STABILITY OF THE ESSENTIAL SPECTRUM OF TRAVELING WAVES IN DIFFUSIVE HOLLING TYPE-II PREDATOR-PREY MODEL

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In this study, we investigate the results obtained by adding the diffusion term to a model consisting of predator and prey. The existence of traveling waves has been proved for this model in previous studies. In the present work, spectrum analysis of traveling waves under the Holling type II functional response is performed. For this, critical points that make our model biologically meaningful are given first. Then the model is linearized around a traveling wave by moving it to the moving coordinates. The essential spectrum is determined by identifying the operator in the linear system and applying the Fourier transform to the operator. The existence of a weight function is investigated to stabilize the essential spectrum.

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1. INTRODUCTION

The interaction dynamics between predators and their prey have been a prominent and widely studied subject in ecology and mathematical ecology, owing to its universal presence and significance [2]. Although these problems might seem mathematically simple at first glance, they are often highly challenging and complex.

The Lotka–Volterra equations describe the dynamics of biological systems involving interactions between two species: a predator and a prey. These equations are a pair of first-order nonlinear differential equations frequently used to model predator-prey dynamics.

Many studies have focused on the Lotka–Volterra system, which forms the basis for modeling complex environmental and ecological systems [1,11,20]. One notable contributor to this field is C. S. Holling, who developed the concept of functional response, which describes the relationship between prey density and the rate at which prey are consumed. This concept is fundamental

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to modern population ecology. Holling proposed three different types of functional responses to make the standard Lotka–Volterra system more realistic [9]: Type I, where there is a linear relationship between prey density and the maximum number of prey killed; Type II, where the rate of prey consumption decreases monotonically with increasing prey density; and Type III, characterized by a sigmoid relationship where the rate of prey consumption initially increases positively with prey density.

This study focuses on traveling waves in diffusive Holling type-II predator-prey model. There has been a growing interest in traveling wave solutions of partial differential equations (see, for example, [6–8, 12, 13, 15–17]). A traveling wave is a solution to a PDE that preserves its shape as it moves at a constant speed. This form represents a wave profile that moves invariantly in a particular direction at a constant velocity. In the context of biological models, traveling wave solutions can illustrate the spread of species across a spatial habitat.

In the predator-prey model with functional response, extensive investigation into the existence of traveling waves was conducted in the studies referenced in [3–5, 10]. In this study, analysis of stability of essential spectra is conducted for the traveling waves in diffusive Holling functional response type-II predator-prey model. Unlike many prior works that focus on the existence of traveling waves, this paper uniquely investigates the essential spectrum stability of these waves when diffusion terms are added. This study extends the classical predator-prey models by incorporating spatial dynamics, which is particularly relevant for ecological and biological systems. While the Holling type-II functional response has been explored extensively in models [14, 19, 22], this study provides a deeper investigation into spectral analysis under this response, connecting it with stability criteria. Other papers typically emphasize broader functional responses or equilibrium stability rather than the specific stability of the traveling waves' spectrum.

A novel feature of this work is the use of a weight function to shift the spectrum from the right to the left half-plane, stabilizing the traveling waves. This approach addresses instability in the right state of the spectrum, which has not been the focus of other works [18, 21]. The study combines operatorial methods and Fourier transform techniques to derive the essential spectrum and establish stability conditions. This blend of methods contributes new insights to research.

Section 2 introduces the mathematical model and obtains critical points. In Section 3, the model is translated into moving coordinates, and the operator around the traveling wave is linearized for spectral analysis. The essential spectrum is determined by identifying the operator in the linear system and applying the Fourier transform to the operator. The existence of a weight function that shifts the essential spectrum to the left half-plane is investigated. Section 4 presents the results.

