COMPLEX WEYL CORRESPONDENCE AND METAPLECTIC REPRESENTATION OF THE JACOBI GROUP

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We study the metaplectic representation of the Jacobi group (the semi-direct product of the Heisenberg group by $SU(1,1)$) by using the complex Weyl correspondence. In particular, we give explicit formulas for the complex Weyl symbols of the metaplectic representation operators and we prove that the complex Weyl correspondence is a Stratonovich–Weyl correspondence for the metaplectic representation.

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1. INTRODUCTION

Let H be the 3-dimensional real Heisenberg group. The Jacobi group is the semi-direct product $G = H \rtimes SU(1,1)$, which plays a prominent role in various areas of Mathematics and Physics, see [4, 5] and references therein.

The Jacobi group is a non-reductive Lie group of Harish-Chandra type which admits highest weight representations on Hilbert spaces of holomorphic functions, see [20,21] and also [4,5]. The (extended) metaplectic representation of G is obtained by combining a non-degenerate unitary irreducible representation of H with the metaplectic representation of $SU(1,1)$ [15]. Let us recall that the metaplectic representation of $SU(1,1)$ is a projective unitary representation σ of $SU(1,1)$ which appears naturally when we consider the action of $SU(1,1)$ on the non-degenerate unitary irreducible representations of H, see [14, 15].

The complex Weyl calculus W is a correspondence between operators on the Fock space $\mathcal{F}$ and functions on $\mathbb{C}$ which can be obtained by translating the usual Weyl correspondence by means of the Bargmann transform or by taking the unitary component in the polar decomposition of the Berezin correspondence on $\mathcal{F}$ [6, 19].

In [12, 13], we studied the metaplectic representation of the homogeneous symplectic group and of the non-homogeneous symplectic group in connection to the complex Weyl correspondence. In particular, we give formulas for the Berezin symbols and for the complex Weyl symbols of the metaplectic representation operators. Here, we treat in detail the case of the Jacobi group. In this case, the technicalities of [12, 13] can be avoided to a large extent. This allows us to present the basic ideas more clearly and also to obtain more explicit formulas than in the case of the (non-homogeneous) symplectic group. Our hope is to make the present note more readable than [12] and [13] and, therefore, accessible to a wider audience.

More precisely, the content of this note is as follows. We first introduce the Fock model of the non-degenerate unitary irreducible representation of H (Section 2), the complex Weyl correspondence (Section 3) and the metaplectic representation of $SU(1,1)$ (Section 4). Then we consider the Jacobi group and its holomorphic representations (Section 5) and, following [11, 12, 20], we use them to obtain some formulas for the kernels of the metaplectic representation operators (Section 6). From this, we deduce explicit formulas for the complex Weyl symbols of the (extended) metaplectic representation operators (Section 7). We study covariance of W for the metaplectic representation and prove that W is a Stratonovich–Weyl correspondence (Section 8). Finally, we compute the complex Weyl symbols of the differential metaplectic representation operators in connection to some coadjoint orbits of the Jacobi group (Section 9).

2. BEREZIN QUANTIZATION FOR HEISENBERG GROUP

Here, we introduce the Bargmann–Fock model of the unitary irreducible (non-degenerate) representations of the Heisenberg group [15, 23]. We follow more or less [9, 10, 12].

For each $z, z', w, w' \in \mathbb{C}$, let

$$\omega\big((z,w),(z',w')\big) = \frac{i}{2}(zw' - z'w).$$

Then the 3-dimensional real Heisenberg group is

$$H := \big\{((z,\bar{z}),c) \,:\, z \in \mathbb{C}, c \in \mathbb{R}\big\}$$

with the multiplication law

$$\big((z,\bar{z}),c\big)\cdot\big((z',\bar{z}'),c'\big) = \left((z+z',\bar{z}+\bar{z}'),c+c'+\frac{1}{2}\omega\big((z,\bar{z}),(z',\bar{z}')\big)\right).$$

Fix $\lambda > 0$. By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation ρ_λ of H whose restriction to the center of H is the character $((0,0),c) \to e^{i\lambda c}$ [15, 23]. The Bargmann–Fock realization of ρ_λ is defined as follows [23].

Let $\mathcal{F}_\lambda$ be the Hilbert space of all holomorphic functions f on $\mathbb{C}$ such that

$$\|f\|^2_{\mathcal{F}_\lambda} := \int_{\mathbb{C}} |f(z)|^2 \, e^{-\lambda |z|^2/2} \, dm_\lambda(z) < +\infty$$

where $dm_\lambda(z) := (2\pi)^{-1} \lambda \, dm(z)$. Here, $z = x + iy$ with x and y in $\mathbb{R}$ and $dm(z) := dx\, dy$ denotes the Lebesgue measure on $\mathbb{C}$.

Then we have

$$\big(\rho_\lambda(h) f\big)(z) = \exp\left(i\lambda c_0 + \frac{\lambda}{2} \bar{z}_0 z - \frac{\lambda}{4} |z_0|^2 \right) f(z - z_0)$$

for each $h = ((z_0, \bar{z}_0), c_0) \in H$ and $z \in \mathbb{C}$.

For each $z \in \mathbb{C}$, define the *coherent state* e_z by $e_z(w) = \exp(\lambda \bar{z} w/2)$. Then we can verify the reproducing property $f(z) = \langle f, e_z \rangle_{\mathcal{F}_\lambda}$ for each $f \in \mathcal{F}_\lambda$.

Let us introduce the Berezin calculus on $\mathcal{F}_\lambda$ [2, 3, 6]. The Berezin (covariant) symbol of an operator A on $\mathcal{F}_\lambda$ is the function $S_\lambda(A)$ defined on $\mathbb{C}$ by

$$S_\lambda(A)(z) := \frac{\langle A\, e_z \, , \, e_z \rangle_{\mathcal{F}_\lambda}}{\langle e_z \, , \, e_z \rangle_{\mathcal{F}_\lambda}}$$

and the double Berezin symbol s_λ is defined by

$$s_\lambda(A)(z, w) := \frac{\langle A\, e_w \, , \, e_z \rangle_{\mathcal{F}_\lambda}}{\langle e_w \, , \, e_z \rangle_{\mathcal{F}_\lambda}}$$

for each $(z, w) \in \mathbb{C}^2$.

The function $s_\lambda(A)(z, w)$ is holomorphic in z and anti-holomorphic in w, then $s_\lambda(A)$ is determined by $S_\lambda(A)$. Moreover, we have

$$(A\, f)(z) = \langle A\, f \, , \, e_z \rangle_{\mathcal{F}_\lambda} = \langle f \, , \, A^*\, e_z \rangle_{\mathcal{F}_\lambda}$$

hence

$$\begin{aligned}(A\, f)(z) &= \int_{\mathbb{C}} \overline{(A^* e_z)(w)} f(w) \, e^{-\lambda |w|^2/2} \, dm_\lambda(w) \\ &= \int_{\mathbb{C}} \langle A e_w, e_z \rangle_{\mathcal{F}_\lambda} f(w) \, e^{-\lambda |w|^2/2} \, dm_\lambda(w).\end{aligned}$$

This implies that the kernel of A which is the function $k_A(z, w)$ on $\mathbb{C}^2$ such that

$$(A\, f)(z) = \int_{\mathbb{C}} k_A(z, w) f(w) \, e^{-\lambda |w|^2/2} \, dm_\lambda(w) \tag{1}$$

is given by

$$k_A(z, w) = \langle A e_w, e_z \rangle_{\mathcal{F}_\lambda} = s_\lambda(A)(z, w) \langle e_w, e_z \rangle_{\mathcal{F}_\lambda} \tag{2}$$

and that the map $A \to S_\lambda(A)$ is injective.

