THE CO-LOCALIZATION AND COSUPPORT OF ARTINIAN MODULES

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Let R be a commutative noetherian ring. For an artinian R-module M and $\mathfrak{p} \in \mathrm{Spec}R$, the set $\mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M)$ of associated primes is studied and the following equality is obtained:

$$\mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M) = \{\mathfrak{q} \in \mathrm{Cosupp}_R M | \mathfrak{q} \subseteq \mathfrak{p}\}$$

whenever $\mathrm{cosupp}_R M = \mathrm{Cosupp}_R M$. Some characterizations of cosupport of artinian R-modules are provided. As consequences, the dual versions of some classical equalities about support are given.

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1. INTRODUCTION

Unless stated to the contrary, we assume throughout this paper that R is a commutative noetherian ring with non-zero identity.

The theory of cosupport, developed by Benson, Iyengar and Krause [2] in the context of compactly generated R-linear triangulated categories, was partially motivated by work of Neeman [11], who classified the colocalizing subcategories of the derived category $\mathrm{D}(R)$. Despite the many ways in which cosupport is dual to the more established notion of support introduced by Benson, Iyengar and Krause [1], Foxby [5], Sather-Wagstaff and Wicklein [14], the theory of cosupport is not completely satisfactory since this construction is not as well understood as support. For example, the cosupport of the ring of integers is empty.

We write $\mathrm{Spec}R$ for the set of prime ideals of R and $\mathrm{Max}R$ for the set of maximal ideals of R. For $\mathfrak{p} \in \mathrm{Spec}R$, set $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. For an ideal $\mathfrak{a}$ in R, set

$$\mathrm{V}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}R | \mathfrak{a} \subseteq \mathfrak{p}\}.$$

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Let M be an R-module. Following [14], the *"big" support* of M is the set

$$\mathrm{Supp}_R M := \{\mathfrak{p} \in \mathrm{Spec}R | M_\mathfrak{p} \neq 0\}.$$

The *"big" cosupport* of M is the set

$$\mathrm{Cosupp}_R M := \{\mathfrak{p} \in \mathrm{Spec}R | \mathrm{Ext}^i_R(R_\mathfrak{p}, M) \neq 0 \text{ for some } i\}.$$

The *"small" support* of M is the set

$$\mathrm{supp}_R M := \{\mathfrak{p} \in \mathrm{Spec}R | \mathrm{Tor}^R_i(\kappa(\mathfrak{p}), M) \neq 0 \text{ for some } i\}.$$

The *"small" cosupport* of M is the set

$$\mathrm{cosupp}_R M = \left\{\mathfrak{p} \in \mathrm{Spec}R | \mathrm{Ext}^i_R(\kappa(\mathfrak{p}), M) \neq 0 \text{ for some } i\right\}.$$

Let S be a multiplicatively closed set of R. It is well known that if N is a finitely generated R-module, then the localization $S^{-1}N$ is a finitely generated $S^{-1}R$-module. Thus

$$\mathrm{supp}_R N = \mathrm{Supp}_R N = \mathrm{V}(\mathrm{Ann}_R N)$$

by Nakaymam's lemma. However, the co-localization $\mathrm{Hom}_R(S^{-1}R, A)$ of an artinian R-module A is almost never an artinian $S^{-1}R$-module, and it may even have infinite Goldie dimension (see [4, Example 3.8]).

Remark 1.1. (1) If R is complete, then for an artinian R-module A, $\mathrm{cosupp}_R A = \mathrm{Cosupp}_R A = \mathrm{V}(\mathrm{Ann}_R A)$ by using Matlis duality and [14, Proposition 6.1].

(2) If $(R, \mathfrak{m})$ is local and $A = E(R/\mathfrak{m})$, then the equalities in (1) hold true.

(3) For an artinian R-module A, the inclusion $\mathrm{cosupp}_R A \subseteq \mathrm{Cosupp}_R A$ may be strict (see Remark 2.4).