2. MODEL

The predator-prey model with a Holling type II functional response is the ordinary differential equation system

$$u_t = u(1-u) - \nu f(u), \tag{1}$$

$$\nu_t = \beta \nu f(u) - \gamma \nu, \tag{2}$$

where $u(x,t)$ and $\nu(x,t)$ represent the population densities of prey and predator, respectively. The parameters β and γ are strictly positive numbers. $f(u)$ is the functional response that is a C^2 function, $f(0) = 0$, $\lim_{u\to\infty} f(u) = 1$ and $f(\cdot)$ is strictly increasing on $[0, \infty)$. According to the classification of Holling [9], type II functional response with positive α is $\frac{u}{u+\alpha}$. This function $f(\cdot)$ describes the rate at which predators consume prey as a proportion of the maximum possible consumption rate. Notably, this model considers the interaction where predators invade the prey population but excludes stochastic effects and environmental influences.

In this study, we investigate the results obtained when diffusive terms are added to the equations (1) and (2). Reaction-diffusion equations that model predator-prey interactions exhibit a broad range of ecologically significant behaviors driven solely by intrinsic factors.

Adding diffusion terms with non-negative coefficients ϵ_u and ϵ_ν to the model and substituting functional response $f(u) = \frac{u}{u+\alpha}$ into equations (1) and (2) yields

$$u_t = \epsilon_u u_{xx} + u(1-u) - \frac{u}{u+\alpha}\nu, \tag{3}$$

$$\nu_t = \epsilon_\nu \nu_{xx} + \beta \frac{u}{u+\alpha}\nu - \gamma\nu. \tag{4}$$

The system is considered in a bounded domain Ω with appropriate initial and zero-flux boundary conditions.

2.1. Critical points

The critical points of equations (1) and (2) are also the spatially homogeneous critical points of equations (3) and (4). To be biologically meaningful, the critical points must satisfy $u \geq 0$ and $\nu \geq 0$. By solving the critical points, we get

$$u(1-u) - \frac{u}{u+\alpha}\nu = 0 \quad \text{and} \quad \beta\frac{u}{u+\alpha}\nu - \gamma\nu = 0.$$

Yielding the critical points $(u,\nu)=(0,0)$, $(u,\nu)=(1,0)$, $(u,\nu)=(-\alpha,0)$ and $(\hat{u},\hat{\nu})$ where

$$\hat{u}=\frac{\alpha\gamma}{\beta-\gamma}\quad(\gamma<\beta),\quad \text{and}\quad \hat{\nu}=\frac{\beta\alpha(\beta-\gamma-\alpha\gamma)}{(\beta-\alpha)^2}\quad\left(\alpha<\frac{\beta-\gamma}{\gamma}\right).$$

We get four critical points but only $(u,\nu)=(1,0)$ and the equilibrium point $(u,\nu)=(\hat{u},\hat{\nu})$ are biologically meaningful. $(1,0)$ represents a state where the prey population is at carrying capacity and predators are absent and $(\hat{u},\hat{\nu})$ corresponds to the coexistence of prey and predators under the conditions $\gamma<\beta$ and $\alpha<\frac{\beta-\gamma}{\gamma}$. In the remaining study, we analyze the spectrums for the saddle point $(1,0)$ and the equilibrium point $(\hat{u},\hat{\nu})$.

3. SPECTRAL ANALYSIS FOR PREDATOR-PREY MODEL

A traveling wave with left state (u^-,ν^-), right state (u^+,ν^+) and moving with velocity c can be demonstrated as $(u^-,\nu^-)\xrightarrow{c}(u^+,\nu^+)$. In this section, we derive a traveling wave by transforming the model (3)–(4) into moving coordinates. Then, we analyze the linearized operator's spectrum after linearizing the equations around the traveling wave.