Let us introduce the action of H on $\mathbb{C}$ defined by $h \cdot z = z + z_0$ for $h = ((z_0, \bar{z}_0), c_0) \in H$ and $z \in \mathbb{C}$. From the formula

$$\rho_\lambda(h)e_z = \exp\left(i\lambda c_0 - \frac{\lambda}{2}\bar{z}z_0 - \frac{\lambda}{4}|z_0|^2\right) e_{h\cdot z} \tag{3}$$

for $h = ((z_0, \bar{z}_0), c_0) \in H$ and $z \in \mathbb{C}$, we deduce that S_λ is covariant with respect to ρ_λ, that is, for each operator A on $\mathcal{F}_\lambda$, each $h \in H$ and each $z \in \mathbb{C}$, we have

$$S_\lambda\big(\rho_\lambda(h)^{-1}A\rho_\lambda(h)\big)(z) = S_\lambda(A)(h \cdot z).$$

Let us note that S_λ is a bounded operator from the space $\mathcal{L}_2(\mathcal{F}_\lambda)$ of all Hilbert–Schmidt operators on $\mathcal{F}_\lambda$ (equipped with the Hilbert–Schmidt norm) to $L^2(\mathbb{C}, m_\lambda)$ which is one-to-one and has dense range [24]. Let S_λ^* be the adjoint operator of S_λ. Then the Berezin transform is the operator B_λ on $L^2(\mathbb{C}, m_\lambda)$ defined by $B_\lambda := S_\lambda S_\lambda^*$ and we have

$$(B_\lambda f)(z) = \int_{\mathbb{C}} f(w)\, e^{-\lambda|z-w|^2/2}\, dm_\lambda(w),$$

see [2,3,24]. It is also known that we have the equality $B_\lambda = \exp(\Delta/2\lambda)$ where $\Delta = 4\partial^2/\partial z\partial\bar{z}$, see [19,24].

3. COMPLEX WEYL CORRESPONDENCE FOR HEISENBERG GROUP

In this section, we first recall the definition of the complex Weyl correspondence from a *Stratonovich–Weyl quantizer* [1,9,16,22].

We begin by introducing the parity operator R on $\mathcal{F}_\lambda$

$$(Rf)(z) = 2f(-z).$$

Then we can define the *Stratonovich–Weyl quantizer* Ω by

$$\Omega(z) := \rho_\lambda\big((z, \bar{z}), 0\big)R\rho_\lambda\big((z, \bar{z}), 0\big)^{-1}$$

for each $z \in \mathbb{C}$. We can easily verify that we have

$$\big(\Omega(z)f\big)(w) = 2\exp\big(\lambda(w\bar{z} - |z|^2)\big)\, f(2z - w) \tag{4}$$

for each $z, w \in \mathbb{C}$ and each $f \in \mathcal{F}_\lambda$.

Now, for each trace-class operator A on $\mathcal{F}_\lambda$, we define

$$W(A)(z) := \mathrm{Tr}\big(A\Omega(z)\big)$$

for each $z \in \mathbb{C}$.

By using Mercer's theorem, we can prove the following result, see [1,9,10].

Proposition 3.1. *For each trace-class operator A on $\mathcal{F}_\lambda$ and each $z \in \mathbb{C}$, we have*

$$W(A)(z) = 2\int_{\mathbb{C}} k_A(z+w, z-w)\exp\left(\frac{\lambda}{2}(-z\bar{z} - w\bar{w} + z\bar{w} - \bar{z}w)\right)dm_\lambda(w). \tag{5}$$

Note that this integral formula allows to extend W to operators on $\mathcal{F}_\lambda$ which are not necessarily trace-class, for instance Hilbert–Schmidt operators. Note also that $W : \mathcal{L}_2(\mathcal{F}_\lambda) \to L^2(\mathbb{C}, m_\lambda)$ is the unitary part in the polar decomposition of S_λ, that is, we have $S_\lambda = B_\lambda^{1/2}W$ [10, 19].

The connection between W and the usual Weyl correspondence via the Bargmann transform is detailed in [12].

Proposition 3.2. *The map W is covariant for ρ_λ, that is, for each operator A on $\mathcal{F}_\lambda$, each $h \in H$ and each $z \in \mathbb{C}$, we have*

$$W\big(\rho_\lambda(h)^{-1}A\rho_\lambda(h)\big)(z) = W(A)(h \cdot z).$$

Proof. From the definition of Ω, we can deduce that for each $h \in H$ and each $z \in \mathbb{C}$, we have

$$\Omega(h \cdot z) = \rho_\lambda(h)\Omega(z)\rho_\lambda(h)^{-1}.$$

The result follows immediately. For an alternate proof, we can use the fact that $W = B_\lambda^{-1/2}S_\lambda$ and the covariance of S_λ for ρ_λ. □

4. THE METAPLECTIC REPRESENTATION

We first introduce the metaplectic representation of $SU(1,1)$. From the action of $SU(1,1)$ on $\mathbb{C}$ given by

$$kz := az + b\bar{z}, \quad k = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

we define an action of $SU(1,1)$ on H by

$$k \cdot \big((z, \bar{z}), c\big) = \big((kz, \overline{kz}), c\big)$$

for $k \in SU(1,1)$, $z \in \mathbb{C}$ and $c \in \mathbb{R}$.

In the rest of the paper, we fix $\lambda > 0$ and denote $\rho := \rho_\lambda$ and $\mathcal{F} := \mathcal{F}_\lambda$. For each $k \in SU(1,1)$, let ρ_k be defined by $\rho_k(h) := \rho(k \cdot h)$ for each $h \in H$. Then ρ_k is also a non-degenerate unitary irreducible representation of H which has the same central character as ρ. Hence, by the Stone-von Neumann theorem, ρ_k and ρ are unitarily equivalent and there exists a unitary operator A_k on $\mathcal{F}$ (defined up to a unit complex number) such that

$$\rho_k(h) = A_k\rho(h)A_k^{-1} \tag{6}$$

for each $h \in H$.

Proposition 4.1. *Let $k \in SU(1,1)$. Then A_k satisfies equation (6) if and only if its kernel $b_k(z,w) := k_{A_k}(z,w)$ satisfies the functional equation*

(7)
$$\exp\left(-\frac{\lambda}{4}|kz_0|^2+\frac{\lambda}{2}\overline{kz_0}z\right)b_k(z-kz_0,w)=\exp\left(-\frac{\lambda}{4}|z_0|^2-\frac{\lambda}{2}z_0\bar{w}\right)b_k(z,w+z_0)$$

for each z_0, z and w in $\mathbb{C}$.

Proof. The result follows immediately form the definition of the kernel of an operator on $\mathcal{F}$ (see equation 2) and from the definition of ρ (Section 2). □

As we see later, it is possible to choose the operators A_k, $k \in SU(1,1)$ so that we have $A_kA_{k'} = \pm A_{kk'}$ for each $k, k' \in SU(1,1)$ and with this choice, we then denote $\sigma(k) := A_k$ for $k \in SU(1,1)$. The (projective) representation σ of $SU(1,1)$ on $\mathcal{F}$ is called the *metaplectic representation* of $SU(1,1)$.

The Jacobi group is the semi-direct product $G := H \rtimes SU(1,1)$ with respect to the above action of $SU(1,1)$ on H. By using equation 6, we can verify that the map π defined by

(8) $$\pi(h,k) := \rho(h)\sigma(k), \quad h \in H,\, k \in SU(1,1)$$

is a (projective) unitary representation of G on $\mathcal{F}$, called the *extended metaplectic representation* of G [15, p. 196].

Proposition 4.2. *Let $g = (h,k) \in G$ where $h = ((z_0,\bar{z}_0),c_0) \in H$. Then the kernel $B_g(z,w)$ of $\pi(g)$ is given by*

$$B_g(z,w)=\exp\left(i\lambda c_0+\frac{\lambda}{2}\bar{z}_0z-\frac{\lambda}{4}|z_0|^2\right)b_k(z-z_0,w).$$

Proof. Let $g = (h,k) \in G$ where $h = ((z_0,\bar{z}_0),c_0) \in H$. Then we have

$$B_g(z,w)=\langle \pi(g)e_w,e_z\rangle_{\mathcal{F}}=\langle \rho_\lambda(h)\sigma(k)e_w,e_z\rangle_{\mathcal{F}}=\langle \sigma(k)e_w,\rho_\lambda(h)^{-1}e_z\rangle_{\mathcal{F}}.$$

From equation 3, we deduce

$$\rho_\lambda(h)^{-1}e_z=\exp\left(-i\lambda c_0+\frac{\lambda}{2}\bar{z}z_0-\frac{\lambda}{4}|z_0|^2\right)e_{z-z_0}.$$

This gives

$$\begin{aligned}B_g(z,w)&=\exp\left(i\lambda c_0+\frac{\lambda}{2}\bar{z}_0z-\frac{\lambda}{4}|z_0|^2\right)\langle \sigma(k)e_w,e_{z-z_0}\rangle_{\mathcal{F}}\\&=\exp\left(i\lambda c_0+\frac{\lambda}{2}\bar{z}_0z-\frac{\lambda}{4}|z_0|^2\right)b_k(z-z_0,w).\end{aligned}$$ □

5. THE HOLOMORPHIC REPRESENTATIONS OF THE JACOBI GROUP

The material of this section is essentially taken from [7] in which we constructed the holomorphic representations of G by using the method of [20], see also [8].