(4) For an artinian R-module A, Melkersson and Schenzel [10, Lemma 7.3] showed the equality $\mathrm{Cosupp}_R A = \mathrm{V}(\mathrm{Ann}_R A)$, which does not hold for non-artinian modules (see Remark 2.4).

For $\mathfrak{p} \in \mathrm{Spec}R$ and an artinian R-module M, we determine the associated prime of the co-localization $\mathrm{Hom}_R(R_\mathfrak{p}, M)$ in Section 2, and show that if $\mathrm{cosupp}_R M = \mathrm{Cosupp}_R M$, then

$$\mathrm{Ass}_R \mathrm{Hom}_R(R_\mathfrak{p}, M) = \{\mathfrak{q} \in \mathrm{Cosupp}_R M \,|\, \mathfrak{q} \subseteq \mathfrak{p}\}.$$

In Section 3, some characterizations of cosupport of artinian modules are provided, which are dual to those of classical support.

Next, we recall some notions which we need later.

Let $\mathfrak{m} \in \mathrm{Max}R$ and M be an R-module. We denote $\mathrm{Hom}_R(-, E(R/\mathfrak{m}))$ by $D_\mathfrak{m}(-)$ and $\mathrm{RHom}_R(\oplus_\mathfrak{m} D_\mathfrak{m}(M), E(R/\mathfrak{p}))$ by ${}^\mathfrak{p}M$. Let M be an R-module. The set $\mathrm{Ass}_R M$ of *associated primes* of M is the set of prime ideals $\mathfrak{p}$ of R such that there exists a cyclic submodule N of M such that $\mathfrak{p} = \mathrm{Ann}_R N$. An R-module L is *cocyclic* if L is a submodule of $E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \mathrm{Max}R$. An R-module M is said to be *secondary* if $M \neq 0$ and for any $x \in R$, the multiplication by x on M is either surjective or nilpotent. The radical of the annihilator of M is then a prime ideal $\mathfrak{p}$ and we say that M is $\mathfrak{p}$-*secondary.* A *secondary representation* of M is an expression of M as a finite sum $M = M_1 + \cdots + M_n$ of $\mathfrak{p}_i$-secondary submodules. This representation is said to be *minimal* if the prime ideals $\mathfrak{p}_i$ are all distinct and none of the summands M_i are redundant. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of minimal representation of M. This set is denoted by $\mathrm{Att}_R M$ and called the set of *attached prime* ideals of M.

The concept of linear compactness was first introduced by Lefschetz [6] for vector spaces of arbitrary dimension and extended for modules by Zelinsky [16] and Leptin [7]. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized R-module M is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection (see [8]). A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. The class of semi-discrete linearly compact modules contains all artinian modules and all finitely generated modules over a complete ring.

2. ASSOCIATED PRIME OF THE CO-LOCALIZATION OF ARTINIAN MODULES

This section determines the associated prime of $\mathrm{Hom}_R(R_\mathfrak{p}, M)$ for an artinian module M. We begin with the following lemma.

LEMMA 2.1. *Let $\mathfrak{p}$ be a non-maximal prime ideal of R and M an artinian R-module. One has an inclusion*

$$\mathrm{Ass}_R \mathrm{Hom}_R(R_\mathfrak{p}, M) \subseteq \{\mathfrak{q} \in \mathrm{Cosupp}_R M | \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. Since M is an artinian R-module, it follows from [15, Theorem 2.13] that M has a composition series