By performing the transformation $\xi=x-ct$ on equations (3) and (4), we get

$$u_t=\epsilon_u u_{\xi\xi}+cu_\xi+u(1-u)-\frac{u}{u+\alpha}\nu, \tag{5}$$

$$\nu_t=\epsilon_\nu \nu_{\xi\xi}+c\nu_\xi+\beta\frac{u}{u+\alpha}\nu-\gamma\nu. \tag{6}$$

The stationary solutions of (5)–(6) are the traveling waves with velocity c of (3)–(4). Let $D^*(\xi)=(u^*(\xi),\nu^*(\xi))$ with velocity c be the traveling wave solution of (3)–(4) connecting the coexistence equilibrium $(\hat{u},\hat{\nu})$ and the saddle point $(1,0)$. $D^*(\xi)$ describes the invading front of the predator and satisfies the following boundary conditions and the ODE system:

$$\lim_{\xi\to-\infty}D^*(\xi)=D^-=(\hat{u},\hat{\nu}),\qquad \lim_{\xi\to+\infty}D^*(\xi)=D^+=(1,0). \tag{7}$$

$$0=\epsilon_u u^*_{\xi\xi}+cu^*_\xi+u^*(1-u^*)-\frac{u^*}{u^*+\alpha}\nu^*, \tag{8}$$

$$0=\epsilon_\nu \nu^*_{\xi\xi}+c\nu^*_\xi+\beta\frac{u^*}{u^*+\alpha}\nu^*-\gamma\nu^*. \tag{9}$$

Linearizing equations (5)–(6) around $D^*(\xi)$, we obtain

$$\tilde{u}_{\mathrm{t}}=\epsilon_u\tilde{u}_{\xi\xi}+c\tilde{u}_\xi+\left(1-2u^*-\frac{\alpha\nu^*}{(u^*+\alpha)^2}\right)\tilde{u}-\frac{u^*}{u^*+\alpha}\tilde{\nu}, \tag{10}$$

$$\tilde{\nu}_{\rm t} = \epsilon_\nu \tilde{\nu}_{\xi\xi} + c\tilde{\nu}_\xi + \frac{\beta\alpha\nu^*}{(u^*+\alpha)^2}\tilde{u} + \left(\frac{\beta u^*}{u^*+\alpha} - \gamma\right)\tilde{\nu}. \tag{11}$$

The system (10)–(11) can be written as $U_t = \mathcal{L}U$ where $U = (\tilde{u}, \tilde{\nu})$ and $\mathcal{L}$ is a matrix operator

$$\mathcal{L} = \begin{pmatrix} \epsilon_u\partial_{\xi\xi} + c\partial_\xi + 1 - 2u^* - \frac{\alpha\nu^*}{(u^*+\alpha)^2} & -\frac{u^*}{u^*+\alpha} \\ \frac{\beta\alpha\nu^*}{(u^*+\alpha)^2} & \epsilon_\nu\partial_{\xi\xi} + c\partial_\xi + \frac{\beta u^*}{u^*+\alpha} - \gamma \end{pmatrix}.$$

Definition 3.1. If the spectrum of $\mathcal{L}$ satisfies the following conditions:

- For $\mu > 0$, if it lies within the half-plane $\{\mathrm{Re}\lambda < -\mu\}$, then D^* is stable.
- If it lies within the left half-plane including the imaginary axis, then D^* is marginally stable.
- If it includes points where the real part of λ is greater than 0, then D^* is unstable.

LEMMA 3.2. *In the L^2 space, for $D^*(\xi)$ to be spectrally stable certain conditions must be met. These conditions are defined as follows:*

1. *$\mathcal{L}$ has eigenvalue $\lambda = 0$. The corresponding eigenfunction is $D^*(\xi)'$.*
2. *For $\mu > 0$, the rest of the spectrum of $\mathcal{L}$ lies within $Re\lambda < -\mu$.*

3.1. **The spectrum of traveling waves in the predator-prey model**

The spectrum of $\mathcal{L}$, denoted as $\mathrm{Sp}(\mathcal{L})$, can be divided into two parts: $\mathrm{Sp}_{\rm ess}(\mathcal{L})$ called essential spectrum and $\mathrm{Sp}_{\rm d}(\mathcal{L})$ called discrete spectrum. In this section, we focus on the study of $\mathrm{Sp}_{\rm ess}(\mathcal{L})$.