We can write the elements of G as $((z,\bar{z}),c,k)$ where $z\in\mathbb{C}$, $c\in\mathbb{R}$ and $k\in SU(1,1)$. The multiplication of G is then given by

$$\big((z,\bar{z}),c,k\big)\cdot\big((z',\bar{z}'),c',k'\big)=\Big((z,\bar{z})+(kz',\overline{kz'}),c+c'+\frac{1}{2}\omega\big((z,\bar{z}),(kz',\overline{kz'})\big),kk'\Big).$$

The complexification G^c of G is then the semi-direct product $G^c=H^c\rtimes SL(2,\mathbb{C})$ whose multiplication is

$$\big((z,w),c,k\big)\cdot\big((z',w'),c',k'\big)=\Big((z,w)+k(z',w'),c+c'+\frac{1}{2}\omega\big((z,w),k(z',w')\big),kk'\Big)$$

where $z,w,z',w,c,c'\in\mathbb{C}$ and $k,k'\in SL(2,\mathbb{C})$. Here, we denote by

$$k(z,w):=(az+bw,cz+dw),\quad k=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{C}),$$

the natural action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$.

Let $\mathfrak{g}$ and $\mathfrak{g}^c$ be the Lie algebras of G and G^c. For each

$$X=\left((z,w),c,\begin{pmatrix}a&b\\c&-a\end{pmatrix}\right)\in\mathfrak{g}^c,$$

let

$$X^*=\left((-\bar{w},-\bar{z}),-\bar{c},\begin{pmatrix}\bar{a}&-\bar{c}\\-\bar{b}&-\bar{a}\end{pmatrix}\right).$$

Also, let $g\to g^*$ be the involutive anti-automorphism of G^c obtained by exponentiating $X\to X^*$.

Let K be the subgroup of G consisting of all elements $((0,0),c,\left(\begin{smallmatrix}a&0\\0&\bar{a}\end{smallmatrix}\right))$ where $c\in\mathbb{R}$ and $|a|=1$. Let $\mathfrak{k}$ be the Lie algebra of K.

Let P^+ and P^- be the subgroups of G^c defined by

$$P^+=\left\{\left((z,0),0,\begin{pmatrix}1&u\\0&1\end{pmatrix}\right)\ :\ z\in\mathbb{C},u\in\mathbb{C}\right\}$$

and

$$P^-=\left\{\left((0,w),0,\begin{pmatrix}1&0\\v&1\end{pmatrix}\right)\ :\ w\in\mathbb{C},v\in\mathbb{C}\right\}.$$

Let $\mathfrak{p}^+$ and $\mathfrak{p}^-$ be the Lie algebras of P^+ and P^-. We denote by $a(z,u)$ the element $((z,0),0,\left(\begin{smallmatrix}0&u\\0&0\end{smallmatrix}\right))$ of $\mathfrak{p}^+$. Note that we have

$$\exp_{G^c}\big(a(z,u)\big)=\left((z,0),0,\begin{pmatrix}1&u\\0&1\end{pmatrix}\right).$$

The group G is a group of the Harish-Chandra type [20, p. 507], that is, we have

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\mathfrak{k}^+, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$;

2. The multiplication map $P^+K^cP^- \to G^c$, $(z,k,y) \to zky$ is a biholomorphic diffeomorphism onto its open image;

3. $G \subset P^+K^cP^-$ and $G \cap K^cP^- = K$.

We can easily verify that each element $g = ((z_0,w_0), c_0, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ in G^c has a $P^+K^cP^-$-decomposition

$$g = \left((z,0),0,\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\cdot\left((0,0),\tilde{c},\begin{pmatrix}p & 0\\ 0 & p^{-1}\end{pmatrix}\right)\cdot\left((0,w),0,\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix}\right)$$

if and only if $d \neq 0$ and, in this case, we have $z = z_0 - bd^{-1}w_0$, $u = bd^{-1}$, $v = cd^{-1}$, $w = d^{-1}w_0$, $p = d^{-1}$ and $\tilde{c} = c_0 - (1/4)i(z_0 - bd^{-1}w_0)w_0$.

We denote by the following $\zeta : P^+K^cP^- \to P^+$, $\kappa : P^+K^cP^- \to K^c$ and $\eta : P^+K^cP^- \to P^-$ the projections onto P^+-, K^c- and P^--components.

We can introduce an action (defined almost everywhere) of G^c on $\mathfrak{p}^+$ as follows. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, let $g \cdot Z$ the element of $\mathfrak{p}^+$ defined by $g \cdot Z := \log \zeta(g \exp Z)$. We can verify that the action of $g = ((z_0,w_0), c_0, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \in G^c$ on $a(z,u) \in \mathfrak{p}^+$ is given by $g \cdot a(z,u) = a(z',u')$ where $u' = (au+b)(cu+d)^{-1}$ and

$$z' = z_0 + az - (au+b)(cu+d)^{-1}(w_0 + cz).$$

Consequently, we have

$$\mathcal{D} := G \cdot 0 = \left\{a(z,u) \in \mathfrak{p}^+ \ : \ |u| < 1\right\} \simeq \mathbb{C} \times \mathbb{D}$$

where $\mathbb{D}$ is the unit open disk of $\mathbb{C}$.

Let χ be a unitary character of the subgroup K. We also denote by χ the extension of χ to K^c. We follow [20] and introduce the next two functions $K_\chi(Z,W) = \chi(\kappa(\exp Z^* \exp Z))^{-1}$ for Z, $W \in \mathcal{D}$ and $J_\chi(g,Z) = \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. We then consider the Hilbert space $\mathcal{H}_\chi$ of all holomorphic functions on $\mathcal{D}$ such that

$$\|f\|^2_\chi := \int_{\mathcal{D}} |f(Z)|^2 \, K_\chi(Z,Z)^{-1} c_\chi d\mu(Z) < +\infty$$

where μ is the G-invariant measure on $\mathcal{D}$ given by

$$d\mu(Z) = (1 - u\bar{u})^{-3} \, dm(z) dm(u)$$

and c_χ is the constant defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z,Z)^{-1}\, d\mu(Z).$$

Therefore, we fix χ as follows. Let $\lambda > 0$ and $m \in \mathbb{Z}$. Then, for each $k = ((0,0), c, \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)) \in K$, we define $\chi(k) = e^{i\lambda c} a^{-m}$.

The following lemma is used several times in this paper. For $z \in \mathbb{C}$, we define $z^{1/2}$ as the principal determination of the square root (with branch cut along the negative real axis).