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M,$$

where M_{i-1}/M_i is cocyclic and $\mathfrak{p}_i = (0 :_R M_{i-1}/M_i) \in \mathrm{Spec}R$ for $i = 1, \ldots, n$. We use induction on n. If $n = 1$ then $M = M_0/M_1$ is $\mathfrak{p}_1$-secondary and so $\mathrm{Hom}_R(R_\mathfrak{p}, M)$ is either 0 or $\mathfrak{p}_1$-secondary by [4, Lemma 3.1]. Let

$$\mathrm{Hom}_R(R_\mathfrak{p}, M) = 0$$

and the assertion holds. Assume that $\mathrm{Hom}_R(R_\mathfrak{p}, M)$ is $\mathfrak{p}_1$-secondary and let $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M)$. Then $\mathfrak{p}_1 = \mathrm{Rad}(0 :_R \mathrm{Hom}_R(R_\mathfrak{p}, M)) \subseteq \mathfrak{q}$ and $\mathrm{Hom}_R(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}, M) \cong \mathrm{Hom}_R(R/\mathfrak{q}, \mathrm{Hom}_R(R_\mathfrak{p}, M)) \neq 0$. Consequently, we notice $\mathfrak{p}_1 \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ and hence, $\mathfrak{q} \in \mathrm{Cosupp}_RM$ by [15, Proposition 2.3]. Now assume that $n > 1$. One has a short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

where M/M_1 is $\mathfrak{p}_1$-secondary and M_1 has the following composition series: $0 = M_n \subset M_{n-1} \subset \cdots \subset M_1$. Also, by [4, Corollary 2.5], we have the next exact sequence

$$0 \to \mathrm{Hom}_R(R_\mathfrak{p}, M_1) \to \mathrm{Hom}_R(R_\mathfrak{p}, M) \to \mathrm{Hom}_R(R_\mathfrak{p}, M/M_1) \to 0.$$

Let $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M)$. Then, we have $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M_1)$ or $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M/M_1)$. If $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M/M_1)$ then $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \mathrm{Cosupp}_RM/M_1 \subseteq \mathrm{Cosupp}_RM$ by the above proof and [15, Theorem 2.7]. If $\mathfrak{q} \in \mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M_1)$ then $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \mathrm{Cosupp}_RM_1 \subseteq \mathrm{Cosupp}_RM$ by the induction and [15, Theorem 2.7], as required. □

The next lemma is a more general version of [10, Lemma 4.1] as

$$\mathrm{cosupp}_RE(R/\mathfrak{m}) = \mathrm{Cosupp}_RE(R/\mathfrak{m}) = \mathrm{Spec}R.$$

THEOREM 2.2. *Let $\mathfrak{p}$ be a non-maximal prime ideal of R and M an artinian R-module. If $\mathfrak{p} \in \mathrm{cosupp}_RM$, then there exists an equality*

$$\mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p}, M) = \{\mathfrak{q} \in \mathrm{Cosupp}_RM | \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. The left-hand side is included in the right-hand side by Lemma 2.1. On the other hand, let $\mathfrak{q} \in \mathrm{Cosupp}_RM$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Since M is artinian, there is an injective resolution $M \to E^\bullet$ with $E^\bullet = 0 \to E(R/\mathfrak{m}_{01}) \oplus \cdots \oplus E(R/\mathfrak{m}_{0n_0}) \to E(R/\mathfrak{m}_{11}) \oplus \cdots \oplus E(R/\mathfrak{m}_{1n_1}) \to \cdots$ with each $\mathfrak{m}_{i_j} \in \mathrm{Max}R$. Then $\mathrm{Hom}_R(R_\mathfrak{p}, M) \to \mathrm{Hom}_R(R_\mathfrak{p}, E^\bullet)$ is also an injective resolution as R-modules. Hence

$$\begin{aligned}\mathrm{Ext}_R^*\big(R/\mathfrak{p}, \mathrm{Hom}_R(R_\mathfrak{p}, M)\big) &= \mathrm{H}^*\big(\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{Hom}_R(R_\mathfrak{p}, E^\bullet))\big)\\ &\cong \mathrm{H}^*\big(\mathrm{Hom}_R(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, E^\bullet)\big)\\ &= \mathrm{Ext}_R^*(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M) \neq 0,\end{aligned}$$