By linearizing (5)–(6) at the left state $D^- = (u^-, \nu^-)$ and the right state $D^+ = (u^+, \nu^+)$, one obtains a set of interrelated constant coefficient linear partial differential equations of the form $U_t = \mathcal{L}^\pm U$. $Sp(\mathcal{L}^-)$ and $Sp(\mathcal{L}^+)$ are computed by applying the Fourier transform in the L^2 space.

$$\hat{\mathcal{L}}^\pm = \begin{pmatrix} -\epsilon_u\mu^2 + i\mu c + 1 - 2u^\pm - \frac{\alpha\nu^\pm}{(u^\pm+\alpha)^2} & -\frac{u^\pm}{u^\pm+\alpha} \\ \frac{\beta\alpha\nu^\pm}{(u^\pm+\alpha)^2} & -\epsilon_\nu\mu^2 + i\mu c + \frac{\beta u^\pm}{u^\pm+\alpha} - \gamma \end{pmatrix}.$$

Since the essential spectrum of $\mathcal{L}$ is the union of $\mathrm{Sp}(\mathcal{L}^+)$ and $\mathrm{Sp}(\mathcal{L}^-)$, we analyze the spectrum at the right state $(u^+, \nu^+) = (1, 0)$ and at the left state $(u^-, \nu^-) = (\hat{u}, \hat{\nu})$.

3.1.1 Spectrum of the right state

At the equilibrium point $(u^+, \nu^+) = (1, 0)$, we obtain

$$\hat{\mathcal{L}}^+ = \begin{pmatrix} -\epsilon_u\mu^2 + i\mu c - 1 & -\frac{1}{1+\alpha} \\ 0 & -\epsilon_\nu\mu^2 + i\mu c + \frac{\beta}{1+\alpha} - \gamma \end{pmatrix}. \tag{12}$$

For some real number μ, collection of all λ that are eigenvalues of (12) creates the spectrum of $\hat{\mathcal{L}}^+$ in L^2. Therefore, eigenvalue problem of (12) is

$$\det(\hat{\mathcal{L}}^+ - \lambda I) = \det\begin{pmatrix} -\epsilon_u\mu^2 + i\mu c - 1 - \lambda & -\frac{1}{1+\alpha} \\ 0 & -\epsilon_\nu\mu^2 + i\mu c + \frac{\beta}{1+\alpha} - \gamma - \lambda \end{pmatrix}.$$

The eigenvalues associated with μ are

$$\lambda(\mu) = -\epsilon_u\mu^2 + i\mu c - 1,$$
$$\lambda(\mu) = -\epsilon_\nu\mu^2 + i\mu c + \frac{\beta}{1+\alpha} - \gamma.$$

Hence, the spectrum of the linearized system at (u^+, ν^+) consists of a parabola in the left half-plane and a parabola in the right half-plane. See Figure 1.

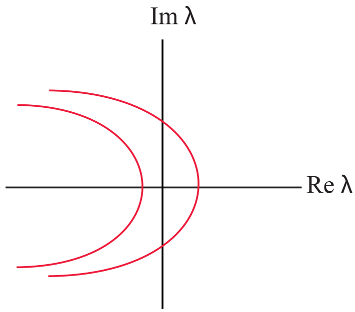

Figure 1 – $Sp(\hat{\mathcal{L}}^+)$ for the equilibrium point $(u^+, \nu^+) = (1, 0)$.

3.1.2 Spectrum of the left state

At the equilibrium point $(u^-, \nu^-) = (\frac{\alpha\gamma}{\beta-\gamma}, \frac{\beta\alpha(\beta-\gamma-\alpha\gamma)}{(\beta-\alpha)^2})$ with $\gamma < \beta$ and $\alpha < \frac{\beta-\gamma}{\gamma}$, we obtain

$$\hat{\mathcal{L}}^- = \begin{pmatrix} -\epsilon_u\mu^2 + i\mu c + 1 - 2\frac{\alpha\gamma}{\beta-\gamma} - \frac{\beta-\gamma-\alpha\gamma}{\beta} & -\frac{\gamma}{\beta} \\ \beta - \gamma - \alpha\gamma & -\epsilon_\nu\mu^2 + i\mu c \end{pmatrix}. \tag{13}$$

