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure.

LEMMA 2.2 ([13]). *Let $f_1(z)$ and $f_2(z)$ be two nonconstant meromophic functions in $|z| < \infty$. Then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right),$$

where $0 < r < \infty$.

LEMMA 2.3 ([13]). *Let $f(z)$ be a nonconstant meromorphic function, and let k be a positive integer. Then*

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r,f).$$

LEMMA 2.4 ([13]). *Let $f(z)$ be a nonconstant meromorphic function, and let n be a positive integer. Suppose that $\psi(f) = a_0 f + a_1 f' + \cdots + a_n f^{(n)}$, where $a_0,\ a_1, \ldots, a_n (\neq 0)$ are small functions of f. Then*

$$m\left(r, \frac{\psi(f)}{f}\right) = S(r,f), \quad T(r,\psi) \leq T(r,f) + k\overline{N}(r,f) + S(r,f).$$

LEMMA 2.5 ([1, 2]). *Let $f(z)$ be a meromorphic function of finite order. Then for any $c \in \mathbb{C} - \{0\}$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

LEMMA 2.6. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $c \in \mathbb{C}$. Then for any periodic small function a of f with period c, we have*

$$m\left(r, \frac{f(z+c) - a(z)}{f(z) - a(z)}\right) = S(r,f).$$

Proof. From the proof of Lemma 2.3 in [2], we have

$$m\left(r, \frac{f(z+c)-f(z)}{f(z)-a(z)}\right) = S(r,f).$$

Using this we can conclude that

$$\begin{aligned} m\left(r, \frac{f(z+c)-a(z)}{f(z)-a(z)}\right) &\le m\left(r, \frac{f(z+c)-f(z)}{f(z)-a(z)}\right) + O(1) \\ &= S(r,f). \end{aligned}$$

This completes the proof of Lemma 2.6. $\square$

LEMMA 2.7 ([1]). *Let $f(z)$ be a transcendental meromorphic function with finite order. Then*

$$T\big(r, f(z)\big) = T\big(r, f(z+c)\big) + S(r,f).$$

LEMMA 2.8. *Let $f(z)$ be a transcendental entire function, and let a_1, a_2, b_1, b_2 be four small functions of f such that $a_1 \not\equiv b_1$ and $a_2 \not\equiv b_2$. Assume that*

$$L\big(f(z+c)\big) = \begin{vmatrix} f(z+c)-a_1 & a_1-b_1 \\ f'(z+c)-a_1' & a_1'-b_1' \end{vmatrix}, \quad L\big(f^{(k)}(z)\big) = \begin{vmatrix} f^{(k)}(z)-a_2 & a_2-b_2 \\ f^{(k+1)}(z)-a_2' & a_2'-b_2' \end{vmatrix}.$$

If $f(z+c)$ and $f^{(k)}(z)$ share (a_1,a_2) CM and (b_1,b_2) IM, then $L(f(z+c)) \not\equiv 0$, $L(f^{(k)}(z)) \not\equiv 0$.

Proof. Suppose that $L(f(z+c)) \equiv 0$. Then we get

$$\frac{f'(z+c)-a_1'}{f(z+c)-a_1} = \frac{a_1'-b_1'}{a_1-b_1}.$$

Integrating, we get $f(z+c)-a_1 = c(a_1-b_1)$, where c is a nonzero constant. Therefore using Lemma 2.7, we obtain

$$\begin{aligned} T\big(r, f(z)\big) &= T\big(r, f(z+c)\big) + S(r,f) \\ &= T\big(r, c(a_1-b_1)+a_1\big) + S(r,f) \\ &= S(r,f), \end{aligned}$$

a contradiction. Hence, $L(f(z+c)) \not\equiv 0$.

Using the same argument as proving $L(f(z+c)) \not\equiv 0$, it is easy to obtain $L(f^{(k)}(z)) \not\equiv 0$. $\square$

LEMMA 2.9. *Let $f(z)$ be a transcendental entire function and a_1, b_1 be two distinct small functions of f. Suppose that for $j = 1, 2, \ldots, q$, $d_j = a_1 - l_j(a_1 - b_1)$ where l_j's are positive integers. Then*

1. $m\left(r, \frac{L(f(z+c))}{f(z+c)-a_1}\right) = S(r,f)$,

2. $m\left(r, \frac{L(f(z+c))}{f(z+c)-b_1}\right) = S(r,f)$,

3. $m\left(r, \frac{L(f(z+c))}{f(z+c)-d_j}\right) = S(r,f)$,

4. $m\left(r, \frac{L(f(z+c))f(z+c)}{(f(z+c)-d_1)(f(z+c)-d_2)\cdots(f(z+c)-d_m)}\right) = S(r,f)$,

where $L(f(z+c))$ is defined as in Lemma 2.8 and $2 \leq m \leq q$.