it implies that $\operatorname{Ext}^*_R(R/\mathfrak{q}, \operatorname{Hom}_R(R_\mathfrak{p}, M)) \neq 0$. Thus, we have the next isomorphisms

$$\begin{aligned}\operatorname{Ext}^*_{R/\mathfrak{q}}\big((R/\mathfrak{q})_\mathfrak{p}, \operatorname{Hom}_R(R/\mathfrak{q}, M)\big) &\cong \operatorname{Ext}^*_R(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}, M)\\ &= \mathrm{H}^*\big(\operatorname{Hom}_R(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}, E^\bullet)\big)\\ &\cong \mathrm{H}^*\big(\operatorname{Hom}_R(R/\mathfrak{q}, \operatorname{Hom}_R(R_\mathfrak{p}, E^\bullet))\big)\\ &= \operatorname{Ext}^*_R\big(R/\mathfrak{q}, \operatorname{Hom}_R(R_\mathfrak{p}, M)\big) \neq 0.\end{aligned}$$

As $\operatorname{Hom}_R(R/\mathfrak{q}, M)$ is a non-zero artinian $R/\mathfrak{q}$-module by [15, Theorem 4.3], it follows from [4, Theorem 2.4] that $\operatorname{Hom}_{R/\mathfrak{q}}((R/\mathfrak{q})_\mathfrak{p}, \operatorname{Hom}_R(R/\mathfrak{q}, M)) \neq 0$. Now, by analogy with the proof of [10, Lemma 4.1], one can obtain the desired inclusion. □

Corollary 2.3. *Let $\mathfrak{p}$ be a non-maximal prime ideal of R and M an artinian R-module. If* $\operatorname{cosupp}_R M = \operatorname{Cosupp}_R M$*, then there exists an equality*

$$\operatorname{Ass}_R \operatorname{Hom}_R(R_\mathfrak{p}, M) = \{\mathfrak{q} \in \operatorname{Cosupp}_R M | \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. If $\operatorname{Hom}_R(R_\mathfrak{p}, M) = 0$, then $\operatorname{Ann}_R M \nsubseteq \mathfrak{p}$. So

$$\{\mathfrak{q} \in \operatorname{Cosupp}_R M | \mathfrak{q} \subseteq \mathfrak{p}\} = \emptyset$$

and the equality holds. If $\operatorname{Hom}_R(R_\mathfrak{p}, M) \neq 0$, then $\mathfrak{p} \in \operatorname{cosupp}_R M$ by assumption. The assertion follows by Theorem 2.2. □

Remark 2.4. (1) For an artinian R-module M, the next inclusion $\operatorname{cosupp}_R M \subseteq \operatorname{Cosupp}_R M$ may be strict. In fact, let R be a domain with $\operatorname{Max} R = \{\mathfrak{m}, \mathfrak{n}\}$. Set $M = E(R/\mathfrak{m})$. As R is domain, $\operatorname{Ann}_R M = 0$, so $\operatorname{Cosupp}_R M = \mathrm{V}(\operatorname{Ann}_R M) = \operatorname{Spec} R$. Since $\operatorname{supp}_R M = \operatorname{Supp}_R M = \{\mathfrak{m}\}$, it follows by [2, Theorem 4.13] that $\mathfrak{n} \notin \operatorname{cosupp}_R M$.

(2) If $(R, \mathfrak{m})$ is a local ring, then $\operatorname{cosupp}_R E(R/\mathfrak{m}) = \operatorname{Cosupp}_R E(R/\mathfrak{m}) = \operatorname{Spec} R$. However, the equality may not hold true when R is not local by the above example.

