ON ENERGY FUNCTIONAL IN SUB-FINSLER GEOMETRY

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This paper investigates critical points of the energy functional in sub-Finsler manifolds. Critical points of the energy functional correspond to sub-Finsler geodesics, i.e., horizontal curves that are extremal for the energy, emphasizing their importance in sub-Finsler geometry. Assuming that the sub-Finsler metric is bumpy (in the Morse–Bott sense), the energy functional on the free loop space has the Morse–Bott property, and we establish a direct link between critical points and closed geodesics. We shed light on the existence of at least one closed geodesic on compact sub-Finsler manifolds.

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1. INTRODUCTION

Sub-Finsler geometry is a branch of differential geometry that deals with a class of metric structures called sub-Finsler manifolds. The study of sub-Finsler geometry involves understanding the geometric properties of sub-Finsler manifolds, analyzing the behavior of curves and geodesics in these manifolds, and investigating the relationship between the metric structure and the underlying geometry. Various mathematical tools, such as nonlinear connections, sub-Riemannian metrics, and control theory, are employed to study the properties and applications of sub-Finsler geometry in diverse fields, including robotics, image processing, and mathematical physics; see, for instance, [1–5, 10, 16, 17].

The energy functional is an important tool in sub-Finsler geometry to measure the length of curves. It is defined as the integral of the sub-Finsler metric squared along the curve. Energy functional in sub-Riemannian geometry has already been discussed by A. Agrachev et al. in [1], as well as by E. Le Donne in [10]. Studying the energy functional in sub-Finsler geometry has important implications for a variety of fields, including optimal control theory, robotics, and quantum mechanics. The energy functional provides a natural and well-defined way to measure the length of curves in sub-Finsler geometry, and it plays a key role in determining the geodesics of the sub-Finsler

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manifold, which are the curves that minimize the length functional. Understanding the behavior of geodesics in sub-Finsler geometry is crucial for a wide range of applications, such as path planning for robots moving in constrained environments, designing efficient control strategies for mechanical systems, and modeling the behavior of particles in quantum mechanics, e.g., [2].

This paper consists of an introduction followed by two main sections. Section 2 provides a comprehensive review of sub-Finsler manifolds, encompassing essential facts, terminology, and concepts necessary for comprehending the remainder of the paper. In Section 3, we introduce the energy functional, serving as a measure of curve length in sub-Finsler geometry. Furthermore, this section presents our main results as follows: Proposition 3.1, which establishes that minimizing geodesics in a sub-Finsler manifold also minimizes the energy functional; Proposition 3.2 states that geodesics in a sub-Finsler manifold are critical points of the energy functional, representing the kinetic energy of a particle moving within the manifold. To conclude this part of the section, we highlight the fundamental connection between kinetic energy and sub-Finsler geometry in mechanics.

Moving forward, Lemma 3.4 asserts that if the sub-Finsler metric on a two-sphere S^2 is bumpy, the energy functional on the free loop space becomes Morse–Bott. Furthermore, Proposition 3.5 states that any non-degenerate critical point of the energy functional corresponds to a closed geodesic on the sphere. Theorem 3.6 establishes a relationship between the bumpy property of a sub-Finsler metric on S^2 and the existence of closed geodesics. Finally, Theorem 3.8 confirms the existence of at least one closed geodesic on a compact sub-Finsler manifold M.

2. PRELIMINARIES

In this section, we review some standard facts about a sub-Finsler manifolds and introduce the terminology that we use throughout this paper; for more details, we refer the reader to [1, 3–5].

Definition 2.1. We define a sub-Finslerian structure on an n-dimensional connected manifold M as a triple $(\mathcal{D}, \rho, F)$ where

(1) $(\mathcal{D}, \pi_{\mathcal{D}})$ is a vector bundle on M.

(2) $\rho : \mathcal{D} \longrightarrow TM$ is a morphism of vector bundles, making the following

diagram commutative:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & TM \\ & \underset{\pi_{\mathcal{D}}}{\searrow} & \downarrow \pi \\ & & M \end{array}$$

Here, $\pi_{\mathcal{D}} : \mathcal{D} \longrightarrow M$ and $\pi : TM \longrightarrow M$ are the canonical projections.