Proof. By Lemmas 2.3 and 2.5, we have

1.
$$\begin{aligned} m&\left(r, \frac{L(f(z+c)}{f(z+c)-a_1}\right) \\ &\leq m(r, a_1' - b_1') + m\left(r, \frac{(a_1-b_1)(f'(z+c)-a_1')}{f(z+c)-a_1}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

2.
$$\begin{aligned} m&\left(r, \frac{L(f(z+c))}{f(z+c)-b_1}\right) \\ &= m\left(r, \frac{(a_1'-b_1')(f(z+c)-a_1)-(a_1-b_1)(f'(z+c)-a_1')}{f(z+c)-b_1}\right) \\ &= m\left(r, \frac{(a_1'-b_1')(f(z+c)-b_1)-(a_1-b_1)(f'(z+c)-b_1')}{f(z+c)-b_1}\right) \\ &\leq m(r, a_1'-b_1') + m\left(r, \frac{(a_1-b_1)(f'(z+c)-b_1')}{f(z+c)-b_1}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

3.
$$\begin{aligned} m&\left(r, \frac{L(f(z+c)}{f(z+c)-d_j}\right) \\ &= m\left(r, \frac{(a_1'-b_1')(f(z+c)-a_1)-(a_1-b_1)(f'(z+c)-a_1')}{f(z+c)-d_j}\right) \\ &\leq m(r, a_1'-b_1') + m\left(r, \frac{(a_1-b_1)(f'(z+c)-(a_1'-l_j(a_1'-b_1')))}{f(z+c)-d_j}\right) \\ &\quad + S(r,f) \\ &\leq m(r, a_1'-b_1') + m\left(r, \frac{(a_1-b_1)(f'(z+c)-d_j')}{f(z+c)-d_j}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

4. We have

$$\frac{L(f(z+c))f(z+c)}{(f(z+c)-d_1)(f(z+c)-d_2)\cdots(f(z+c)-d_m)}=\sum_{i=1}^{m}\frac{c_iL(f(z+c))}{f(z+c)-d_i},$$

where $c_i=\frac{d_i}{\prod_{j\neq i}(d_i-d_j)},(i=1,2,\ldots,q)$ are small functions of f. Therefore we obtain

$$\begin{aligned}
&m\left(r,\frac{L(f(z+c))f(z+c)}{(f(z+c)-d_1)(f(z+c)-d_2)\cdots(f(z+c)-d_m)}\right)\\
&\quad=m\left(r,\sum_{i=1}^{m}\frac{c_iL(f(z+c))}{f(z+c)-d_i}\right)\\
&\quad\leq\sum_{i=1}^{m}m\left(r,\frac{L(f(z+c))}{f(z+c)-d_i}\right)+S(r,f)\\
&\quad=S(r,f).
\end{aligned}$$

□

LEMMA 2.10. *Let $f(z)$ be a transcendental entire function and a_2,b_2 be two distinct small functions of f. Suppose that for $j=1,2,\ldots,q$, $d_j=a_2-l_j(a_2-b_2)$, where l_j's are positive integers. Then*

1. $m\left(r,\frac{L(f^{(k)}(z))}{f^{(k)}(z)-a_2}\right)=S(r,f)$,
2. $m\left(r,\frac{L(f^{(k)}(z))}{f^{(k)}(z)-b_2}\right)=S(r,f)$,
3. $m\left(r,\frac{L(f^{(k)}(z))}{f^{(k)}(z)-d_j}\right)=S(r,f)$,
4. $m\left(r,\frac{L(f^{(k)}(z))f^{(k)}(z)}{(f^{(k)}(z)-d_1)(f^{(k)}(z)-d_2)\cdots(f^{(k)}(z)-d_m)}\right)=S(r,f)$,

where $L(f^{(k)}(z))$ is defined as in Lemma 2.8 and $2\leq m\leq q$.

Proof. Using the same arguments as in the proof of Lemma 2.9, the above relations can be proved easily. □

3. PROOF OF THEOREM 1.10

Proof. Assume that

$$(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1\not\equiv 0.$$

Let $L(f(z+c))$ and $L(f^{(k)}(z))$ be defined as in Lemma 2.8. Therefore by Lemma 2.8, we have $L(f(z+c))\not\equiv 0$ and $L(f^{(k)}(z))\not\equiv 0$.