(3) For any R-module M, the inclusion $\operatorname{Cosupp}_R M \subseteq \mathrm{V}(\operatorname{Ann}_R M)$ may be strict. For example, let $(R, \mathfrak{m})$ be a local domain with $\dim R > 0$. Next, we set $M = \operatorname{Hom}_R(\oplus_{n>0} R/\mathfrak{m}^n, E(R/\mathfrak{m}))$. Then

$$\mathrm{V}(\operatorname{Ann}_R M) = \mathrm{V}\big(\operatorname{Ann}_R(\oplus_{n>0} R/\mathfrak{m}^n)\big) = \operatorname{Spec} R$$

by [3, p. 139] as $\operatorname{Ass}_R(\oplus_{n>0} R/\mathfrak{m}^n) = \{\mathfrak{m}\}$. But $\operatorname{Cosupp}_R M \neq \operatorname{Spec} R$ by Proposition 3.1.

Corollary 2.5. *Let N be a finitely generated R-module and M an artinian R-module. One has an equality*

$$\mathrm{cosupp}_R\mathrm{Hom}_R(N,M)=\mathrm{supp}_R N\cap\mathrm{cosupp}_R M.$$

Proof. By [15, Theorem 3.4],

$$\mathrm{cosupp}_R\mathrm{Hom}_R(N,M)\subseteq\mathrm{supp}_R N\cap\mathrm{cosupp}_R M.$$

Conversely, let $\mathfrak{p}\in\mathrm{supp}_R N\cap\mathrm{cosupp}_R M$. If $\mathfrak{p}$ is maximal, then

$$\mathrm{Hom}_R\big(R/\mathfrak{p},\mathrm{Hom}_R(N,M)\big)\neq 0,$$

and so $\mathfrak{p}\in\mathrm{cosupp}_R\mathrm{Hom}_R(N,M)$. We assume that $\mathfrak{p}\neq\mathfrak{m}$. It follows from Theorem 2.2 that $\mathfrak{p}\in\mathrm{Ass}_R\mathrm{Hom}_R(R_\mathfrak{p},M)$, so $\mathfrak{p}\in\mathrm{Ass}_R\mathrm{Hom}_R(N,\mathrm{Hom}_R(R_\mathfrak{p},M))$. Consequently,

$$\mathrm{Hom}_R\big(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},\mathrm{Hom}_R(N,M)\big)\cong\mathrm{Hom}_R\big(R/\mathfrak{p},\mathrm{Hom}_R(N,\mathrm{Hom}_R(R_\mathfrak{p},M)\big)\neq 0$$

and $\mathfrak{p}\in\mathrm{cosupp}_R\mathrm{Hom}_R(N,M)$, as desired. □

The following corollary is a dual version of [13, Exercise 9.23] or [3, Chap. II, p. 106, Corollary].

Corollary 2.6. *Let $\mathfrak{a}$ be a proper ideal of R and M an artinian R-module. If M is a cogenerator, then*

$$\mathrm{Rad}\big(0:_R\mathrm{Hom}_R(R/\mathfrak{a},M)\big)=\mathrm{Rad}\big(\mathfrak{a}+(0:_R M)\big).$$

Proof. As M is a cogenerator, $\mathrm{cosupp}_R M=\mathrm{Cosupp}_R M=\mathrm{V}(\mathrm{Ann}_R M)$. By [15, Theorem 3.4], one has $\mathrm{Cosupp}_R\mathrm{Hom}_R(R/\mathfrak{a},M)\subseteq\mathrm{V}(\mathfrak{a})\cap\mathrm{V}(\mathrm{Ann}_R M)$. On the other hand, it follows by Corollary 2.5 that

$$\mathrm{cosupp}_R\mathrm{Hom}_R(R/\mathfrak{a},M)=\mathrm{V}(\mathfrak{a})\cap\mathrm{cosupp}_R M=\mathrm{V}(\mathfrak{a})\cap\mathrm{V}(\mathrm{Ann}_R M).$$

Therefore,

$$\begin{aligned}\mathrm{V}\big(\mathrm{Ann}_R\mathrm{Hom}_R(R/\mathfrak{a},M)\big)&=\mathrm{Cosupp}_R\mathrm{Hom}_R(R/\mathfrak{a},M)\\&=\mathrm{cosupp}_R\mathrm{Hom}_R(R/\mathfrak{a},M)=\mathrm{V}(\mathfrak{a}+\mathrm{Ann}_R M),\end{aligned}$$

as claimed. □

3. COSUPPORT OF ARTINIAN MODULES

In this section, we give some characterizations of cosupport of artinian R-modules, which are similar to those of classical support.

