ON CHEN'S THEOREM, GOLDBACH'S CONJECTURE AND ALMOST PRIME TWINS II

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Let N denote a sufficiently large even integer and x denote a sufficiently large integer, we define $D_{1,2}(N)$ as the number of primes p such that $N-p$ has at most 2 prime factors. In this paper, we show that $D_{1,2}(N) \geqslant 1.9728\frac{C(N)N}{(\log N)^2}$, which is rather near to the asymptotic constant 2 in Hardy–Littlewood conjecture for Goldbach's conjecture. We also get similar results on twin prime problem and additive representations of integers. The proof combines various techniques in sieve methods, such as weighted sieve, Chen's switching principle, new distribution levels proved by Lichtman and Pascadi, Chen's double sieve and Harman's sieve.

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1. INTRODUCTION

One of the most famous open problems in number theory is the Goldbach's conjecture, which states that any even integers can be written as the sum of two primes. Since the original conjecture is so hard, mathematicians try to consider the problem of writing a large even integer as a sum of a prime and a number with few prime factors. Let N denote a sufficiently large even integer, p denote a prime, and P_r denote an integer with at most r prime factors counted with multiplicity. We define

$$D_{1,2}(N) = \left|\{p : p \leqslant N, N-p = P_2\}\right|. \tag{1}$$

In 1973, Chen [5] established his remarkable theorem

$$D_{1,2}(N) \geqslant 0.67\frac{C(N)N}{(\log N)^2}, \tag{2}$$

where

$$C(N) = \prod_{\substack{p\mid N\\ p>2}} \frac{p-1}{p-2} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{3}$$

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Chen's constant 0.67 was improved successively to

$$0.689,\ 0.7544,\ 0.81,\ 0.8285,\ 0.836,\ 0.867,\ 0.899$$

by Halberstam and Richert [10,11], Chen [6,7], Cai and Lu [4], Wu [21], Cai [2] and Wu [22], respectively. Chen [8] announced a better constant 0.9, but this work has not been published.

In our 2024 preprint [13], we increase this constant to 1.733, which almost doubles Wu's 0.899. In the proof, we use the distribution levels of Lichtman (see [15], and [16] for an earlier development of this kind of results) and complicated techniques in sieves. In this paper, by modifying the parameters used in [13] and inserting more advanced techniques, we obtain the following sharper result.

THEOREM 1.1.

$$D_{1,2}(N) \geqslant 1.9728 \frac{C(N)N}{(\log N)^2}.$$

Our new constant 1.9728 gives a 13.8% improvement over our previous result 1.733 and a 119% refinement of Wu's prior record 0.899. An important meaning of our new constant is that it is very close to the conjectured asymptotic constant 2 for $D_{1,1}(N)$, the number of primes p such that $N-p$ is also a prime.

Furthermore, for two relatively prime square-free positive integers a, b, let M denote a sufficiently large integer that is relatively prime to both a and b, $a, b < M^{\varepsilon}$ and let M be even if a and b are both odd. Let $R_{a,b}(M)$ denote the number of primes p such that ap and $M-ap$ are both square-free, $b \mid (M-ap)$, and $\frac{M-ap}{b} = P_2$. In 1976, Ross [[19], Chapter 3] established that

$$R_{a,b}(M) \geqslant 0.608 \frac{C(abM)M}{ab(\log M)^2}, \tag{4}$$

and in [13] the constant 0.608 was improved successively to 1.733 by essentially the same process. Now, by using the new sieve process and methods in [14], we have the following sharper.

THEOREM 1.2.

$$R_{a,b}(M) \geqslant 1.9728 \frac{C(abM)M}{ab(\log M)^2}.$$

Another famous problem in number theory is the twin prime problem, which states that there are infinitely many prime pairs that differ by 2. Again, mathematicians consider the problem that there are infinitely many prime p such that $p+2$ has few prime factors. For the twin prime problem, let x denote a sufficiently large integer and define

$$\pi_{1,2}(x) = \left|\{p : p \leqslant x, p+2 = P_2\}\right|. \tag{5}$$

In 1973, Chen [5] showed simultaneously that

$$\pi_{1,2}(x) \geqslant 0.335\frac{C_2 x}{(\log x)^2}, \tag{6}$$

where

$$C_2 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right), \tag{7}$$

and the constant 0.608 was improved successively to

$$0.3445,\ 0.3772,\ 0.405,\ 0.71,\ 1.015,\ 1.05,\ 1.0974,\ 1.104,\ 1.123,\ 1.13$$

by Halberstam [10], Chen [6,7], Fouvry and Grupp [9], Liu [17], Wu [20], Cai [1], Wu [21], Cai [2] and Cai [3], respectively.

In [13], we increase this constant to 1.238 by similar methods. Recently, Pascadi [18] got a powerful new distribution level for primes, which is quite helpful in improving the lower bound for $\pi_{1,2}(x)$. Using his new distribution results together with sieve inputs in [13], we get the following sharper.