Set

$$U=\frac{f(z+c)-a_1}{f^{(k)}(z)-a_2}. \tag{1}$$

Since f is a transcendental entire function and $f(z+c)$ and $f^{(k)}(z)$ share (a_1,a_2) CM, by (1), we have

$$N(r,U)=S(r,f),\quad N\left(r,\frac{1}{U}\right)=S(r,f). \tag{2}$$

As $f(z+c)$ and $f^{(k)}(z)$ share (a_1,a_2) CM and (b_1,b_2) IM and f is a transcendental entire function, by Lemmas 2.1, 2.3, 2.5 and 2.7, we obtain

$$\begin{aligned}
&T\big(r,f(z+c)\big)\\
&\quad\le \overline{N}\left(r,\frac{1}{f(z+c)-a_1}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+\overline{N}(r,f(z+c))+S(r,f)\\
&\quad=\overline{N}\left(r,\frac{1}{f^{(k)}(z)-a_2}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}(z)-b_2}\right)+S(r,f)\\
&\quad\le N\left(r,\frac{1}{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}\right)+S(r,f)\\
&\quad\le T\big(r,(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1\big)+S(r,f)\\
&\quad=m\big(r,(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1\big)+S(r,f)\\
&\quad\le m\big(r,f(z)\big)+m\left(r,\frac{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)}{f(z)}\right)+S(r,f)\\
&\quad\le T\big(r,f(z)\big)+S(r,f)\\
&\quad=T\big(r,f(z+c)\big)+S(r,f).
\end{aligned}$$

Thus, we have

$$T\big(r,f(z+c)\big)=\overline{N}\left(r,\frac{1}{f(z+c)-a_1}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+S(r,f) \tag{3}$$

and

$$\begin{aligned}
T\big(r,f(z+c)\big)&=T\big(r,(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1\big)\\
&\qquad+S(r,f)\\
&=N\left(r,\frac{1}{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}\right)\\
&\qquad+S(r,f).
\end{aligned} \tag{4}$$

Now using Lemmas 2.3, 2.6 and (2), we get

$$T(r,U)=m\left(r,\frac{1}{U}\right)+S(r,f)$$

$$
\begin{aligned}
&= m\left(r, \frac{f^{(k)}(z)-a_2}{f(z+c)-a_1}\right) + S(r,f)\\
&\le m\left(r, \frac{f^{(k)}(z)-a_1^{(k)}}{f(z)-a_1}\right) + m\left(r, \frac{f(z)-a_1}{f(z+c)-a_1}\right)\\
&\quad + m\left(r, \frac{a_1^{(k)}-a_2}{f(z+c)-a_1}\right) + S(r,f)\\
&\le m\left(r, \frac{1}{f(z+c)-a_1}\right) + S(r,f). \qquad (5)
\end{aligned}
$$

From (1) and (4), we have

$$
\begin{aligned}
m\left(r, \frac{1}{f(z+c)-a_1}\right) &= m\left(r, \frac{a_2-b_2-(a_1-b_1)U^{-1}}{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}\right)\\
&\le m\left(r, \frac{1}{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}\right)\\
&\quad + m\left(r, a_2-b_2-\frac{a_1-b_1}{U}\right) + S(r,f)\\
&\le T(r,U) + S(r,f). \qquad (6)
\end{aligned}
$$

Combining (5) and (6), we have

$$
T(r,U) = m\left(r, \frac{1}{f(z+c)-a_1}\right) + S(r,f). \qquad (7)
$$

Now (1) can be rewritten as

$$
\frac{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}{f(z+c)-a_1} = a_2-b_2-(a_1-b_1)U^{-1}.
$$

Since $f(z+c)$ and $f^{(k)}(z)$ share (b_1,b_2) IM, from above we get

$$
\begin{aligned}
\overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right) &\le \overline{N}\left(r, \frac{1}{a_2-b_2-(a_1-b_1)U^{-1}}\right)\\
&\le T(r,U) + S(r,f). \qquad (8)
\end{aligned}
$$

From (3), (7) and (8), we have

$$
\begin{aligned}
&m\left(r, \frac{1}{f(z+c)-a_1}\right) + N\left(r, \frac{1}{f(z+c)-a_1}\right)\\
&\quad = \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right) + S(r,f)\\
&\quad \le \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + m\left(r, \frac{1}{f(z+c)-a_1}\right) + S(r,f).
\end{aligned}
$$

By the above relation and (7), we obtain

$$N\left(r, \frac{1}{f(z+c)-a_1}\right) = \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + S(r,f) \tag{9}$$

and

$$\overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right) = T(r,U) + S(r,f). \tag{10}$$