LEMMA 5.1. 1. *Let A be a $n \times n$ symmetric complex matrix such that* $\mathrm{Re}(A)$ *is definite positive and $z \in \mathbb{C}^n$. Then we have*

$$\int_{\mathbb{R}^n} \exp(-xAx + zx)\, dx = (\mathrm{Det}\, A)^{-1/2} \pi^{n/2} \exp\left(\frac{1}{4} z(A^{-1}z)\right).$$

2. *Let $M = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$ and $N = U^t M U$. Assume that* $\mathrm{Re}(N)$ *is positive definite. Let $u, v \in \mathbb{C}^n$. Then we have*

$$\int_{\mathbb{C}} \exp\big(-(aw^2 + 2b|w|^2 + d\bar{w}^2)\big) \exp(uw + v\bar{w})\, dm(w)$$
$$= \pi (\mathrm{Det}\, N)^{-1/2} \exp\left(\frac{1}{4} \begin{pmatrix} u & v \end{pmatrix} M^{-1} \begin{pmatrix} u \\ v \end{pmatrix}\right).$$

Proof. (1) is well known. To prove (2), we note that, writing $w = x + iy$ with $x, y \in \mathbb{R}$, we have $\left(\begin{smallmatrix} w \\ \bar{w} \end{smallmatrix}\right) = U \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Then we get

$$aw^2 + 2b|w|^2 + d\bar{w}^2 = (w, \bar{w}) M \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = (x, y) N \begin{pmatrix} x \\ y \end{pmatrix}$$

and $uw + v\bar{w} = \left(\begin{smallmatrix} u & v \end{smallmatrix}\right) U \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. The result hence follows by using (1). □

PROPOSITION 5.2 ([7]). *We have the following equalities:*

1. *Let $Z = a(z,u) \in \mathcal{D}$ and $W = a(w,v) \in \mathcal{D}$. Then we have*

$$K_\chi(Z, W) = (1 - u\bar{v})^{-m} \exp\left(\frac{\lambda}{4}\left(\frac{2\bar{w}z + \bar{v}z^2 + u\bar{w}^2}{1 - u\bar{v}}\right)\right).$$

2. *For each $g = ((z_0, \bar{z_0}), c_0, \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)) \in G$ and each $Z = a(z,u) \in \mathcal{D}$, we have*

$$J_\chi(g, Z) = e^{i\lambda c_0} (\bar{b}u + \bar{a})^m \exp\left(\frac{\lambda}{4}\left(|z_0|^2 + a\bar{b}z^2 + 2a\bar{z}_0 z\right)\right)$$
$$\times \exp\left(-\frac{\lambda}{4} \frac{au + b}{\bar{b}u + \bar{a}} (\bar{z_0} + \bar{b}z)^2\right).$$

3. We have $\mathcal{H}_\chi \neq (0)$ if and only if $m > 3/2$. In this case, we also have $c_\chi = \frac{\lambda}{2\pi^2}(m - \frac{3}{2})$.

Proof. (1) and (2) follow from simple computations using the $P^+K^cP^-$-decomposition.

(3) By [20, Theorem XII.5.6], we know that $\mathcal{H}_\chi \neq (0)$ if and only

$$c_\chi^{-1} := \int_{\mathcal{D}} K_\chi(Z,Z)^{-1}\, d\mu(Z) < +\infty.$$

But, by (1) and Lemma 5.1, we have

$$\begin{aligned} c_\chi^{-1} &= \int_{\mathbb{C}\times\mathbb{D}} (1-u\bar{u})^{m-3} \exp\left(-\frac{\lambda}{4}\left(\frac{2\bar{z}z + \bar{u}z^2 + u\bar{z}^2}{1-u\bar{u}}\right)\right) dm(z)dm(u) \\ &= \frac{2\pi}{\lambda}\int_{\mathbb{D}} (1-u\bar{u})^{m-\frac{5}{2}}\, dm(u) = \frac{2\pi^2}{\lambda}\int_0^1 (1-t)^{m-\frac{5}{2}}\, dt. \end{aligned}$$

The result follows. $\square$

From now on, we assume that $m > 3/2$. Then the formula

$$\big(\pi_\chi(g)\big)f(Z) = J_\chi(g^{-1}, Z)^{-1}\, f(g^{-1}\cdot Z)$$

defines a unitary representation of G on $\mathcal{H}_\chi$ [20, p. 540].

Moreover, $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, if we define the equality $e_Z(W) := K_\chi(W,Z)$ then we have the reproducing property $f(Z) = \langle f, e_Z\rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [20, p. 540]. Here $\langle\cdot,\cdot\rangle_\chi$ stands for the inner product on $\mathcal{H}_\chi$.

Then, by using the coherent states $(e_Z)_{Z\in\mathcal{D}}$, we can define the Berezin symbol $S_\chi(A)(Z)$ and the double Berezin symbol $s_\chi(A)(Z,W)$ of an operator A on $\mathcal{H}_\chi$ as in Section 2. Note also that the kernel $k_A(Z,W)$ of A which is the function on $\mathcal{D}^2$ such that

$$f(Z) = \int_{\mathcal{D}} k_A(Z,W) f(W) K_\chi(W,W)^{-1} c_\chi d\mu(W)$$

for each $f \in \mathcal{H}_\chi$ is given by

$$k_A(Z,W) = \langle A\, e_W\,,\, e_Z\rangle_\chi = s_\chi(A)(Z,W)\langle e_W\,,\, e_Z\rangle_\chi.$$

Proposition 5.3. *For each* $g = ((z_0,\bar{z}_0), c_0, \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right)) \in G$ *and for each* $Z = a(z,u) \in \mathcal{D}$, *we have*

$$\begin{aligned} \big(\pi_\chi(g)f\big)(Z) &= e^{i\lambda c_0}(-\bar{b}u + a)^{-m} \\ &\quad\times \exp\left(\frac{\lambda}{4}\left(\overline{ab}z^2 + 2\bar{a}(a\bar{z}_0 - \bar{b}z_0)z - |\bar{a}z_0 - b\bar{z}_0|^2\right)\right) \\ &\quad\times \exp\left(\frac{\lambda}{4}\frac{\bar{a}u - b}{-\bar{b}u + a}(\bar{b}z_0 - a\bar{z}_0 - \bar{b}z)^2\right) f\left(a\left(\frac{u\bar{z}_0 - z_0 + z}{-\bar{b}u + a}, \frac{\bar{a}u - b}{-\bar{b}u + a}\right)\right). \end{aligned}$$

In particular, for each $g=((0,0),0,\left(\begin{smallmatrix} a & b\\ \bar b & \bar a\end{smallmatrix}\right))$*, we have*

$$\big(\pi_\chi(g)f\big)(Z)=(-\bar b u+a)^{-m}\exp\left(\frac{\lambda}{4}\frac{\bar b z^2}{-\bar b u+a}\right)\times f\left(a\left(\frac{z}{-\bar b u+a},\frac{\bar a u-b}{-\bar b u+a}\right)\right)$$

and for each $g=((z_0,\bar z_0),0,I_2))$,

$$\big(\pi_\chi(g)f\big)(Z)=\exp\left(\frac{\lambda}{4}(-|z_0|^2+2\bar z_0 z+u\bar z_0{}^2)\right)f\big(a(z+u\bar z_0-z_0,u)\big).$$

Proof. This follows from (2) of Proposition 5.2. □

6. DETERMINATION OF THE KERNELS OF METAPLECTIC OPERATORS

In this section, we follow closely [11, 12, 20] and use π_χ in order to find a solution to equation 7. The idea is to translate in terms of kernels the relation

$$\pi_\chi\big((kz_0,\overline{kz_0}),0,I_2\big)\pi_\chi\big((0,0),0,k\big)=\pi_\chi\big((0,0),0,k\big)\pi_\chi\big((z_0,\bar z_0),0,I_2\big)$$

which is a consequence of the equality

$$\big((kz_0,\overline{kz_0}),0,I_2\big)\cdot\big((0,0),0,k\big)=\big((0,0),0,k\big)\cdot\big((z_0,\bar z_0),0,I_2\big)$$

for $k\in SU(1,1)$ and $z_0\in\mathbb{C}$.

Let us denote by $b'_k(Z,W)$ the kernel of $\pi_\chi((0,0),0,k)$ for $k\in SU(1,1)$.

Proposition 6.1. *For each* $k\in SU(1,1)$, b'_k *satisfies the functional equation*

$$\exp\left(\frac{\lambda}{4}(-|kz_0|^2+2(\overline{kz_0})z+u(\overline{kz_0})^2)\right)b'_k\big(a(z-kz_0+\overline{kz_0}u,u),W\big)K_\chi(W,W)^{-1}$$
$$=\exp\left(\frac{\lambda}{4}(|z_0|^2+2\bar z_0 w-v\bar z_0{}^2)\right)b'_k\big(Z,a(w+z_0-v\bar z_0,v)\big)$$
$$\times K_\chi\big(a(w+z_0-v\bar z_0,v),a(w+z_0-v\bar z_0,v)\big)^{-1}.$$

for each $Z=a(z,u),W=a(w,v)\in\mathcal{D}$ *and each* $z_0\in\mathbb{C}$.

Proof. We just have to write that the operators

$$\pi_\chi\big((kz_0,\overline{kz_0}),0,I_2\big)\pi_\chi\big((0,0),0,k\big)$$

and

$$\pi_\chi\big((0,0),0,k\big)\pi_\chi\big((z_0,\bar z_0),0,I_2\big)$$

have the same kernel.