For some real number μ, collection of all λ that are eigenvalues of (13) creates the spectrum of $\hat{\mathcal{L}}^-$ in L^2. Therefore, eigenvalue problem of (13) is

$$\det(\hat{\mathcal{L}}^- - \lambda I) = \det\begin{pmatrix} -\epsilon_u\mu^2 + i\mu c + 1 - 2\frac{\alpha\gamma}{\beta-\gamma} - \frac{\beta-\gamma-\alpha\gamma}{\beta} - \lambda & -\frac{\gamma}{\beta} \\ \beta-\gamma-\alpha\gamma & -\epsilon_\nu\mu^2 + i\mu c - \lambda \end{pmatrix}.$$

Then, the characteristic equation is

$$\begin{aligned} \lambda^2 &+ \left((\epsilon_u+\epsilon_\nu)\mu^2 - 2i\mu c - 1 + \frac{2\alpha\gamma}{\beta-\gamma} + \frac{\beta-\gamma-\alpha\gamma}{\beta}\right)\lambda \\ &+ \epsilon_u\epsilon_\nu\mu^4 - (\epsilon_u+\epsilon_\nu)i\mu^3 c \\ &- \left(\epsilon_\nu - \epsilon_\nu\frac{2\alpha\gamma}{\beta-\gamma} - \epsilon_\nu\frac{\beta-\gamma-\alpha\gamma}{\beta} + c^2\right)\mu^2 \\ &+ \left(1 - \frac{2\alpha\gamma}{\beta-\gamma} - \frac{\beta-\gamma-\alpha\gamma}{\beta}\right)i\mu c \\ &+ \gamma\frac{\beta-\gamma-\alpha\gamma}{\beta} = 0. \end{aligned}$$

To show the spectrum of the linearized system at (u^-, ν^-) lies in the left half-plane, we give the following proposition.

PROPOSITION 3.3. *Given $\alpha, \beta, \gamma > 0$ with $\alpha \geq 1$, $\beta-\gamma-\alpha\gamma > \frac{\alpha\beta}{2}$, and $\frac{4\beta(\beta-\gamma)}{\gamma} \geq 1$. In this case, the spectrum of the operator* (13) *consists of two parabolas located in the left half-plane.*

Proof. Given $\alpha \geq 1$, $\beta-\gamma-\alpha\gamma > \frac{\alpha\beta}{2}$, and $\frac{4\beta(\beta-\gamma)}{\gamma} \geq 1$, let the eigenvalues of (13) be $\lambda_1(\mu)$ and $\lambda_2(\mu)$. In this case, let us take $\lambda_1 = -\epsilon_u\mu^2 + i\mu c + a$ and $\lambda_2 = -\epsilon_\nu\mu^2 + i\mu c + b$. Equate the characteristic equation of (13) to the product of $\lambda_1(\mu) - \lambda$ and $\lambda_2(\mu) - \lambda$ and solve for the values of a and b. Then we obtain

$$\begin{aligned} a + b &= 1 - 2\frac{\alpha\gamma}{\beta-\gamma} - \frac{\beta-\gamma-\alpha\gamma}{\beta}, \\ a.b &= \gamma\frac{\beta-\gamma-\alpha\gamma}{\beta}. \end{aligned}$$

After solving a and b, we get the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ as follows

$$\begin{aligned} \lambda_1 &= -\epsilon_u\mu^2 + i\mu c \\ &\quad - \frac{(-1 + 2\frac{\alpha\gamma}{\beta-\gamma} + \frac{\beta-\gamma-\alpha\gamma}{\beta}) + \sqrt{(-1 + 2\frac{\alpha\gamma}{\beta-\gamma} + \frac{\beta-\gamma-\alpha\gamma}{\beta})^2 - 4\gamma\frac{\beta-\gamma-\alpha\gamma}{\beta}}}{2}, \\ \lambda_2 &= -\epsilon_\nu\mu^2 + i\mu c \\ &\quad - \frac{(-1 + 2\frac{\alpha\gamma}{\beta-\gamma} + \frac{\beta-\gamma-\alpha\gamma}{\beta}) - \sqrt{(-1 + 2\frac{\alpha\gamma}{\beta-\gamma} + \frac{\beta-\gamma-\alpha\gamma}{\beta})^2 - 4\gamma\frac{\beta-\gamma-\alpha\gamma}{\beta}}}{2}. \end{aligned}$$