Set

$$\zeta = \frac{L(f(z+c))((a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1)}{(f(z+c)-a_1)(f(z+c)-b_1)} \tag{11}$$

and

$$\eta = \frac{L(f^{(k)}(z))((a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1)}{(f^{(k)}(z)-a_2)(f^{(k)}(z)-b_2)}. \tag{12}$$

From our assumption and Lemma 2.8, we see that $\zeta \not\equiv 0$ and obviously, $N(r,\zeta) = S(r,f)$. Using Lemmas 2.3, 2.5 and 2.9, we obtain

$$\begin{aligned}
T(r,\zeta) &= m(r,\zeta) + S(r,f) \\
&= m\left(r, \frac{L(f(z+c))((a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1)}{(f(z+c)-a_1)(f(z+c)-b_1)}\right) \\
&\quad + S(r,f) \\
&\le m\left(r, \frac{L(f(z+c))f(z+c)}{(f(z+c)-a_1)(f(z+c)-b_1)}\right) \\
&\quad + m\left(r, \frac{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)}{f(z+c)}\right) \\
&\quad + m\left(r, \frac{L(f(z+c))(a_1b_2-a_2b_1)}{(f(z+c)-a_1)(f(z+c)-b_1)}\right) + S(r,f) \\
&\le m\left(r, \frac{f^{(k)}(z)}{f(z+c)}\right) + S(r,f) \\
&\le m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) + S(r,f) \\
&= S(r,f).
\end{aligned}$$

Let $d_1 = a_1 - k(a_1-b_1)$ and $d_2 = a_2 - k(a_2-b_2)$, $k \neq 0,1$. By Lemma 2.1 and (3), we get

$$\begin{aligned}
2T\big(r, f(z+c)\big) \le \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right) \\
+ \overline{N}\left(r, \frac{1}{f(z+c)-d_1}\right) + S(r,f)
\end{aligned}$$

$$\leq T\big(r, f(z+c)\big) + \overline{N}\left(r, \frac{1}{f(z+c)-d_1}\right) + S(r,f)$$
$$\leq 2T\big(r, f(z+c)\big) - m\left(r, \frac{1}{f(z+c)-d_1}\right) + S(r,f),$$

which gives

$$m\left(r, \frac{1}{f(z+c)-d_1}\right) = S(r,f). \tag{13}$$

Equation (12) can be rewritten as

(14)
$$\eta = \left[\frac{a_2-d_2}{a_2-b_2}\frac{L(f^{(k)}(z))}{f^{(k)}(z)-a_2} - \frac{b_2-d_2}{a_2-b_2}\frac{L(f^{(k)}(z))}{f^{(k)}(z)-b_2}\right]\left[\frac{(a_2-b_2)(f(z+c)-d_1)}{f^{(k)}(z)-d_2} - a_1 + b_1\right].$$

Set

$$\xi = \frac{L(f(z+c))}{(f(z+c)-a_1)(f(z+c)-b_1)} - \frac{L(f^{(k)}(z))}{(f^{(k)}(z)-a_2)(f^{(k)}(z)-b_2)}. \tag{15}$$

Now we consider the following two cases.

Case 1. Let $\xi = 0$. Integrating both sides of (15), we get

$$\frac{f(z+c)-a_1}{f(z+c)-b_1} = A\frac{f^{(k)}(z)-a_2}{f^{(k)}(z)-b_2}, \tag{16}$$

where A is a nonzero constant.

Subcase 1.1. If $A = 1$, then from (16), we get

$$(a_2-b_2)f(z+c) - (a_1-b_1)f^{(k)}(z) + a_1b_2 - a_2b_1 = 0,$$

which contradicts our initial assumption.

Subcase 1.2. If $A \neq 1$, then from (16), we have

$$\frac{a_2-b_2}{f^{(k)}(z)-a_2} = \frac{(A-1)f(z+c)-(Ab_1-a_1)}{f(z+c)-a_1} \tag{17}$$

and

$$T\big(r, f(z+c)\big) = T\big(r, f^{(k)}(z)\big) + S(r,f).$$

Now it is easy to see that $\frac{Ab_1-a_1}{A-1} \neq a_1$, $\frac{Ab_1-a_1}{A-1} \neq b_1$. Since $f(z+c)$ and $f^{(k)}(z)$ share (a_1,a_2) CM, by (17), we get

$$N\left(r, \frac{1}{f(z+c)-\frac{Ab_1-a_1}{A-1}}\right) = N\left(r, \frac{1}{a_2-b_2}\right) = S(r,f).$$

Then by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}2T\big(r,f(z+c)\big) &\leq \overline{N}\left(r,\frac{1}{f(z+c)-a_1}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{f(z+c)-\frac{Ab_1-a_1}{A-1}}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f(z+c)-a_1}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+S(r,f),\end{aligned}$$

a contradiction to (3).