Let $f \in \mathcal{H}_\chi$ and $Z = a(z,u) \in \mathcal{D}$. On the one hand, we have

$$\big(\pi_\chi((kz_0,\overline{kz_0}),0,I_2)\pi_\chi((0,0),0,k)f\big)(Z)$$
$$= \exp\left(\frac{\lambda}{4}(-|kz_0|^2 + 2\overline{kz_0}z + u(\overline{kz_0})^2)\right)$$
$$\times \int_{\mathcal{D}} b'_k\big(a(z-kz_0+u\overline{kz_0},u),W\big)f(W)K_\chi(W,W)^{-1}c_\chi d\mu(W).$$

On the other hand, we have

$$\big(\pi_\chi((0,0),0,k)\pi_\chi((z_0,\bar{z}_0),0,I_2)f\big)(Z)$$
$$= \int_{\mathcal{D}} b'_k(Z,W)\big(\pi_\chi((z_0,\bar{z}_0),0,I_2)f\big)(W)K_\chi(W,W)^{-1}c_\chi d\mu(W)$$
$$= \int_{\mathcal{D}} b'_k(Z,W)\exp\left(\frac{\lambda}{4}(-|z_0|^2 + 2\bar{z}_0 w + v\bar{z}_0{}^2)\right)$$
$$\times f\big(a(w-z_0+v\bar{z}_0,v)\big)K_\chi(W,W)^{-1}c_\chi d\mu(W).$$

We have used the notation $W = a(v,V)$. By making in the last integral the change of variables $w \to w + z_0 - v\bar{z}_0$ and writing that the kernels are equal, we obtain the result. $\square$

Corollary 6.2. *For each $k \in SU(1,1)$, a solution of equation (7) is given by*

$$b_k(z,w) = b'_k\big(a(z,0),a(w,0)\big).$$

Proof. Let $k \in SU(1,1)$. By taking $u = v = 0$ in the functional equation for b'_k (Proposition 6.1), we obtain the result by using the relation

$$K_\chi\big(a(w,0),a(w,0)\big) = \exp\left(\frac{\lambda}{2}|w|^2\right)$$

for each $w \in \mathbb{C}$ (Proposition 5.2). $\square$

Proposition 6.3. *Let $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right) \in SU(1,1)$. Let $z,w \in \mathbb{C}$. We have*

$$b'_k\big(a(z,0),a(w,0)\big) = a^{-m}\exp\left(\frac{\lambda}{4}(\bar{b}a^{-1}z^2 + 2a^{-1}z\bar{w} - ba^{-1}\bar{w}^2)\right). \tag{9}$$

Proof. Let $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right) \in SU(1,1)$. Let $Z = a(z,u), W = a(w,0) \in \mathcal{D}$. By Proposition 5.2, we have

$$e_W(Z) = K_\chi(Z,W) = \exp\left(\frac{\lambda}{4}(u\bar{w}^2 + 2z\bar{w})\right). \tag{10}$$

Now, let $Z_0 = a(z,0)$. Then we have

$$b'_k(Z_0,W) = \big\langle \pi_\chi((0,0),0,k)e_W, e_{Z_0}\big\rangle_\chi$$

$$= \big(\pi_\chi((0,0),0,k)e_W\big)(Z_0)$$
$$= a^{-m}\exp\left(\frac{\lambda}{4}\bar{b}a^{-1}z^2\right)e_W\left(a(a^{-1}z,-a^{-1}b)\right)$$
$$= a^{-m}\exp\left(\frac{\lambda}{4}(\bar{b}a^{-1}z^2+2a^{-1}z\bar{w}-ba^{-1}\bar{w}^2)\right)$$

by the reproducing property and Proposition 5.3. □

Now, for each $k\in SU(1,1)$, we denote by A_k the operator on $\mathcal{F}_\lambda$ with kernel
$$b_k(z,w)=a^{-m}\exp\left(\frac{\lambda}{4}(\bar{b}a^{-1}z^2+2a^{-1}z\bar{w}-ba^{-1}\bar{w}^2)\right).$$
Since b_k then satisfies equation 7, A_k intertwines the representations ρ and ρ_k (see Section 3). Hence, by Schur's lemma, there exists, for each $k,k'\in S$ a scalar $\alpha(k,k')$ such that $A_{kk'}=\alpha(k,k')A_kA_{k'}$.

Proposition 6.4. *Let* $k=\left(\begin{smallmatrix} a & b\\ \bar{b} & \bar{a}\end{smallmatrix}\right)\in SU(1,1)$, $k'=\left(\begin{smallmatrix} a' & b'\\ \bar{b}' & \bar{a}'\end{smallmatrix}\right)\in SU(1,1)$ *and* $k''=kk'=\left(\begin{smallmatrix} a'' & b''\\ \bar{b}'' & \bar{a}''\end{smallmatrix}\right)$. *Then one has*
$$a''^{-m}=\alpha(k,k')a^{-m}a'^{-m}(a^{-1}a'^{-1}a'')^{-1/2}$$
where for $z\in\mathbb{C}\backslash]-\infty,0]$, $z^{1/2}$ *denotes the principal determination of the square root.*

Proof. The kernel of $A_{kk'}$ is given by
$$b_{kk'}(z,w)=\alpha(k,k')\int_{\mathbb{C}}b_k(z,u)b_{k'}(u,w)e^{-\lambda|u|^2/2}\,dm_\lambda(u).$$
By taking $z=w=0$ and using Lemma 5.1, we obtain the desired result. □

For $k\in SU(1,1)$, we denote by $\sigma(k)$ the operator A_k with kernel $b_k(z,w)$ corresponding to $m=1/2$. Then σ is called the *metaplectic representation* of $SU(1,1)$.

Proposition 6.5 ([15, Theorem 4.37]). *We have the following facts:*

1. *For each* $k,k'\in S$, *we have* $\sigma(kk')=\pm\sigma(k)\sigma(k')$.

2. *For each* $k\in K$, *the operator* $\sigma(k)$ *is a unitary operator.*

Proof. (1) This follows from Proposition 6.4.

(2) Clearly, we have $b_{k^{-1}}(z,w)=\overline{b_k(z,w)}$ for each $z,w\in\mathbb{C}^n$. This implies that $A_{k^{-1}}=A_k^*$ hence $A_kA_k^*=\pm Id$. Since $A_kA_k^*$ is positive, we can conclude that $A_kA_k^*=Id$. Similarly, we get $A_k^*A_k=Id$. □

7. COMPLEX WEYL SYMBOLS OF METAPLECTIC OPERATORS

In this section, we compute $W(\sigma(k))$ for $k \in SU(1,1)$ and $W(\pi(g))$ for $g \in G$ starting from the expressions of the kernel of $\sigma(k)$ (Section 6) and of the kernel of $\pi(g)$ (Proposition 4.2).

We denote by $\operatorname{Arg}(z)$ the principal argument of $z \in \mathbb{C}\backslash]-\infty, 0]$.

THEOREM 7.1. *Let $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in SU(1,1)$ such that $a \neq 0$ and $a+\bar{a}+2 \neq 0$. Then we have*

$$W\big(\sigma(k)\big)(z) = c(k) \exp\left(\lambda(a+\bar{a}+2)^{-1}\begin{pmatrix} z & \bar{z}\end{pmatrix}\begin{pmatrix} \bar{b} & \frac{1}{2}(\bar{a}-a) \\ \frac{1}{2}(\bar{a}-a) & -b \end{pmatrix}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)$$

for each $z \in \mathbb{C}$, where

$$c(k) = 2(a+\bar{a}+2)^{-1/2} \text{ if } a+\bar{a}+2 > 0;$$
$$c(k) = -2i|a+\bar{a}+2|^{-1/2} \text{ if } a+\bar{a}+2 < 0 \text{ and } \operatorname{Arg}(a) \in]0,\pi[;$$
$$c(k) = 2i|a+\bar{a}+2|^{-1/2} \text{ if } a+\bar{a}+2 < 0 \text{ and } \operatorname{Arg}(a) \in]-\pi,0[.$$

Proof. Let $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in SU(1,1)$. We write $a = a_1 + ia_2$, $b = b_1 + ib_2$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Recall that the kernel of $\sigma(k)$ is

$$b_k(z,w) = a^{-1/2} \exp\left(\frac{\lambda}{4}(\bar{b}a^{-1}z^2 + 2a^{-1}z\bar{w} - ba^{-1}\bar{w}^2)\right).$$