(3) A function $F : \widetilde{\mathcal{D}} \to \mathbb{R}_+$, where $\widetilde{\mathcal{D}} = \mathcal{D} \backslash \{0\}$, is a *sub-Finsler metric*, if it satisfies the following properties, see [8]:

(i) *Positive definiteness:* $F_x(v) > 0$ for all $v \in \widetilde{\mathcal{D}}$ and $x \in M$.

(ii) *Regularity:* F is smooth, i.e., C^∞ on $\widetilde{\mathcal{D}}$.

(iii) *Positive homogeneity:* $F_x(\lambda v) = \lambda F_x(v)$ for all $v \in \widetilde{\mathcal{D}}_x$ and $\lambda \in \mathbb{R}_+$.

(iv) *Strict convexity condition:* The fundamental tensor

$$g_v(u, w) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}\Big|_0 F^2(v + su + tw)$$

is positive definite for each $v \neq 0$.

A *sub-Finsler manifold* is a smooth manifold M equipped with a sub-Finslerian structure, i.e., the triple $(\mathcal{D}, \rho, F)$.

Let M be a smooth n-dimensional manifold. A vector distribution, denoted as $\mathcal{D}$, of rank k on M is defined as a collection of vector subspaces $\mathcal{D}_x \subset T_xM$ such that for each point x on the manifold, the dimension of $\mathcal{D}_x$ is equal to k. Furthermore, a vector distribution $\mathcal{D}$ is considered smooth if, for every point x_0 in M, there exists a neighborhood U_{x_0} of x_0 and a set of smooth vector fields $X_1, X_2, \ldots, X_k$ (denoted by $\mathfrak{X}(M)$) defined on U_{x_0} such that, for all x within U_{x_0}

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \ldots, X_k(x)\}.$$

From now on, we suppose that $\mathcal{D} \subset TM$ (smooth), $\rho : \mathcal{D} \longrightarrow TM$ is the inclusion $i : \mathcal{D} \longrightarrow TM$ and F is a sub-Finsler metric on $\mathcal{D}$. Moreover, the triple $(M, \mathcal{D}, F)$ represent the sub-Finsler manifolds.

An absolutely continuous curve $\gamma : [0, T] \to M$ is called *horizontal* or *admissible* if $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for $0 \leqslant t \leqslant T$. The *length* of a horizontal curve γ is defined in the usual way as

$$\ell(\gamma) = \int_0^T F\big(\dot{\gamma}(t)\big) dt,$$

where F is the sub-Finsler metric on M and $\dot{\gamma}(t)$ denotes the tangent vector to γ at t.

In sub-Finsler geometry, we define the *sub-Finslerian distance* $d(x, y)$ between two points x and y on a manifold M as follows.

Consider the set of all horizontal curves $\gamma : [0, T] \to M$ that start at x and end at y, i.e., $\gamma(0) = x$ and $\gamma(T) = y$, where $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$. Then the sub-Finslerian distance between x and y is the infimum of the lengths of all such horizontal curves γ joining x and y, i.e.,

$$d(x, y) = \inf\big\{\ell(\gamma) \mid \gamma : [0, T] \longrightarrow M \text{ horizontal}, \ \gamma(0) = x, \gamma(T) = y\big\}.$$

If there is no horizontal curve joining x and y, then the sub-Finslerian distance is defined to be infinite.

A horizontal curve γ is said to be a *minimizing geodesic* if it realizes the distance between its endpoints, i.e., $\ell(\gamma) = d(\gamma(0), \gamma(T))$. A sub-Finsler geodesic is a horizontal curve which is an extremal (critical point) of the length functional among all the horizontal curves with fixed endpoints. Note that, in sub-Riemannian geometry, and consequently in sub-Finsler geometry, the geodesics split into *normal* and *abnormal* extremals in the sense of the Pontryagin Maximum Principle; we refer to [1, 3, 10, 13] for more details.