It is well known that $\mathfrak{p}\in\mathrm{Supp}_R M$ if and only if there is a cyclic submodule N of M with $\mathrm{Ann}_R N\subseteq\mathfrak{p}$. The next lemma is a dual version of [14, Fact 3.3] and the above fact for "big" cosupport, which is proved by Nam when R is a local ring (see [11, Theorem 3.8]).

Proposition 3.1. *Let M be a linearly compact R-module. One has*

$$\begin{aligned}\mathrm{Cosupp}_R M &= \{\mathfrak{p} \in \mathrm{Spec}R \mid \mathrm{Hom}_R(R_\mathfrak{p}, M) \neq 0\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}R \mid {}^{\mathfrak{p}}M \neq 0\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}R \mid \exists \textit{ a cocyclic quotient } L \textit{ of } M \textit{ such that } \mathrm{Ann}_R L \subseteq \mathfrak{p}\}.\end{aligned}$$

Proof. The first equality follows by [4, Theorem 2.4(ii)]. Hence [15, Corollary 2.16] implies that $\mathrm{Cosupp}_R M \subseteq \{\mathfrak{p} \in \mathrm{Spec}R \mid {}^{\mathfrak{p}}M \neq 0\}$. On the other hand, let ${}^{\mathfrak{p}}M \neq 0$ for $\mathfrak{p} \in \mathrm{Spec}R$. Then there is an exact sequence

$$0 \to N \to M \to A \to 0,$$

where A is artinian with $\mathrm{Ann}_R A \subseteq \mathfrak{p}$, and so $\mathfrak{p} \in \mathrm{Cosupp}_R A$ by [15, Corollary 2.5]. Since A is a linearly compact R-module with the discrete topology, N is an open submodule of M, and hence N is closed in M. It means that N is a linearly compact submodule of M. Hence [4, Corollary 2.5] implies that the sequence

$$0 \to \mathrm{Hom}_R(R_\mathfrak{p}, N) \to \mathrm{Hom}_R(R_\mathfrak{p}, M) \to \mathrm{Hom}_R(R_\mathfrak{p}, A) \to 0$$

is exact. As $\mathrm{Hom}_R(R_\mathfrak{p}, A) \neq 0$, we have $\mathrm{Hom}_R(R_\mathfrak{p}, M) \neq 0$ and $\mathfrak{p} \in \mathrm{Cosupp}_R M$, the second equality holds. The third equality is proved by using the remark after [15, Theorem 3.8]. □

If $M = 0$ or M has a secondary representation, then we say that M is *representable*. It was shown by Macdonald [9] and Sharp [12] that artinian modules and injective modules are representable.

Corollary 3.2. (1) *Let N be a finitely generated R-module and M a representable R-module. One has an inequality*

$$\mathrm{Cosupp}_R(N \otimes_R M) = \mathrm{Supp}_R N \cap \mathrm{Cosupp}_R M.$$

(2) *Let M be a linearly compact R-module. One has*

$$\mathrm{Cosupp}_R \mathrm{Hom}_R(R_\mathfrak{p}, M) = \{\mathfrak{q} \in \mathrm{Cosupp}_R M \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. (1) Let $\mathfrak{p} \in \mathrm{Cosupp}_R(N \otimes_R M)$. It follows from [10, Lemma 5.1] that $\mathrm{Hom}_R(R_\mathfrak{p}, M) \otimes_R N_\mathfrak{p} \cong \mathrm{Hom}_R(R_\mathfrak{p}, N \otimes_R M) \neq 0$. Hence

$$\mathrm{Cosupp}_R(N \otimes_R M) \subseteq \mathrm{Supp}_R N \cap \mathrm{Cosupp}_R M.$$