Then, by using the formula for W given in Proposition 3.1, we obtain immediately

$$\begin{aligned} W\big(\sigma(k)\big)(z) &= \frac{\lambda}{\pi}a^{-1/2}\exp\left(\frac{\lambda}{4}(a^{-1}\bar{b}z^2 - a^{-1}b\bar{z}^2 + 2(a^{-1}-1)|z|^2)\right) \\ &\times \int_{\mathbb{C}} \exp\left(\frac{\lambda}{4}(a^{-1}\bar{b}w^2 - a^{-1}b\bar{w}^2 - 2(a^{-1}+1)|w|^2)\right) \\ &\times \exp\left(\frac{\lambda}{2}\big((a^{-1}\bar{b}z + (a^{-1}-1)\bar{z})w + ((1-a^{-1})z + a^{-1}b\bar{z})\bar{w}\big)\right) dm(w). \end{aligned} \tag{11}$$

Thus, we see that the computation of $W(\sigma(k))(z)$ can be performed by using Lemma 5.1 with

$$M = \frac{\lambda}{4}\begin{pmatrix} -a^{-1}\bar{b} & 1+a^{-1} \\ 1+a^{-1} & a^{-1}b \end{pmatrix}$$

and

$$u = \frac{\lambda}{2}\big(a^{-1}\bar{b}z + (a^{-1}-1)\bar{z}\big);\; v = \frac{\lambda}{2}\big((1-a^{-1})z + a^{-1}b\bar{z}\big),$$

provided that the hypothesis of this lemma are satisfied.

Let $N = U^t M U$ with $U = \left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$ as in Lemma 5.1. By an easy calculation, we get

$$\mathrm{Re}(N) = \frac{\lambda}{2}\begin{pmatrix} 1+|a|^{-2}(a_1+a_2b_2) & -|a|^{-2}b_1a_2 \\ -|a|^{-2}b_1a_2 & 1+|a|^{-2}(a_1-a_2b_2) \end{pmatrix}.$$

Since we have

$$\mathrm{Det}\big(\mathrm{Re}(N)\big) = \frac{\lambda^2}{4}|a|^{-2}(a_1+1)^2 > 0$$

and

$$\mathrm{Tr}\big(\mathrm{Re}(N)\big) = \lambda(1+|a|^{-2}a_1) = \lambda\big(1+\mathrm{Re}(a^{-1})\big) \geq \lambda(1-|a^{-1}|) \geq 0,$$

we see that $\mathrm{Re}(N)$ is positive definite and the hypothesis of Lemma 5.1 are satisfied. Note that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\lambda}{2}\begin{pmatrix} a^{-1}\bar{b} & a^{-1}-1 \\ 1-a^{-1} & a^{-1}b \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}.$$

By a routine computation, we get

$$\begin{aligned} &\begin{pmatrix} u & v \end{pmatrix} M^{-1}\begin{pmatrix} u \\ v \end{pmatrix} \\ =&\frac{\lambda}{a(a+\bar{a}+2)}\begin{pmatrix} z & \bar{z} \end{pmatrix}\begin{pmatrix} (3a-\bar{a}-2)\bar{b} & a-\bar{a}-a^2+3|a|^2-2 \\ a-\bar{a}-a^2+3|a|^2-2 & (-3a+\bar{a}+2)b \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}. \end{aligned}$$

Consequently, the value of the Gaussian integral in equation 11 is given by

$$\pi(\mathrm{Det}\, N)^{-1/2} \times \exp\Bigg(\frac{\lambda}{4a(a+\bar{a}+2)}\begin{pmatrix} z \bar{z} \end{pmatrix} \begin{pmatrix} (3a-\bar{a}-2)\bar{b} & a-\bar{a}-a^2+3|a|^2-2 \\ a-\bar{a}-a^2+3|a|^2-2 & (-3a+\bar{a}+2)b \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\Bigg).$$

On the other hand, the factor in front of the integral in equation 11 can be written as

$$\frac{\lambda}{\pi}a^{-1/2}\exp\left(\frac{\lambda}{4}\begin{pmatrix} z & \bar{z} \end{pmatrix}\begin{pmatrix} a^{-1}\bar{b} & a^{-1}-1 \\ a^{-1}-1 & -a^{-1}b \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right).$$

Hence, putting these expressions together, we obtain

$$\begin{aligned} &W\big(\sigma(k)\big)(z) \\ =&\lambda a^{-1/2}(\mathrm{Det}\, N)^{-1/2}\exp\left(\lambda(a+\bar{a}+2)^{-1}\begin{pmatrix} z & \bar{z} \end{pmatrix}\begin{pmatrix} \bar{b} & \frac{1}{2}(\bar{a}-a) \\ \frac{1}{2}(\bar{a}-a) & -b \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right). \end{aligned}$$

Since $\mathrm{Det}(N) = \frac{\lambda^2}{4}a^{-1}(a+\bar{a}+2)$, the result follows. □

We can also obtain a slightly different expression for $W(\sigma(k))(z)$ which is close to the formulas given in [12, 14]. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Corollary 7.2. *Let* $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in SU(1,1)$, $a \neq 0$ *and* $a + \bar{a} + 2 \neq 0$. *Then we have*

$$W\big(\sigma(k)\big)(z) = c(k) \exp\left(\frac{\lambda}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k+I_2)^{-1}(k-I_2) \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right)$$

where $c(k)$ *is defined as in Theorem* 7.1.

Proof. Let $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in SU(1,1)$. The result is a direct consequence of the relation

$$\frac{1}{2} J(k+I_2)^{-1}(k-I_2) = (a+\bar{a}+2)^{-1} \begin{pmatrix} \bar{b} & \frac{1}{2}(\bar{a}-a) \\ \frac{1}{2}(\bar{a}-a) & -b \end{pmatrix}. \qquad \square$$

Similarly, we can compute the complex Weyl symbol of $\pi(g)$ for $g \in G$.

Theorem 7.3. *Let* $g = ((z_0, \bar{z}_0), c_0, k) \in G$ *with* $k = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in SU(1,1)$. *Then, for each* $z \in \mathbb{C}$, *we have*

$$\begin{aligned} W\big(\pi(g)\big)(z) = {}& c(k) e^{i\lambda c_0} \exp\left(\lambda (a+\bar{a}+2)^{-1} \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} \bar{b} & \frac{1}{2}(\bar{a}-a) \\ \frac{1}{2}(\bar{a}-a) & -b \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \\ & \times \exp\left(\lambda (a+\bar{a}+2)^{-1} \begin{pmatrix} z & \bar{z} \end{pmatrix} \begin{pmatrix} -\bar{b} & a+1 \\ -\bar{a}-1 & b \end{pmatrix} \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix} \right) \\ & \times \exp\left(\frac{\lambda}{4} (a+\bar{a}+2)^{-1} \begin{pmatrix} z_0 & \bar{z}_0 \end{pmatrix} \begin{pmatrix} \bar{b} & \frac{1}{2}(\bar{a}-a) \\ \frac{1}{2}(\bar{a}-a) & -b \end{pmatrix} \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix} \right), \end{aligned}$$

or, equivalently,

$$\begin{aligned} W\big(\pi(g)\big)(z) = {}& c(k) e^{i\lambda c_0} \exp\left(\frac{\lambda}{2} \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k+I_2)^{-1}(k-I_2) \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \right) \\ & \times \exp\left(\lambda \begin{pmatrix} z & \bar{z} \end{pmatrix} J(k+I_2)^{-1} \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix} \right) \\ & \times \exp\left(\frac{\lambda}{8} \begin{pmatrix} z_0 & \bar{z}_0 \end{pmatrix} J(k+I_2)^{-1}(k-I_2) \begin{pmatrix} z_0 \\ \bar{z}_0 \end{pmatrix} \right). \end{aligned}$$

Proof. Let $g \in G$. Taking into account the formula for the kernel of $\pi(g)$ (Proposition 4.2), the proof follows the same lines as that of Theorem 7.1 and Corollary 7.2. In particular, we apply Lemma 5.1 with

$$M = \frac{\lambda}{4} \begin{pmatrix} -a^{-1}\bar{b} & 1+a^{-1} \\ 1+a^{-1} & a^{-1}b \end{pmatrix}$$

and

$$u = \frac{\lambda}{2}\big(a^{-1}\bar{b}z + (a^{-1}-1)\bar{z} - a^{-1}\bar{b}z_0 + \bar{z}_0\big); \; v = \frac{\lambda}{2}\big((1-a^{-1})z + a^{-1}b\bar{z} + a^{-1}z_0\big). \; \square$$

8. COVARIANCE PROPERTIES OF *W*

We have seen in Section 3 that W is covariant with respect to ρ (Proposition 3.2). Here we prove that W is also covariant with respect to σ and then covariant with respect to π.