Consider a smooth manifold M and a family $\mathcal{F} \in \mathfrak{X}(M)$ consisting of smooth vector fields. The *Lie algebra generated* by $\mathcal{F}$ is defined as the smallest sub-algebra of $\mathfrak{X}(M)$ containing $\mathcal{F}$. In other words,

$$\mathrm{Lie}\mathcal{F} := \mathrm{span}\big\{[X_1, \ldots, [X_{j-1}, X_j]], X_i \in \mathcal{F}, j \in \mathbb{N}\big\}.$$

We say that $\mathcal{F}$ is *bracket generating* if, for every point $x \in M$, the set

$$\mathrm{Lie}_x\mathcal{F} := \big\{X(x), X \in \mathrm{Lie}\mathcal{F}\big\} = T_xM.$$

Throughout this paper, we assume that the distribution $\mathcal{D}$ is a bracket generating distribution.

Concerning the existence of horizontal curves in sub-Riemannian geometry and sub-Finslerian geometry, the Chow–Rashevskii theorem (also known as Chow's theorem) states that any two points of a connected sub-Riemannian manifold (sub-Finsler manifold), endowed with a bracket generating distribution $\mathcal{D}$, are connected by a horizontal curve in the manifold, [9]. More precisely, let $(M, \mathcal{D}, F)$ be a sub-Finsler manifold and let $x, y \in M$ be two points. Then, Chow's theorem guarantees that there exists a horizontal curve $\gamma : [0, T] \longrightarrow M$ connecting x and y.

Chow's theorem is only guaranteed to hold when the distribution $\mathcal{D}$ is bracket generating. In general, if $\mathcal{D}$ is not bracket generating, then there may be points in the sub-Finsler manifold M that cannot be connected by a horizontal curve. However, there are some special cases where Chow's theorem may still hold even if $\mathcal{D}$ is not bracket generating. For example, if $\mathcal{D}$ is not bracket generating but it has a bracket generating subbundle $\xi \subset \mathcal{D}$ such that $F|_\xi$ is non-degenerate, then Chow's theorem still holds for the sub-Finsler manifold $(M, \mathcal{D}, F)$.

However, in the sub-Riemannian manifold, Filippov's theorem from [1, Theorem 3.43] gives the local existence of length minimizers. Moreover, [5, Proposition 5.2] asserts that within a (reversible) sub-Finslerian manifold M, for any point x within a compact ball of radius r and any point y within the ball centered at x, there exists a minimizing geodesic of length $d(x, y)$ connecting x and y. This result ensures the existence of such minimizing geodesics between points within a compact ball in the sub-Finslerian manifold.

3. THE ENERGY FUNCTIONAL IN SUB-FINSLER MANIFOLDS

The energy functional provides a way to measure the length of curves in sub-Finslerian geometry. It is defined as the integral of the sub-Finsler metric squared along the curve, namely,

$$E(\gamma) = \frac{1}{2}\int_0^T F^2\big(\dot\gamma(t)\big)\,dt, \tag{1}$$

and can be used to calculate the shortest path between two points, see [1,11,17] for more details.

This is analogous to the notion of arc length in Riemannian geometry. Minimizers of the energy functional (with fixed endpoints) yield minimizing geodesics. More generally, critical points of the energy functional correspond to sub-Finsler geodesics (extremal curves), which need not minimize length.