For the obtained eigenvalues λ to be in the left half-plane, the real part of $\lambda_{1,2}$ must be negative. To have this, we need to show

$$1+2\frac{\alpha\gamma}{\beta-\gamma}+\frac{\beta-\gamma-\alpha\gamma}{\beta}>0$$

$$\left(-1+2\frac{\alpha\gamma}{\beta-\gamma}+\frac{\beta-\gamma-\alpha\gamma}{\beta}\right)^2-4\gamma\frac{\beta-\gamma-\alpha\gamma}{\beta}<0.$$

After rearranging the first inequality, we obtain $\gamma(1+\alpha)+\beta(\alpha-1)>0$. Given that $\alpha\geq 1$, the inequality is satisfied. For the second inequality, let $k=\beta-\gamma-\alpha\gamma$. After making the necessary arrangements, we get

$$0<\frac{4k\beta(\beta-\gamma)^2}{\gamma}-\alpha^2\beta^2+2\alpha\beta k-k^2.$$

For this inequality to hold, we must have

$$2\alpha\beta k>\alpha^2\beta^2 \quad \text{and} \quad \frac{4k\beta(\beta-\gamma)^2}{\gamma}>k^2.$$

Given that $k=\beta-\gamma-\alpha\gamma>\frac{\alpha\beta}{2}$ and $\frac{4\beta(\beta-\gamma)}{\gamma}\geq 1$, the desired inequality is satisfied.

Thus, the spectrum of $\hat{\mathcal{L}}^-$ lies in the left half-plane and consists of two parabolas. See Figure 2. $\square$

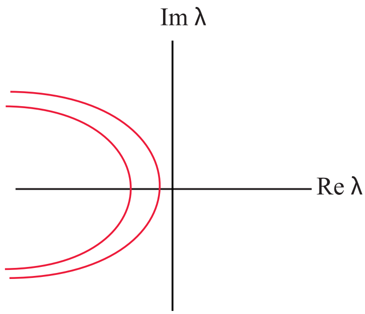

Figure 2 – $Sp(\hat{\mathcal{L}}^-)$ for the equilibrium point $(u^-,\nu^-)=(\hat{u},\hat{\nu})$.

Since $\mathrm{Sp}(\hat{\mathcal{L}}^+)$ has one parabola in the right half-plane, we do not have stability. It can be achieved if a weight function is identified that relocates these spectrum to the left half-plane.

3.2. **Weight function for the right state**

Let $w = (w_-, w_+) \in \mathbb{R}^2$ and let $G_w : \mathbb{R} \to \mathbb{R}$ be a weight function

$$G_w(\xi) = \begin{cases} e^{w_-\xi}, & \xi \le 0, \\ e^{w_+\xi}, & \xi \ge 0. \end{cases}$$

Let $X_0 \in L^2(\mathbb{R}, \mathbb{R}^2)$ be one of the standard Banach spaces with the norm $\|\cdot\|_0$. Let X_w denote a weighted space with the weight function $G_w(\xi)$. That is, for $u(\xi) \in X_w$, $G_w(\xi)u(\xi) \in X_0$ and $\|u\|_w = \|G_w(\xi)u(\xi)\|_0$.

Let us examine next $Sp(\mathcal{L})$ on X_w, and sets $U = (\tilde{u}(\xi), \tilde{\nu}(\xi)) \in X_w$ and $W = G_w(\xi)U = (y(\xi), z(\xi)) \in X_0$.