Case 2. Consider $\xi\not\equiv 0$. Clearly, $m(r,\xi)=S(r,f)$. Since $f(z+c)$ and $f^{(k)}(z)$ share (a_1,a_2) CM and (b_1,b_2) IM, all poles of ξ must come from the zeros of $f(z+c)-b_1$. So we have

$$\begin{aligned}T(r,\xi)&=m(r,\xi)+N(r,\xi)\\ &\leq \overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+S(r,f).\end{aligned}\tag{18}$$

Using (11), (12), (15), (18) and noting that $T(r,\zeta)=S(r,f)$, we have

$$\begin{aligned}m\big(r,f(z+c)\big)&=m\big(r,(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1\big)\\ &\quad+S(r,f)\\ &=m\left(r,\frac{\xi((a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1)}{\xi}\right)\\ &\quad+S(r,f)\\ &=m\left(r,\frac{\zeta-\eta}{\xi}\right)+S(r,f)\\ &\leq T\left(r,\frac{\xi}{\zeta-\eta}\right)+S(r,f)\\ &\leq T(r,\zeta-\eta)+T(r,\xi)+S(r,f)\\ &\leq T(r,\eta)+\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+S(r,f).\end{aligned}\tag{19}$$

Again, it is easy to obtain that $N(r,\eta)=S(r,f)$. By Lemma 2.10, (1), (9) and (10), we have

$$\begin{aligned}T(r,\eta)&=m\left(r,\frac{L(f^{(k)}(z))((a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1)}{(f^{(k)}(z)-a_2)(f^{(k)}(z)-b_2)}\right)\\ &\quad+S(r,f)\\ &\leq m\left(r,\frac{L(f^{(k)}(z))}{f^{(k)}(z)-b_2}\right)\end{aligned}$$

$$
\begin{aligned}
&+m\left(r, \frac{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}{f^{(k)}(z)-a_2}\right)+S(r,f)\\
&\leq m\left(r, \frac{(a_2-b_2)f(z+c)-(a_1-b_1)f^{(k)}(z)+a_1b_2-a_2b_1}{f(z+c)-a_1}\frac{f(z+c)-a_1}{f^{(k)}(z)-a_2}\right)\\
&\quad +S(r,f)\\
&\leq m\big(r,(a_2-b_2)U-(a_1-b_1)\big)+S(r,f)\\
&\leq m\left(r, \frac{1}{f(z+c)-a_1}\right)+S(r,f)
\end{aligned}
$$

(20)

$$
= \overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right)+S(r,f).
$$

Combining (19) and (20), we get

$$
T\big(r, f(z+c)\big) \leq 2\overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right)+S(r,f). \tag{21}
$$

Now, if $a_1^{(k)} \equiv a_2$, then using Lemmas 2.3, 2.6, (1) and (2), we get

$$
\begin{aligned}
T(r,U) &= m\left(r, \frac{1}{U}\right)+S(r,f)\\
&= m\left(r, \frac{f^{(k)}-a_1^{(k)}}{f(z+c)-a_1}\right)+S(r,f)\\
&\leq m\left(r, \frac{f^{(k)}-a_1^{(k)}}{f(z)-a_1}\right)+m\left(r, \frac{f(z)-a_1}{f(z+c)-a_1}\right)+S(r,f)\\
&= S(r,f).
\end{aligned}
$$

Therefore from (10) and (21), we have $T(r,f(z+c)) = S(r,f)$, a contradiction. If $a_1^{(k)} \equiv b_2$, then from Lemma 2.6, (7), (10) and (21), we obtain

$$
\begin{aligned}
T\big(r, f(z+c)\big) &\leq m\left(r, \frac{1}{f(z+c)-a_1}\right)+\overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right)+S(r,f)\\
&\leq m\left(r, \frac{1}{f(z)-a_1}\right)+m\left(r, \frac{f(z)-a_1}{f(z+c)-a_1}\right)+\overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right)\\
&\quad +S(r,f)\\
&\leq m\left(r, \frac{1}{f^{(k)}(z)-b_2}\right)+N\left(r, \frac{1}{f^{(k)}(z)-b_2}\right)+S(r,f)\\
&\leq T\big(r, f^{(k)}(z)\big)+S(r,f).
\end{aligned} \tag{22}
$$