PROPOSITION 8.1. 1. *For each $k \in SU(1,1)$ and each $z \in \mathbb{C}$, we have*

$$\Omega(kz) = \sigma(k)\Omega(z)\sigma(k)^{-1}.$$

2. *For each operator A on $\mathcal{F}$, each $k \in SU(1,1)$ and each $z \in \mathbb{C}$, we have*

$$W\big(\sigma(k)^{-1}A\sigma(k)\big)(z) = W(A)(kz).$$

Proof. Recall that for each $z \in \mathbb{C}$, we have

$$\Omega(z) := \rho\big((z,\bar{z}),0\big)R\rho\big((z,\bar{z}),0\big)^{-1}$$

where the parity operator R is defined on $\mathcal{F}$ by $(Rf)(z) = 2f(-z)$.

First, we verify that $\sigma(k)R = R\sigma(k)$ for each $k \in SU(1,1)$. Indeed, for each $f \in \mathcal{F}$ and each $z \in \mathbb{C}$, we have

$$\begin{aligned}
\big(R\sigma(k)f\big)(z) &= 2\big(\sigma(k)f\big)(-z)\\
&= \frac{\lambda}{\pi}\int_{\mathbb{C}} b_k(-z,w)f(w)\,e^{-\lambda|w|^2/2}\,dm(w)\\
&= \frac{\lambda}{\pi}\int_{\mathbb{C}} b_k(z,-w)f(w)\,e^{-\lambda|w|^2/2}\,dm(w)\\
&= \frac{\lambda}{\pi}\int_{\mathbb{C}} b_k(z,w)f(-w)\,e^{-\lambda|w|^2/2}\,dm(w)\\
&= \big(\sigma(k)Rf\big)(z)
\end{aligned}$$

since $b_k(-z,w) = b_k(z,-w)$ for each $z, w \in \mathbb{C}$, see Section 6.

Now, recall that

$$\rho\big(k\cdot((z,\bar{z}),0)\big) = \sigma(k)\rho\big((z,\bar{z}),0\big)\sigma(k)^{-1}$$

for each $k \in SU(1,1)$ and each $z \in \mathbb{C}$, see equation 6. Then we have

$$\begin{aligned}
\sigma(k)\Omega(z)\sigma(k)^{-1} &= \sigma(k)\rho\big((z,\bar{z}),0\big)R\rho\big((z,\bar{z}),0\big)^{-1}\sigma(k)^{-1}\\
&= \rho\big(k\cdot((z,\bar{z}),0)\big)\sigma(k)R\sigma(k)^{-1}\rho\big(k\cdot((z,\bar{z}),0)\big)^{-1}\\
&= \rho\big(k\cdot((z,\bar{z}),0)\big)R\rho\big(k\cdot((z,\bar{z}),0)\big)^{-1}\\
&= \rho\big((kz,\overline{kz}),0\big)R\rho\big((kz,\overline{kz}),0\big)^{-1}\\
&= \Omega(kz).
\end{aligned}$$

This proves the first assertion of the proposition. Since W is defined from the quantizer Ω, the second assertion follows immediately. □

Note that the Berezin correspondence S_λ is not covariant with respect to σ (S_λ is nevertheless covariant with respect to ρ, see Section 2). In order to prove this, let us introduce for each $z \in \mathbb{C}$, the orthogonal projection operator $P(z)$ on the line generated by e_z. Then $P(z)$ is a quantizer for S_λ, that is, we have

$$S_\lambda(A)(z) = \operatorname{Tr}\big(AP(z)\big)$$

for each trace-class operator A on $\mathcal{F}$ [2,3].

Now, on the one hand, for each $k \in SU(1,1)$, each $f \in \mathcal{F}$ and each $z \in \mathbb{C}$, we have

$$\begin{aligned}\big(\sigma(k)P(0)f\big)(z) &= f(0)\big(\sigma(k)1\big)(z)\\ &= \frac{\pi}{\lambda} f(0) \int_{\mathbb{C}} b_k(z,w)\, e^{-\lambda|w|^2/2}\, dm(w)\\ &= f(0)b_k(z,0)\end{aligned}$$

by the reproducing property of $\mathcal{F}$. On the other hand, we have

$$\big(P(0)\sigma(k)f\big)(z) = \frac{\langle \sigma(k)f\,,\, e_0\rangle_{\mathcal{F}}}{\langle e_0\,,\, e_0\rangle_{\mathcal{F}}}\, e_0(z) = (\sigma(k)f)(0).$$

So, we see that, if $f(0) \neq 0$, $(P(0)\sigma(k)f)(z)$ does not depend on z as $(\sigma(k)P(0)f)(z)$ does. Then we can conclude that $P(0)\sigma(k) \neq \sigma(k)P(0)$. This proves that S_λ is not covariant with respect to σ. In particular, we cannot deduce the covariance of W with respect to σ from that of S_λ as it can be done for the covariance of W with respect to ρ, see Section 3.

Let us consider the action of G on $\mathbb{C}$ defined by $g \cdot z = h \cdot (kz)$ for $g = (h,k) \in G$ and $z \in \mathbb{C}$.

Proposition 8.2. 1. *For each $g \in G$ and each $z \in \mathbb{C}$, we have*

$$\Omega(g \cdot z) = \pi(g)\Omega(z)\pi(g)^{-1}.$$

2. *For each operator A on $\mathcal{F}$, each $g \in G$ and each $z \in \mathbb{C}$, we have*

$$W\big(\pi(g)^{-1}A\pi(g)\big)(z) = W(A)(g\cdot z).$$

Proof. We just have to use the covariance of W with respect to ρ (Proposition 3.2) and with respect to σ (Proposition 8.2). □

As a consequence of these covariance properties, we can interpret W in terms of the so-called Stratonovich–Weyl correspondence [6,10,16,22,25]. The following definition is adapted from [16].

Definition 8.3. Let G_0 be a Lie group and π_0 be a unitary representation of G_0 on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G_0-space and μ_0 a (suitably normalized) G_0-invariant measure on M. Then a Stratonovich–Weyl correspondence for the triple (G_0, π_0, M) is an isomorphism W_0 from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on M satisfying the following properties:

1. Reality: the function $W_0(A^*)$ is the complex-conjugate of $W_0(A)$;
2. Covariance: we have $W_0(\pi_0(g)\,A\,\pi_0(g)^{-1})(x) = W_0(A)(g^{-1}\cdot x)$;
3. Unitarity: we have

$$\int_M W_0(A)(x)W_0(B)(x)\,d\mu_0(x) = \mathrm{Tr}(AB).$$

Recall that $\mathcal{L}_2(\mathcal{F})$ denotes the space of all Hilbert–Schmidt operators on $\mathcal{F}$.

THEOREM 8.4. 1. *The next map $W : \mathcal{L}_2(\mathcal{F}) \to L^2(\mathbb{C}, m_\lambda)$ is a Stratonovich–Weyl correspondence for the triple $(H, \rho, \mathbb{C})$;*

2. *The map $W : \mathcal{L}_2(\mathcal{F}) \to L^2(\mathbb{C}, m_\lambda)$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi, \mathbb{C})$.*

Proof. (1) is well known, see for instance [10]. In particular, the map $W : \mathcal{L}_2(\mathcal{F}) \to L^2(\mathbb{C}, m_\lambda)$ is unitary. (2) follows from Proposition 8.2. □

9. COMPLEX WEYL SYMBOLS OF DIFFERENTIAL OPERATORS

Here, we first compute the complex Weyl symbols of $d\rho(X_0)$ for X_0 in the Lie algebra $\mathfrak{h}$ of H and of $d\sigma(X_1)$ for X_1 in the Lie algebra $su(1,1)$ of $SU(1,1)$.