If the equality $W = G_w U$ is multiplied by G_w^{-1}, then $U = G_w^{-1}W$. Substituting U in the equation $U_t = \mathcal{L}U$, we get $G_w^{-1}W_t = \mathcal{L}G_w^{-1}W$. By multiplying $G_w^{-1}W_t = \mathcal{L}G_w^{-1}W$ by G_w, we get $W_t = G_w\mathcal{L}G_w^{-1}W$. Here, $G_w\mathcal{L}G_w^{-1}$ is on X_0. Instead of finding $Sp(\mathcal{L})$ in X_w, we determine the spectrum of the isomorphic operator $\mathcal{L}_w = G_w\mathcal{L}G_w^{-1}$ on X_0. Since $W = G_w(\xi)U = e^{w\xi}U$, $(y(\xi), z(\xi)) = e^{w\xi}(\tilde{u}(\xi), \tilde{\nu}(\xi))$. For clarity of exposition, denote by

$$w_1 = \epsilon_u\partial_{\xi\xi} + (c - 2w\epsilon_u)\partial_\xi + \epsilon_u w^2 - cw + 1 - 2u - \frac{\alpha\nu}{(u+\alpha)^2},$$

$$w_2 = \epsilon_\nu\partial_{\xi\xi} + (c - 2w\epsilon_\nu)\partial_\xi + \epsilon_\nu w^2 - cw + \frac{\beta u}{u+\alpha} - \gamma.$$

In this case,

$$\mathcal{L}_w = \begin{pmatrix} w_1 & -\frac{u}{u+\alpha} \\ \frac{\beta\alpha\nu}{(u+\alpha)^2} & w_2 \end{pmatrix}.$$

In the equation $W_t = \mathcal{L}_w W$, if $\xi \to \pm\infty$, the equation $W_t = \mathcal{L}_w^{\pm} W$ becomes a linear differential equation with constant coefficients. The spectrum of $\mathcal{L}_w^+$ for the equilibrium point $(u^+, \nu^+) = (1, 0)$ is analyzed by applying the Fourier transform in the L^2 space. In order to shift the spectrum to the left side of the imaginary axis, it is necessary for all eigenvalues to have negative real parts. By a similar computation, this happens if $\frac{4\epsilon_\nu(\beta-\gamma-\alpha\gamma)}{1+\alpha} < c^2$, then we obtain

$$\frac{c - \sqrt{c^2 - \frac{4\epsilon_\nu(\beta-\gamma-\alpha\gamma)}{1+\alpha}}}{2\epsilon_\nu} < w_+ < \frac{c + \sqrt{c^2 - \frac{4\epsilon_\nu(\beta-\gamma-\alpha\gamma)}{1+\alpha}}}{2\epsilon_\nu}.$$

4. **CONCLUSION**

In this study, we investigated the stability of the essential spectrum of traveling waves under the Holling type II functional response in a predator-prey model. First, we identified the critical points that make our model biologically meaningful, considering scenarios where the prey, the predator, or both are present in the environment, along with appropriate initial conditions. Next, we transformed our model into moving coordinates to linearize it around a traveling wave. We expressed the linearized system in an operatorial form and applied the Fourier transform to the operator. During the spectrum analysis, we discussed the conditions that the operator's spectrum must satisfy and obtained the eigenfunctions of the operator for the essential spectrum. The results indicated that the operator's right state exhibited spectrum positioned to the right of the imaginary axis, whereas the left state showed two parabolic regions in the left side of the imaginary axis. Finally, research aimed to discover a weight function capable of shifting the essential spectrum from the right half-plane to the left. Upon applying this identified weight function, the essential spectrum was successfully relocated and stabilized in the left half-plane.

REFERENCES

[1] D.S. Abadi and C. Ummah, *Stability analysis of Lotka–Volterra model with Holling type* II *functional response*. Sci. Res. J. **22** (2013), 22–26.

[2] A.A. Berryman, *The origins and evolution of predator-prey theory*. Ecology **73** (1992), 1530–1535.