Again, it follows from Lemmas 2.4 and 2.7 that

$$
T\big(r, f^{(k)}(z)\big) \leq T\big(r, f(z)\big)+S(r,f)
$$

$$= T\big(r, f(z+c)\big) + S(r,f). \tag{23}$$

Combining (22) and (23), we get

$$T\big(r, f(z+c)\big) = T\big(r, f^{(k)}(z)\big) + S(r,f). \tag{24}$$

Therefore by Nevanlinna's second fundamental theorem, (3) and (24), we have

$$\begin{aligned}
2T\big(r, f(z+c)\big) &= 2T\big(r, f^{(k)}(z)\big) + S(r,f)\\
&\leq \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a_2}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-b_2}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) + S(r,f)\\
&\leq \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right)\\
&\quad + T\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) - m\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) + S(r,f)\\
&\leq T\big(r, f(z+c)\big) + T\big(r, f^{(k)}(z)\big) - m\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) + S(r,f)\\
&\leq 2T\big(r, f(z+c)\big) - m\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) + S(r,f).
\end{aligned}$$

This implies

$$m\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) = S(r,f). \tag{25}$$

Now, by Nevanlinna's first fundamental theorem, Lemmas 2.2, 2.3, 2.6, (13), (24) and (25), we obtain

$$\begin{aligned}
m\left(r, \frac{f(z+c)-d_1}{f^{(k)}(z)-d_2}\right) &= T\left(r, \frac{f^{(k)}(z)-d_2}{f(z+c)-d_1}\right) - N\left(r, \frac{f(z+c)-d_1}{f^{(k)}(z)-d_2}\right) + O(1)\\
&\leq m\left(r, \frac{f^{(k)}(z)-d_1^{(k)}}{f(z)-d_1}\right) + m\left(r, \frac{f(z)-d_1}{f(z+c)-d_1}\right)\\
&\quad + m\left(r, \frac{d_1^{(k)}-d_2}{f(z+c)-d_1}\right) + N\left(r, \frac{f^{(k)}(z)-d_2}{f(z+c)-d_1}\right)\\
&\quad - N\left(r, \frac{f(z+c)-d_1}{f^{(k)}(z)-d_2}\right) + O(1)\\
&\leq N\left(r, \frac{1}{f(z+c)-d_1}\right) - N\left(r, \frac{1}{f^{(k)}(z)-d_2}\right) + S(r,f)\\
&\leq T\big(r, f(z+c)\big) - T\big(r, f^{(k)}(z)\big) + S(r,f)\\
&= S(r,f).
\end{aligned}$$

Thus, using the above inequality and Lemma 2.10, we get

$$\begin{aligned}
T(r,\eta) &= m(r,\eta)+N(r,\eta)\\
&\le m\left(r,\frac{a_2-d_2}{a_2-b_2}\frac{L(f^{(k)}(z))}{f^{(k)}(z)-a_2}\right)+m\left(r,\frac{b_2-d_2}{a_2-b_2}\frac{L(f^{(k)}(z))}{f^{(k)}(z)-b_2}\right)\\
&\quad+m\left(r,\frac{(a_2-b_2)(f(z+c)-d_1)}{f^{(k)}(z)-d_2}-a_1+b_1\right)+S(r,f)\\
&= S(r,f). \qquad (26)
\end{aligned}$$

Now it follows from (3), (9), (19) and (26) that

$$N\left(r,\frac{1}{f(z+c)-a_1}\right)=\overline{N}\left(r,\frac{1}{f(z+c)-a_1}\right)+S(r,f)=S(r,f). \qquad (27)$$

Again by (3), (24) and (27), we obtain

$$\begin{aligned}
m\left(r,\frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) &= m\left(r,\frac{1}{f^{(k)}(z)-b_2}\right)\\
&\le T\big(r,f^{(k)}(z)\big)-\overline{N}\left(r,\frac{1}{f^{(k)}(z)-b_2}\right)+S(r,f)\\
&= T\big(r,f(z+c)\big)-\overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)+S(r,f)\\
&= S(r,f). \qquad (28)
\end{aligned}$$

Using Lemma 2.6 and (28), we have

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) &= \overline{N}\left(r,\frac{1}{f^{(k)}(z)-b_2}\right)\\
&= \overline{N}\left(r,\frac{1}{f(z+c)-b_1}\right)\\
&= m\left(r,\frac{1}{f(z+c)-a_1}\right)+S(r,f)\\
&\le m\left(r,\frac{1}{f(z)-a_1}\right)+m\left(r,\frac{f(z)-a_1}{f(z+c)-a_1}\right)+S(r,f)\\
&\le m\left(r,\frac{1}{f^{(k)}(z)-a_1^{(k)}}\right)+S(r,f)\\
&= S(r,f). \qquad (29)
\end{aligned}$$

Hence by (3), (27) and (29), we get $T(r,f(z+c))=S(r,f)$, a contradiction.