On the other hand, let $\mathfrak{p} \in \mathrm{Supp}_R N \cap \mathrm{Cosupp}_R M$. Then there is

$$\mathfrak{q} \in \mathrm{Att}_R(N \otimes_R M) = \mathrm{Supp}_R N \cap \mathrm{Att}_R M$$

such that $\mathfrak{q} \subseteq \mathfrak{p}$. Thus, $\mathrm{Hom}_R(R_\mathfrak{q}, \mathrm{Hom}_R(R_\mathfrak{p}, N \otimes_R M)) \cong \mathrm{Hom}_R(R_\mathfrak{q}, N \otimes_R M) \neq 0$ by [15, Theorems 1.21 and 1.14], and so $\mathrm{Hom}_R(R_\mathfrak{p}, N \otimes_R M) \neq 0$, as desired.

(2) This is an immediate consequence of Proposition 3.1. $\square$

The following result is a dual version of [14, Fact 3.3] for "small" cosupport.

Theorem 3.3. *Let M be a semi-discrete linearly compact R-module. One has*

$$\begin{aligned}\mathrm{cosupp}_R M &= \{\mathfrak{p} \in \mathrm{Spec}R \mid {}^{\mathfrak{p}}\mathrm{Ext}^i_R(R/\mathfrak{p}, M) \neq 0 \textit{ for some } i\}\\ &= \{\mathfrak{p} \in \mathrm{Spec}R \mid \mathrm{Ext}^i_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \neq 0 \textit{ for some } i\}.\end{aligned}$$

Proof. First assume that M is artinian. There is an injective resolution $M \to E^\bullet$ with

$$E^\bullet = 0 \to E(R/\mathfrak{m}_{01}) \oplus \cdots \oplus E(R/\mathfrak{m}_{0n_0}) \to E(R/\mathfrak{m}_{11}) \oplus \cdots \oplus E(R/\mathfrak{m}_{1n_1}) \to \cdots,$$

where each $\mathfrak{m}_{ij} \in \mathrm{Max}R$. Hence [8] and [4, Corollary 2.5] yield the next isomorphisms

$$\begin{aligned}\mathrm{Ext}^i_R\big(\kappa(\mathfrak{p}), M\big) &= \mathrm{H}^i\big(\mathrm{Hom}_R(\kappa(\mathfrak{p}), E^\bullet)\big)\\ &\cong \mathrm{H}^i\big(\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{Hom}_R(R/\mathfrak{p}, E^\bullet))\big)\\ &\cong \mathrm{Hom}_R\big(R_{\mathfrak{p}}, \mathrm{H}^i(\mathrm{Hom}_R(R/\mathfrak{p}, E^\bullet))\big)\\ &= \mathrm{Hom}_R\big(R_{\mathfrak{p}}, \mathrm{Ext}^i_R(R/\mathfrak{p}, M)\big).\end{aligned}$$

Consequently, we have the following equivalences:

$$\begin{aligned}\mathfrak{p} \in \mathrm{cosupp}_R M &\Longleftrightarrow \mathfrak{p} \in \mathrm{Cosupp}_R \mathrm{Ext}^i_R(R/\mathfrak{p}, M) \text{ for some } i\\ &\Longleftrightarrow {}^{\mathfrak{p}}\mathrm{Ext}^i_R(R/\mathfrak{p}, M) \neq 0 \text{ for some } i,\end{aligned}$$

where the first one is determined by the above isomorphism, and the second one by Proposition 3.1. As

$${}^{\mathfrak{p}}\mathrm{Ext}^i_R(R/\mathfrak{p}, M) \cong \mathrm{Hom}_R\Big(\mathrm{Tor}^R_i(R/\mathfrak{p}, \bigoplus_{\mathfrak{m}} D_{\mathfrak{m}}(M)), E(R/\mathfrak{p})\Big) \cong \mathrm{Ext}^i_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M),$$

it implies that ${}^{\mathfrak{p}}\mathrm{Ext}^i_R(R/\mathfrak{p}, M) \neq 0$ for some i if and only if $\mathrm{Ext}^i_R(R/\mathfrak{p}, {}^{\mathfrak{p}}M) \neq 0$ for some i.