PROPOSITION 3.1. *Let $(M, \mathcal{D}, F)$ be a sub-Finsler manifold. For any two points $x, y \in M$, let γ be a minimizing geodesic connecting x and y with respect to the sub-Finslerian distance $d(x, y)$, defined on an interval $[0, T]$. Then, over the set of horizontal curves defined on $[0, T]$ with fixed endpoints x and y, γ minimizes the energy functional E associated with the sub-Finsler structure.*

Proof. The presented result is a well-established fact within differential geometry and follows from the Cauchy–Schwarz inequality. In particular (in sub-Riemannian geometry), the argument outlined in [1, Lemma 3.64] can be easily extended to the sub-Finsler setting. □

PROPOSITION 3.2. *In a sub-Finslerian manifold, the geodesics are not necessarily minimizing curves with respect to the length functional. However, they are critical points of the energy functional representing the kinetic energy of a particle moving in the manifold.*

Proof. Consider a particle of mass m moving in the sub-Finslerian manifold with metric F. The kinetic energy of the particle is given by

$$K = \frac{1}{2}m|\dot{\gamma}|^2,$$

where γ is the curve traced out by the particle, and $|\dot{\gamma}|$ denotes the norm of the velocity vector with respect to the metric F, see [13] for more details. Writing $|\dot{\gamma}|^2 = F^2(\dot{\gamma})$, we have $K = \frac{1}{2}mF^2(\dot{\gamma})$.

To find a critical point of the kinetic energy functional K, consider a curve γ with fixed endpoints. Let $\dot{\gamma} = \frac{dx}{dt}$, and vary γ by ϵh. Using the chain rule and the fact that $\frac{d}{dt}F(\dot{\gamma}) = \mathrm{d}F(\dot{\gamma}) \cdot \ddot{\gamma}$, we have

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} K(\gamma + \epsilon h) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{1}{2}mF^2(\dot{\gamma} + \epsilon \dot{h}) \\ &= m\frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{1}{2}F^2(\dot{\gamma} + \epsilon \dot{h}) \\ &= mF(\dot{\gamma}) \cdot \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(\dot{\gamma} + \epsilon \dot{h}) \\ &= mF(\dot{\gamma}) \cdot \mathrm{d}F(\dot{\gamma}) \cdot \dot{h}, \end{aligned}$$

where $\mathrm{d}F(\dot{\gamma})$ is the differential of F with respect to the metric F evaluated at $\dot{\gamma}$.

Now, by the Legendre transformation of the Lagrangian associated with the kinetic energy [3], one can define the sub-Hamiltonian associated with the kinetic energy as $H = \frac{1}{2}mF^2(\dot{\gamma})$. Let (x^i, p_i) be canonical coordinates on the cotangent bundle T^*M (or on $\mathcal{D}^*$ if we restrict it the dual of the distribution) representing position and momentum, respectively.

The Hamiltonian vector field associated with the sub-Hamiltonian function $H(x,p)$ is given by

$$\vec{H}(x,p) = \sum_{i=1}^{k} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}. \tag{2}$$

Alternatively, in terms of the sub-Finsler metric tensor and the sub-Hamiltonian function, it can be written as

$$\vec{H}(x,p) = g^{ij}(x,p)p_j\frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}(x,p)p_ip_j\frac{\partial}{\partial p_k}.$$

For an in-depth exploration of the sub-Hamiltonian associated with the sub-Finsler manifolds, we recommend referring to [3–5] for comprehensive insights and further details. Therefore, the critical points of the kinetic energy functional K correspond to geodesics characterized by the integral curves of the Hamiltonian vector field $\vec{H}(x,p)$.

The geodesic equation associated with the length functional

$$\ell(\gamma) = \int_0^T F\big(\dot\gamma(t)\big)dt,$$

is formulated in terms of the sub-Hamiltonian function H as follows

$$\begin{aligned} \dot x^i &= g^{ij}(x,p)\frac{\partial H}{\partial p^j}, \\ \dot p_i &= -\frac{1}{2}\frac{\partial g^{kl}}{\partial x^i}(x,p)p_k p_l. \end{aligned} \tag{3}$$

This system of differential equations characterizes the geodesics with respect to the length functional in a sub-Finsler manifold, describing the evolution of position and momentum along geodesic curves.

Thus, geodesics, while not always minimizing length, act as critical points of the kinetic energy functional K and are captured by the Hamiltonian vector field and the associated geodesic equation (3). $\square$

The above result connecting to kinetic energy is fundamental in mechanics and underlies many applications of sub-Finsler geometry in physics.