Consequently, we have $a_1^{(k)}\not\equiv a_2$ and $a_1^{(k)}\not\equiv b_2$. Therefore, from Nevanlinna's second fundamental theorem, Lemma 2.6, (3), (7), (10), (21) and the fact that $f(z+c)$ and $f^{(k)}(z)$ share (a_1, a_2) CM and (b_1, b_2) IM, we obtain

$$\begin{aligned}
T\big(r, f(z+c)\big) &\leq 2m\left(r, \frac{1}{f(z+c)-a_1}\right) + S(r,f) \\
&\leq 2m\left(r, \frac{1}{f(z)-a_1}\right) + 2m\left(r, \frac{f(z)-a_1}{f(z+c)-a_1}\right) + S(r,f) \\
&\leq 2m\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) + S(r,f) \\
&\leq 2T\big(r, f^{(k)}(z)\big) - 2N\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) + S(r,f) \\
&\leq \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a_2}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-b_2}\right) \\
&+ \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) - 2N\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) + S(r,f) \\
&\leq T\big(r, f(z+c)\big) - N\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) + S(r,f).
\end{aligned}$$

This gives

$$N\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) = S(r,f). \tag{30}$$

Now (1) can be rewritten as

$$f(z+c) - a_1 = U\big(f^{(k)}(z) - a_1^{(k)}\big) + U(a_1^{(k)} - a_2). \tag{31}$$

Let $h = f^{(k)}(z) - a_1^{(k)}$. Differentiating (31) k times, we get

$$f^{(k)}(z+c) - a_1^{(k)} = U^{(k)}h + kU^{(k-1)}h' + \cdots + kU'h^{(k-1)} + Uh^{(k)} + B^{(k)}, \tag{32}$$

where $B = U(a_1^{(k)} - a_2)$. Clearly $h \not\equiv 0$. Rewriting (32), we have

$$hU(gU^{-1} - D) = B^{(k)}, \tag{33}$$

where

$$g = \frac{f^{(k)}(z+c) - a_1^{(k)}}{f^{(k)}(z) - a_1^{(k)}}$$

and

$$D = \frac{U^{(k)}}{U} + \frac{kU^{(k-1)}h'}{hU} + \cdots + \frac{kU'h^{(k-1)}}{hU} + \frac{h^{(k)}}{h}. \tag{34}$$

Again from (30), we get $N(r, \frac{1}{h}) = S(r,f)$. Now it follows from (2) and (34) that

$$T(r,D) \le \sum_{i=1}^{k}\Big(T\Big(r, \frac{U^{(i)}}{U}\Big) + T\Big(r, \frac{h^{(i)}}{h}\Big)\Big) + S(r,f)$$
$$\le \sum_{i=1}^{k}\Big(m\Big(r, \frac{U^{(i)}}{U}\Big) + N\Big(r, \frac{U^{(i)}}{U}\Big) + m\Big(r, \frac{h^{(i)}}{h}\Big) + N\Big(r, \frac{h^{(i)}}{h}\Big)\Big)$$
$$+ S(r,f)$$
$$= S(r,U) + S(r,f). \tag{35}$$

Using Lemmas 2.2, 2.7 and (31), we get

$$T(r,U) \le T\big(r, f(z+c)\big) + T\big(r, f^{(k)}(z)\big) + S(r,f)$$
$$\le 2T(r,f) + S(r,f). \tag{36}$$

So from (35) and (36), we have $T(r,D) = S(r,f)$.

Now we consider the following two subcases.

Subcase 2.1. Let $gU^{-1} - D \not\equiv 0$. We claim that $D \equiv 0$. Otherwise, from (33), we see that $N(r, \frac{1}{U^{-1}-Dg^{-1}}) = S(r,f)$. Now applying Nevanlinna's first and second fundamental theorem, (2) and (36), we obtain

$$T(r,U) = T(r,U^{-1}) + O(1)$$
$$\le \overline{N}(r,U^{-1}) + \overline{N}\Big(r, \frac{1}{U^{-1}}\Big) + \overline{N}\Big(r, \frac{1}{U^{-1} - Dg^{-1}}\Big) + S(r,U)$$
$$\le S(r,f).$$