PROPOSITION 9.1. 1. *For each $X_0 = ((z_0, \bar{z}_0), c_0) \in \mathfrak{h}$ and for each $z \in \mathbb{C}$, we have*

$$W\big(d\rho(X_0)\big)(z) = i\lambda c_0 + \frac{\lambda}{2}(\bar{z}_0 z - z_0 \bar{z});$$

2. *There exists a map $\psi_0 : \mathbb{C} \to \mathfrak{h}^*$ such that*

$$W\big(d\rho(X_0)\big)(z) = i\langle \psi_0(z), X_0\rangle$$

for each $X_0 \in \mathfrak{h}$ and each $z \in \mathbb{C}$. Then we have

$$\psi_0(h\cdot z) = \mathrm{Ad}^*(h)\,\psi_0(z)$$

for each $h \in H$ and each $z \in \mathbb{C}$;

3. *Let* $\xi_0 \in \mathfrak{h}^*$ *be defined by* $\langle \xi_0, ((z_0,\bar{z}_0),c_0)\rangle = i\lambda c_0$. *Then* ψ_0 *is a bijection from* $\mathbb{C}$ *onto the orbit of* ξ_0 *for the coadjoint action of* H *on* $\mathfrak{h}^*$.

Proof. (1) Take $g = ((z_0,\bar{z}_0),c_0,I_2)$ in Theorem 7.3. Since

$$\pi(g) = \rho\big((z_0,\bar{z}_0),c_0\big),$$

we get

$$W\big(\rho((z_0,\bar{z}_0),c_0)\big)(z) = e^{i\lambda c_0}\exp\left(\frac{\lambda}{2}(\bar{z}_0 z - z_0\bar{z})\right)$$

and by differentiating this equation, we obtain the result.

(2) First let us note that for each $z \in \mathbb{C}$ the linear form defined on $\mathfrak{h}$ by $X_0 \to -iW(d\rho(X_0))(z)$ is real-valued by (1). Moreover, by using the covariance of W with respect to ρ, for each $h \in H$, $X_0 \in \mathfrak{h}$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}\big\langle \psi_0(h\cdot z), X_0\big\rangle &= -iW\big(d\rho(X_0)\big)(h\cdot z)\\ &= -iW\big(\rho(h)^{-1}d\rho(X_0)\rho(h)\big)(z)\\ &= -iW\big(d\rho(\mathrm{Ad}(h)^{-1}X_0)\big)(z)\\ &= \big\langle \psi_0(z), \mathrm{Ad}(h)^{-1}X_0\big\rangle\\ &= \big\langle\, \mathrm{Ad}^*(h)\psi_0(z), X_0\big\rangle\end{aligned}$$

hence $\psi_0(h\cdot z) = \mathrm{Ad}^*(h)\psi_0(z)$. The rest of the proposition can be verified easily. □

We identify $SU(1,1)$-equivariantly the dual $su(1,1)^*$ of $su(1,1)$ with $su(1,1)$ by means of the bilinear form defined by $\langle X,Y\rangle = \mathrm{Tr}(XY)$. We denote by $\mathcal{O}$ the minimal (non-trivial) adjoint orbit in $su(1,1)$. Then $\mathcal{O}$ consists of all rank-one matrices of $su(1,1)$.

Proposition 9.2. 1. *For each* $X_1 = \begin{pmatrix} a & b\\ \bar{b} & -a\end{pmatrix} \in su(1,1)$ *and each* $z \in \mathbb{C}$, *we have*

$$W\big(d\sigma(X_1)\big)(z) = \frac{\lambda}{4}(\bar{b}z^2 - b\bar{z}^2 - 2az\bar{z});$$

2. *Let* $\psi_1 : \mathbb{C} \to su(1,1)^*$ *be defined by*

$$W\big(d\sigma(X_1)\big)(z) = i\big\langle \psi_1(z), X_1\big\rangle$$

for each $X_1 \in su(1,1)$ *and each* $z \in \mathbb{C}$. *Then we have*

$$\psi_1(kz) = \mathrm{Ad}(k)\,\psi_1(z)$$

for each $k \in SU(1,1)$ *and each* $z \in \mathbb{C}$;

3. *For each* $z \in \mathbb{C}$, *we have*

$$\psi_1(z) = \frac{i\lambda}{4}\begin{pmatrix} z\bar{z} & -z^2 \\ \bar{z}^2 & -z\bar{z} \end{pmatrix}.$$

Moreover, the image of ψ_1 *is* $\mathcal{O} \cup (0)$ *and the restriction of* ψ_1 *to* $\mathbb{C} \setminus (0)$ *is a covering of degree* 2 *of* $\mathcal{O}$.

Proof. (1) can be proved by differentiating $W(\sigma(k))$. The proof of (2) is similar to that of (2) of the preceding proposition. Finally, (3) can be verified easily. □

Let $u \in \mathbb{C}$, $d \in \mathbb{R}$ and $Y \in su(1,1)$. We denote by $((u,\bar{u}), d, Y)$ the element of $\mathfrak{g}^*$ defined by

$$\big\langle((u,\bar{u}),d,Y),((z_0,\bar{z}_0),c_0,X_0)\big\rangle := \omega\big((u,\bar{u}),(z_0,\bar{z}_0)\big) + dc_0 + \langle Y, X_0\rangle$$

for each $((z_0,\bar{z}_0),c_0,X_0) \in \mathfrak{g}$.

By combining Proposition 9.1 and Proposition 9.2, we obtain the following result.

THEOREM 9.3. 1. *For each* $X = ((z_0,\bar{z}_0), c_0, \left(\begin{smallmatrix} a & b \\ \bar{b} & -a \end{smallmatrix}\right)) \in \mathfrak{g}$ *and each* $z \in \mathbb{C}$, *we have*

$$W\big(d\pi(X)\big)(z) = i\lambda c_0 + \frac{\lambda}{2}(\bar{z}_0 z - z_0\bar{z}) + \frac{\lambda}{4}(\bar{b}z^2 - b\bar{z}^2 - 2az\bar{z});$$

2. *Let* $\psi : \mathbb{C} \to \mathfrak{g}^*$ *be defined by*

$$W\big(d\pi(X)\big)(z) = i\big\langle\psi(z), X\big\rangle$$

for each $X \in \mathfrak{g}$ *and each* $z \in \mathbb{C}$. *Then we have*

$$\psi(g \cdot z) = \mathrm{Ad}^*(g)\,\psi(z)$$

for each $g \in G$ *and each* $z \in \mathbb{C}$;

3. *For each* $z \in \mathbb{C}$, *we have*

$$\psi(z) = \left(-\lambda(z,\bar{z}), \lambda, \frac{i\lambda}{4}\begin{pmatrix} z\bar{z} & -z^2 \\ \bar{z}^2 & -z\bar{z} \end{pmatrix}\right).$$

The map ψ *is a bijection from* $\mathbb{C}$ *onto the orbit* $\mathcal{O}(\xi)$ *of* $\xi := ((0,0),\lambda,0) \in \mathfrak{g}^*$ *for the coadjoint action of* G.

Proof. (1) Let $X \in \mathfrak{g}$. We then write $X = (X_0, X_1)$ where $X_0 \in \mathfrak{h}$ and $X_1 \in su(1,1)$. Then, by the definition of π, we have

$$W\big(d\pi(X)\big)(z) = W\big(d\rho(X_0)\big)(z) + W\big(d\sigma(X_1)\big)(z)$$

for each $z \in \mathbb{C}$ and the result is a consequence of Proposition 9.1 and Proposition 9.2.

(2) The assertion can be proved similarly as (2) of Proposition 9.2.

(3) This follows from (1) and (2). $\square$

Note that in Proposition 9.1 we recover via W the well-known fact that ρ is a unitary irreducible representation of H which is associated with the coadjoint orbit $\mathcal{O}(\xi_0)$ as in the Kirillov-Kostant method of orbits [17]. Similarly, Theorem 9.3 connects π with $\mathcal{O}(\xi)$. Finally, Proposition 9.2 indicates a connection between the minimal (co)adjoint orbit of $su(1,1)$ and the metaplectic representation σ; this point has been emphasized by many authors, see [26] and also [18] and its references.

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