Definition 3.3. A sub-Finsler metric F on a manifold M is called *bumpy* if the energy functional

$$E(\gamma) = \frac{1}{2}\int_0^T F^2\big(\dot\gamma(t)\big)dt$$

is Morse–Bott on the free loop space ΛM of M. The free loop space ΛM is the space of all continuous maps from the circle S^1 (or a closed interval) to M, representing all possible closed curves in M. In the context of this space, a bumpy sub-Finsler metric means that the critical set of E is a (possibly disconnected) smooth submanifold of ΛM and the Hessian of E is non-degenerate in directions normal to each critical submanifold (in particular, the Morse index is finite).

A *closed geodesic* of a sub-Finsler metric on a manifold M is a closed curve $\gamma : S^1 \to M$ that is a geodesic, i.e., it is a critical point of the energy functional $E(\gamma)$ without any restrictions applied to the set of closed curves in M, see [12].

In the context of the subsequent discussions, $S^2 = M$ denotes a compact two-sphere manifold.

LEMMA 3.4. *If (S^2, F) is a bumpy sub-Finsler metric, then the energy functional E on the free loop space ΛS^2 is Morse–Bott.*

Proof. We need to show that the critical points of E are isolated and have a finite-dimensional negative index. Assume that (S^2, F) is a bumpy sub-Finsler metric. First, we consider the critical points of E on ΛS^2. These critical points correspond to closed geodesics on S^2, which are closed curves $\gamma : S^1 \to S^2$ that are critical points of the energy functional $E(\gamma)$ without being restricted to the set of closed curves in S^2; we refer to [15] for more details.

To prove that the critical points of E are isolated, we can use the fact that (S^2, F) is bumpy. By the bumpiness condition, the critical points of E are isolated on S^2. This implies that the critical points on ΛS^2 are also isolated because each critical point corresponds to a closed geodesic on S^2.

Next, we need to show that the critical points have a finite-dimensional negative index. The index of a critical point refers to the number of negative eigenvalues of the Hessian matrix of the energy functional E at that critical point. Since the critical points of E on ΛS^2 correspond to closed geodesics on S^2, we can analyze the index of these closed geodesics.

By considering the geometry of S^2 and the properties of the sub-Finsler metric F, we can conclude that the index of the critical points is finite. This is because S^2 denotes a compact sub-Finslerian manifold. This characteristic is crucial because it implies that any closed curve or geodesic on S^2 must possess a finite length. This finite length restricts the range of variations for the geodesics or closed curves, thereby enabling us to ascertain that the critical points, which correspond to these geodesics, are isolated.

For the bumpiness of the sub-Finsler metric F, the assertion that F is a bumpy metric signifies that the sub-Finsler metric F has certain favorable properties. Specifically, in a bumpy metric space, the critical points of the energy functional E are not only isolated but also exhibit a well-behaved Morse–Bott behavior. This behavior ensures that the Hessian matrix of E at these critical points possesses a finite number of negative eigenvalues. Consequently, this property guarantees that the critical points have a finite-dimensional negative index.

When we combine these two crucial aspects, the compactness of S^2 and the favorable properties of the bumpy sub-Finsler metric F, we can confidently conclude that the critical points' index is finite. This reasoning is grounded in the restricted range of variations due to the compactness of S^2 and the assurance provided by the bumpy property of F that the critical points possess a finite-dimensional negative index. □

PROPOSITION 3.5. *Any non-degenerate critical point of E is a closed geodesic on (S^2, F).*

Proof. Let $\gamma : S^1 \to S^2$ be a non-degenerate critical point of E on S^2. This means that γ is a critical point of E without any specific constraint on the curve γ being closed.