Hence from (10) and (21), we get $T(r, f(z+c)) = S(r,f)$, a contradiction. Thus $D \equiv 0$. Therefore from (33), we get

$$f^{(k)}(z+c) - a_1^{(k)} = B^{(k)}.$$

Integrating, we obtain

$$f(z+c) = U(a_1^{(k)} - a_2) + P(z) + a_1,$$

where $P(z)$ is a polynomial of degree at most $k-1$. Therefore

$$T\big(r, f(z+c)\big) = T(r,U) + S(r,f). \tag{37}$$

Since $f(z+c)$ and $f^{(k)}(z)$ share (a_1, a_2) CM, it follows from (3), (10) and (37) that

$$\overline{N}\Big(r, \frac{1}{f^{(k)}(z) - a_2}\Big) = \overline{N}\Big(r, \frac{1}{f(z+c) - a_1}\Big) = S(r,f). \tag{38}$$

Now applying Nevanlinna's second fundamental theorem to $f^{(k)}(z)$ and using (30) and (38), we obtain

$$\begin{aligned} T\big(r, f^{(k)}(z)\big) &\leq \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a_2}\right) + N\left(r, \frac{1}{f^{(k)}(z)-a_1^{(k)}}\right) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{39}$$

As $f(z+c))$ and $f^{(k)}(z)$ share (a_1, a_2) CM and (b_1, b_2) IM, from (3) and (39), we get

$$\begin{aligned} T\big(r, f(z+c)\big) &= \overline{N}\left(r, \frac{1}{f(z+c)-a_1}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b_1}\right) + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a_2}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-b_2}\right) + S(r,f) \\ &\leq 2T\big(r, f^{(k)}(z)\big) + S(r,f) \\ &= S(r,f), \end{aligned}$$

a contradiction.

Subcase 2.2. Let $gU^{-1} - D \equiv 0$. Then from the equations (35) and (36), we have $T(r,U) \leq T(r,D) + T(r,g) = S(r,f)$. So from (10) and (21), we get $T(r, f(z+c)) = S(r,f)$, which gives a contradiction.

This completes the proof of Theorem 1.10. □

Acknowledgments. The authors would like to thank the referee for the helpful comments.

REFERENCES

[1] Y.-M. Chiang and S.-J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*. Ramanujan J. **16** (2008), *1*, 105–129.

[2] R.G. Halburd and R.J. Korhonen, *Nevanlinna theory for the difference operator*. Ann. Acad. Sci. Fenn. Math. **31** (2006), *2*, 463–478.

[3] W.K. Hayman, *Meromorphic Function*. Oxford Math. Monogr., Clarendon Press, Oxford, 1964.

[4] X.H. Huang, *Uniqueness of meromorphic functions with respect to their shifts concerning derivatives*. Izv. Nats. Akad. Nauk Armenii Mat. **59** (2024), *2*, 35–55.

[5] X.H. Huang, B. Deng and M.L. Fang, *Entire functions that share two pairs of small functions*. Open Math. **19** (2021), *1*, 144–156.

[6] I. Laine, *Nevanlinna Theory and Complex Differential Equations*. De Gruyter Stud. Math. 15, Walter de Gruyter & Co., Berlin, 1993.

[7] P. Li and C.-C. Yang, *Value sharing of an entire function and its derivatives*. J. Math. Soc. Japan **51** (1999), *4*, 781–799.

[8] E. Mues and N. Steinmetz, *Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen*. Manuscripta Math. **29** (1979), *2–4*, 195–206.

[9] X. Qi and L. Yang, *Uniqueness of meromorphic functions concerning their shifts and derivatives*. Comput. Methods Funct. Theory **20** (2020), *1*, 159–178.

[10] G. Qiu, *Uniqueness of entire functions that share some small functions*. Kodai Math. J. **23** (2000), *1*, 1–11.

[11] M. Ru, *Nevanlinna Theory and its Relation to Diophantine Approximation*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[12] L.A. Rubel and C.C. Yang, *Values shared by an entire function and its derivative*. Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, KY, 1976), pp. 101–103. Lecture Notes in Math. 599, Springer, Berlin, New York, 1977.

[13] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*. Mathematics and its Applications 557, Kluwer Academic Publishers, Dordrecht, 2003.

[14] J.H. Zheng and S.P. Wang, *Unicity of meromorphic functions and their derivatives*. Adv. in Math. (China) **21** (1992), *3*, 334–341.

Received 22 December 2023

Pulak Sahoo
Soniya Sultana
Department of Mathematics,
University of Kalyani,
West Bengal-741235, India
sahoopulak1@gmail.com
soniyasultana3@gmail.com