[3] S.R. Dunbar, *Travelling wave solutions of diffusive Lotka–Volterra equations*. J. Math. Biol. **17** (1983), *1*, 11–32.

[4] S.R. Dunbar, *Traveling wave solutions of diffusive Lotka–Volterra equations: A heteroclinic connection in* $\mathbf{R}^4$. Trans. Amer. Math. Soc. **286** (1984), *2*, 557–594.

[5] S.R. Dunbar, *Traveling waves in diffusive predator-prey equations: Periodic orbits and point-to-periodic heteroclinic orbits*. SIAM J. Appl. Math. **46** (1986), *6*, 1057–1078.

[6] S.C. Fu, *Traveling waves for a diffusive SIR model with delay*. J. Math. Anal. Appl. **435** (2016), *1*, 20–37.

[7] A. Ghazaryan, V. Manukian, and S. Schecter, *Travelling waves in the Holling–Tanner model with weak diffusion*. Proc. A **471** (2015), article no. 20150045.

[8] Y. Guo, S.S. Ge, and A. Arbi, *Stability of traveling waves solutions for nonlinear cellular neural networks with distributed delays*. J. Syst. Sci. Complex. **35** (2022), *1*, 18–31.

[9] C.S. Holling, *Some characteristics of simple types of predation and parasitism*. Can. Entomol. **91** (1959), 385–398.

[10] Y. Huang and P. Weng, *Traveling waves for a diffusive predator-prey system with general functional response*. Nonlinear Anal. Real World Appl. **14** (2013), *2*, 940–959.

[11] P.K. Jha and S. Ghorai, *Stability of prey-predator model with Holling type response function and selective harvesting*. J. Appl. Comput. Math. **6** (2017), 1–7.

[12] W.T. Li and S.L. Wu, *Traveling waves in a diffusive predator-prey model with Holling type*-III *functional response*. Chaos Solitons Fractals **37** (2008), *2*, 476–486.

[13] W.T. Li, W.B. Xu, and L. Zhang, *Traveling waves and entire solutions for an epidemic model with asymmetric dispersal*. Discrete Contin. Dyn. Syst. **37** (2017), *5*, 2483–2512.

[14] J. Liu, J. Wu, X. Lin, and Z. Du, *Traveling wave solutions of a diffusive predator-prey system with Holling* II *type functional response*. Proc. Amer. Math. Soc. **153** (2025), *2*, 577–589.

[15] F. Ozbag, *Numerical simulations of traveling waves in a counterflow filtration combustion model*. Turkish J. Math. **46** (2022), *4*, 1424–1435.

[16] F. Ozbag and S. Schecter, *Stability of combustion waves in a simplified gas-solid combustion model in porous media*. Philos. Trans. Roy. Soc. A **376** (2018), article no. 20170185.

[17] E. Pelinovsky and O. Kaptsov, *Traveling waves in shallow seas of variable depths*. Symmetry **14** (2022), *7*, 1448.

[18] A. Satria, A.R. Putri, and M. Syafwan, *Stability analysis and numerical simulation of a diffusive prey-predator Holling type* II *model*. J. Phys. Conf. Ser. **1940** (2021), article no. 012013.

[19] S. Shabani, *Diffusive Holling type*-II *predator-prey system with harvesting of prey*. J. Math. Anal. Appl. **410** (2014), *1*, 469–482.

[20] M.S.W. Sunaryo, Z. Salleh, and M. Mamat, *Mathematical model of three species food chain with Holling type*-III *functional response*. Int. J. Pure Appl. Math. **89** (2013), 647–657.

[21] Y. Tian and P. Weng, *Stability analysis of diffusive predator-prey model with modified Leslie–Gower and Holling-type* II *schemes*. Acta Appl. Math. **114** (2011), *3*, 173–192.

[22] Y. Tian and C. Wu, *Traveling wave solutions of a diffusive predator-prey model with modified Leslie–Gower and Holling-type* II *schemes*. Proc. Indian Acad. Sci. Math. Sci. **128** (2018), *3*, article no. 35.

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