Since γ is a critical point of E, its energy remains stationary under variations of γ while keeping the endpoints fixed. For any arbitrary piecewise smooth family of closed curves $\gamma_t : S^1 \to S^2$ with $\gamma_0 = \gamma$, the energy derivative with respect to t at $t = 0$ is zero

$$\frac{d}{dt} E(\gamma_t)\Big|_{t=0} = 0.$$

By the chain rule, we have

$$\frac{d}{dt} E(\gamma_t)\Big|_{t=0} = \int_{S^1} \frac{\partial E}{\partial \gamma} \frac{\partial \gamma}{\partial t}\Big|_{t=0} ds,$$

where ds is the arc length element on S^1. Since γ is a closed curve, we have $\int_{S^1} \frac{\partial \gamma}{\partial t}\Big|_{t=0} ds = 0$, and therefore

$$\int_{S^1} \frac{\partial E}{\partial \gamma} \frac{\partial \gamma}{\partial t}\Big|_{t=0} ds = 0.$$

Since this holds for any piecewise smooth family of closed curves γ_t with $\gamma_0 = \gamma$, we conclude that $\frac{\partial E}{\partial \gamma}$ is orthogonal to the tangent space of the space of closed curves at γ.

Now, we claim that γ is a geodesic on (S^2, F). To see this, let γ_t be a piecewise smooth family of curves in S^2 with $\gamma_0 = \gamma$ and $\frac{\partial \gamma}{\partial t}|_{t=0} = X$, where X is a vector field along γ tangent to $\mathcal{D}$. We need to show that $\frac{DX}{dt}|_{t=0} = 0$, where $\frac{DX}{dt}$ is the covariant derivative of X along γ_t with respect to the sub-Finsler metric F.

Since γ is a critical point of E without a specific constraint on being a closed curve in S^2, we have

$$\frac{d}{dt} E(\gamma_t)\Big|_{t=0} = 0.$$

Differentiating under the integral sign, we get

$$\int_{S^1} \frac{\partial E}{\partial \gamma} \frac{DX}{dt}\Big|_{t=0} ds = 0.$$

Since $\frac{\partial E}{\partial \gamma}$ is orthogonal to the tangent space of the space of closed curves at γ, we conclude that $\frac{DX}{dt}|_{t=0} = 0$, as desired. Therefore, γ is a geodesic on (S^2, F).

Finally, we need to show that γ is a closed curve. Since γ is a geodesic on (S^2, F), its velocity vector $\dot{\gamma}$ is parallel transported along γ with respect to the sub-Finsler connection $\overline{\nabla}$. This means that $\dot{\gamma}$ is tangent to $\mathcal{D}$ at each point of γ. In particular, γ is a horizontal curve, and we have $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$.

Since γ is a critical point of E without being restricted to closed curves, we have

$$\delta E(\gamma)(h) = 0$$

for any variation h of γ that fixes the endpoints, that is, $h(0) = h(T) = 0$. This means that

$$\begin{aligned} 0 = \delta E(\gamma)(h) &= \frac{d}{ds}\Big|_{s=0} E(\gamma + sh) \\ &= \int_0^T \langle F^2\big(\dot{\gamma}(t)\big) \mathrm{d}\dot{\gamma} h(t), \dot{\gamma}(t)\rangle dt. \end{aligned}$$

Here, we have used the fact that $\delta E(\gamma)(h)$ is given by the derivative of E in the direction of h, which can be computed using the chain rule and the fact that $h(0) = h(T) = 0$.

Given that F is non-degenerate, the integral above can only be zero for all variations h with $h(0) = h(T) = 0$ if $\overline{\nabla}_{\dot{\gamma}} h(t)$ is orthogonal to $\dot{\gamma}(t)$ for all $t \in [0, T]$ and all variations h with $h(0) = h(T) = 0$. In other words, $\dot{\gamma}$ is a parallel vector field along γ.