Next, assume that M is a semi-discrete linearly compact R-module. Then [17, Theorem] yields a short exact sequence $0 \to N \to M \to A \to 0$, where A is artinian and N is finitely generated. Let $N \to E^\bullet_N$ and $A \to E^\bullet_A$ be minimal injective resolutions of N and A, respectively. Then $M \to E^\bullet_N \oplus E^\bullet_A$ is an injective resolution of M. For any $\mathfrak{p} \in \mathrm{Spec}R$, we have

$$\begin{aligned}\mathrm{Ext}^i_R\big(\kappa(\mathfrak{p}), M\big) &= \mathrm{H}^i\big(\mathrm{Hom}_R(\kappa(\mathfrak{p}), E^\bullet_N \oplus E^\bullet_A)\big)\\ &\cong \mathrm{H}^i\big(\mathrm{Hom}_R(R_{\mathfrak{p}}, \mathrm{Hom}_R(R/\mathfrak{p}, E^\bullet_N))\big)\end{aligned}$$

$$\oplus \operatorname{Hom}_R\big(R_{\mathfrak p}, \mathrm{H}^i(\operatorname{Hom}_R(R/\mathfrak p, E_A^\bullet))\big).$$

If $\mathfrak p \in \operatorname{cosupp}_R N$ then $\mathfrak p$ is a maximal ideal, and so

$$\mathrm{H}^i\big(\operatorname{Hom}_R(R_{\mathfrak p}, \operatorname{Hom}_R(R/\mathfrak p, E_N^\bullet))\big) \cong \operatorname{Hom}_R\big(R_{\mathfrak p}, \mathrm{H}^i(\operatorname{Hom}_R(R/\mathfrak p, E_N^\bullet))\big).$$

Therefore, we have the isomorphism $\operatorname{Ext}_R^i(\kappa(\mathfrak p), M) \cong \operatorname{Hom}_R(R_{\mathfrak p}, \operatorname{Ext}_R^i(R/\mathfrak p, M))$. If $\mathfrak p \notin \operatorname{cosupp}_R N$, then $\mathrm{H}^i(\operatorname{Hom}_R(R_{\mathfrak p}, \operatorname{Hom}_R(R/\mathfrak p, E_N^\bullet))) = 0$ and hence

$$\operatorname{Ext}_R^i\big(\kappa(\mathfrak p), M\big) \cong \operatorname{Hom}_R\big(R_{\mathfrak p}, \operatorname{Ext}_R^i(R/\mathfrak p, M)\big).$$

Consequently, we get that $\mathfrak p \in \operatorname{cosupp}_R M$ if and only if

$$\mathfrak p \in \operatorname{Cosupp}_R \operatorname{Ext}_R^i(R/\mathfrak p, M)$$

for some i if and only if ${}^{\mathfrak p}\operatorname{Ext}_R^i(R/\mathfrak p, M) \neq 0$ for some i by Proposition 3.1, as claimed. $\square$

Remark 3.4. The equality in Theorem 3.3 is not true, in general, whenever M is not semi-discrete linearly compact. Let $(R, \mathfrak m)$ be a local domain with $\dim R > 0$ and $M = \prod_{i>0} R/\mathfrak m^i$. By [14, Propositions 4.9 and 4.10], one has $\operatorname{cosupp}_R M = \{\mathfrak m\}$. But ${}^0\operatorname{Ext}_R^i(R/0, M) \neq 0$.

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