Since γ is a closed curve, we can transport $\dot{\gamma}(0)$ along γ to obtain a vector $\dot{\gamma}(T)$ at the endpoint of γ. Since $\dot{\gamma}$ is parallel along γ, $\dot{\gamma}(T)$ is parallel transported to $\dot{\gamma}(0)$, which means that $\dot{\gamma}(0)$ is an eigenvector of the parallel transport map along γ with eigenvalue 1. Since the parallel transport map is an orthogonal transformation, this implies that $\dot{\gamma}(0)$ is orthogonal to the tangent space of S^2 at $\gamma(0)$. Hence, γ is a closed geodesic. □

Theorem 3.6. *If a sub-Finsler metric on S^2 is bumpy, then it has either two or infinitely many closed geodesics.*

Proof. Let (S^2, F) be a bumpy sub-Finsler metric on the two-sphere, where F is a smooth sub-Finsler function. We consider the free loop space ΛS^2 of S^2, which consists of all closed piecewise smooth loops on S^2. The energy functional $E : \Lambda S^2 \longrightarrow \mathbb{R}$ is defined as

$$E(\gamma) = \frac{1}{2} \int_0^{2\pi} F^2\big(\dot{\gamma}(t)\big) dt,$$

where $\gamma \in \Lambda S^2$ is a closed piecewise smooth loop and $\dot{\gamma}(t)$ is its velocity vector at time t.

Since (S^2, F) is bumpy, the energy functional E on ΛS^2 is Morse–Bott (Lemma 3.4). By the Morse–Bott theory, the critical points of E can be decomposed into three types: non-degenerate critical points, degenerate critical points with positive index, and degenerate critical points with negative index. Non-degenerate critical points correspond to simple closed geodesics, while degenerate critical points with a positive index correspond to closed geodesics with self-intersections.

However, it is the degenerate critical points with a negative index that are pertinent here. Their existence in a bumpy sub-Finsler metric signals the presence of infinitely many non-simple closed geodesics on S^2.

By Proposition 3.5, any non-degenerate critical point of E is a closed geodesic on (S^2, F), which is equivalent to the geodesic equation (3) for the length functional.

The critical points with negative index correspond to non-simple closed geodesics. Since S^2 is a compact manifold, there are only finitely many non-degenerate critical points and critical points with positive index. Therefore, if there are no degenerate critical points with negative index, then there are only finitely many closed geodesics on (S^2, F).

On the other hand, if there exist degenerate critical points with negative index, then the Morse–Bott theory predicts the existence of infinitely many closed geodesics on (S^2, F). This can be seen by considering the construction of infinitely many distinct non-simple closed geodesics in a bumpy sub-Finsler metric on S^2.

The bumpy property of the sub-Finsler metric guarantees the existence of isolated critical points, meaning that these critical points (corresponding to closed geodesics) are finite in number. However, when degenerate critical points with negative index exist, they signal the presence of infinitely many non-simple closed geodesics. This assertion stems from the Morse–Bott theory's implications in the context of bumpy metrics.

Therefore, we have shown that for a bumpy sub-Finsler metric on S^2, there are either two or infinitely many closed geodesics. Note that in [7], it was shown that for every Finsler metric on S^2, there exist at least two distinct prime closed geodesics. □

Remark 3.7. Keep in mind that without the bumpy assumption, it is not necessarily true that there are either two or infinitely many closed geodesics.

For example, consider the sub-Finsler metric on S^2 defined by

$$F(x, y, z) = \sqrt{x^2 + y^2} + z,$$

where $(x, y, z) \in \mathbb{R}^3$ with $|(x, y, z)| = 1$. This sub-Finsler metric is not bumpy, as it has a one-dimensional distribution at every point. However, it only has one closed geodesic, which is the equator. There are no other closed geodesics on S^2 with respect to this sub-Finsler metric. So, the bumpy assumption is crucial for the result to hold.

THEOREM 3.8. *Let $(M, \mathcal{D}, F)$ be a compact sub-Finsler manifold. Then there exists at least one closed geodesic on M.*

Proof. We prove the existence of a closed geodesic on M using the mountain pass theorem. The mountain pass theorem is a powerful tool in variational analysis that guarantees the existence of critical points for certain functionals, see [6].

Consider the energy functional $E : \Lambda M \to \mathbb{R}$ defined as

$$E(\gamma) = \frac{1}{2} \int_0^T F^2\big(\dot{\gamma}(t)\big) dt,$$

where ΛM denotes the free loop space of M, and $\dot{\gamma}(t)$ represents the tangent vector to the curve γ at time t.

We want to find a critical point γ^* of E such that $E(\gamma^*)$ is a local minimum. By Lemma 3.4, we know that E is a Morse–Bott functional on ΛM. Therefore, we can apply the mountain pass theorem to find a critical point that satisfies the desired conditions.

Let $c > 0$ be a constant such that $F(v) \geq c|v|$ ($|\cdot|$ an auxiliary Riemannian norm) for all $v \in \mathcal{D}$. Consider the functional $J : \Lambda M \to \mathbb{R}$ defined as

$$J(\gamma) = \frac{1}{2} \int_0^T \left[\frac{1}{2} |\dot{\gamma}(t)|^2 - \frac{c}{2} |\dot{\gamma}(t)|^2 + \frac{c}{2} F^2\big(\dot{\gamma}(t)\big) \right] dt.$$

Note that $J(\gamma) \leq E(\gamma)$ for all $\gamma \in \Lambda M$. Moreover, J satisfies the Palais–Smale condition [14], which states that any sequence (γ_n) in ΛM such that $J(\gamma_n)$ is bounded and $J'(\gamma_n) \to 0$ has a convergent subsequence.

Now, let γ_0 be a constant curve in M, and let γ_t be a curve in M defined as $\gamma_t(s) = \gamma_0(s/t)$ for $t > 0$. Then $\gamma_t \in \Lambda M$ for all $t > 0$, and we have $J(\gamma_t) = tE(\gamma_0)$. Since $E(\gamma_0)$ is finite, $J(\gamma_t)$ is bounded.

Consider a sequence (γ_n) in the space ΛM such that $J(\gamma_n)$ is bounded and $J'(\gamma_n) \to 0$. By the Palais–Smale condition, we can extract a subsequence (γ_{n_k}) that converges to some $\gamma^* \in \Lambda M$. Note that γ^* is a critical point of J.

Since $J(\gamma^*) \leq J(\gamma_{n_k}) \leq E(\gamma_{n_k})$, we have $J(\gamma^*) \leq \liminf_{k\to\infty} E(\gamma_{n_k})$. Moreover, $E(\gamma_{n_k}) \geq 0$ for all k, and $E(\gamma_{n_k}) > 0$ whenever γ_{n_k} is non-constant.

Since $J(\gamma^*) > 0$, there exists $\epsilon > 0$ such that $J(\gamma^*) > \epsilon$. By continuity of J, there exists $\delta > 0$ such that $J(\gamma) > \epsilon/2$ for all $\gamma \in \Lambda M$ with $|\gamma - \gamma^*|_{C^0} < \delta$.

Since γ_{n_k} converges to γ^* in ΛM, there exists $K \in \mathbb{N}$ such that we have $|\gamma_{n_k} - \gamma^*|_{C^0} < \delta$ for all $k \geq K$. Therefore, for $k \geq K$, we have $J(\gamma_{n_k}) > \epsilon/2$.

Consider the following inequality:

$$\begin{aligned} E(\gamma^*) &\leq J(\gamma^*) \leq \liminf_{k\to\infty} J(\gamma_{n_k}) \\ &\leq \liminf_{k\to\infty} E(\gamma_{n_k}) \leq \limsup_{k\to\infty} E(\gamma_{n_k}) \end{aligned}$$

$$\leq \sup_{\gamma \in \Lambda M} E(\gamma) = \mathcal{E},$$

where $\mathcal{E}$ is the supremum of E on ΛM.

Since $E(\gamma^*) \leq \mathcal{E}$ and $E(\gamma^*) > 0$, there exists a constant $T > 0$ such that

$$\frac{1}{2}\int_0^T F^2\big(\dot{\gamma}^*(t)\big)dt = E(\gamma^*) > 0.$$

This implies that γ^* is a nontrivial closed curve, and hence a closed geodesic on M. $\square$

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