THEOREM 1.3.

$$\pi_{1,2}(x) \geqslant 1.2759\frac{C_2 x}{(\log x)^2}.$$

2. NEW DISTRIBUTION LEVELS

In this section, we put $A, B > 0$, $\theta_0 = 0$, $\theta_1 = \frac{7}{32}$ from Kim–Sarnak [12], and we define the functions $\boldsymbol{\vartheta}_\alpha(t_1)$ and $\boldsymbol{\vartheta}_\alpha(t_1, t_2, t_3)$ with $\alpha = 0$ or 1 similar to those in [15], but with $\theta = \theta_\alpha$ here for $\boldsymbol{\vartheta}_\alpha$. We consider the analogous set of well-factorable vectors $\mathbf{D}_r^{\text{well}}$:

$$\mathbf{D}_r^{\text{well}}(D) = \left\{(D_1, \ldots, D_r) : D_1 \cdots D_{m-1} D_m^2 < D \text{ for all } m \leqslant r\right\}. \tag{8}$$

We first state the distribution results for Theorem 1.1, which were proved in [15]. We remark that the maximum possible distribution level here is

$$\frac{19101}{32000} \approx 0.5969.$$

The first one is used when Chen's switching principle is not used, and the second one is used when Chen's switching principle is used.

LEMMA 2.1. *Let* $(D_1, \ldots, D_r) \in \mathbf{D}_r^{\text{well}}(D)$ *and write* $D = N^{\boldsymbol{\vartheta}}, D_i = N^{t_i}$ *for* $i \leqslant r$. *If* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$, *then*

$$\text{(i)} \qquad \sum_{\substack{b = p_1 \cdots p_r \\ D_i < p_i \leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q = bc \leqslant D \\ c \mid P(p_r) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N; q, N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$ *and* $r \geqslant 3$, *then* (i) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 2$, *then*

(ii) $$\sum_{\substack{b=p_1p_2\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D\\ c|P(N^u)\\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$, *then* (ii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 1$, *then*

(iii) $$\sum_{\substack{b=p_1\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D\\ c|P(N^u)\\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$, *then* (iii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, u, u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$, *this simplifies as*

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}}\\ q|P(N^{1/500})\\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N) - \frac{\pi(N)}{\varphi(q)}\right) \ll \frac{N}{(\log N)^A}.$$

Lemma 2.2. *Let* $(D_1, \ldots, D_r) \in \mathbf{D}_r^{\mathrm{well}}(D)$ *and write* $D = N^{\boldsymbol{\vartheta}}, D_i = N^{t_i}$ *for* $i \leqslant r$. *Let* $\varepsilon > 0$ *and real numbers* $\varepsilon_1, \ldots, \varepsilon_k \geqslant \varepsilon$ *such that* $\sum_{i\leqslant k} \varepsilon_i = 1$, *and let* $\Delta = 1 + (\log N)^{-B}$. *If* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$, *then*

$$\sum_{\substack{b=p'_1\cdots p'_r\\ D_i<p'_i\leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D\\ c|P(p'_r)\\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\Bigg(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1\Bigg)$$

(i) $$\ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$ *and* $r \geqslant 3$, *then* (i) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 2$, *then*

$$\sum_{\substack{b=p'_1p'_2\\ D_1<p'_1\leqslant D_1^{1+\varepsilon^9}\\ D_2<p'_2\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D\\ c|P(N^u)\\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\Bigg(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1\Bigg)$$

(ii) $$\ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$, *then* (ii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1) - \varepsilon$ *and* $r = 1$, *then*

$$\text{(iii)} \quad \sum_{\substack{b=p_1' \\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(N^u) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\Bigg(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q) \\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,N)=1 \\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1\Bigg) \ll \frac{N}{(\log N)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_1}{4}$, *then* (iii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_1(t_1, u, u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$, *this simplifies as*

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}} \\ q|P(N^{1/500}) \\ (q,N)=1}} \widetilde{\lambda}^{\pm}(q)\Bigg(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q) \\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,N)=1 \\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}} 1\Bigg) \ll \frac{N}{(\log N)^A}.$$

Next, we state the distribution results for Theorem 1.3, which were proved in [18]. We remark that the maximum possible distribution level here is

$$\frac{2497}{4000} = 0.62425.$$

The first one is used when Chen's switching principle is not used, and the second one is used when Chen's switching principle is used.

LEMMA 2.3. *Let* $(D_1, \ldots, D_r) \in \mathbf{D}_r^{\text{well}}(D)$ *and write* $D = x^{\boldsymbol{\vartheta}}, D_i = x^{t_i}$ *for* $i \leqslant r$. *If* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$, *then*

$$\text{(i)} \quad \sum_{\substack{b=p_1\cdots p_r \\ D_i<p_i\leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(p_r) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x; q, -2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}.$$

Moreover if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$ *and* $r \geqslant 3$, *then* (i) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 2$, *then*

$$\text{(ii)} \quad \sum_{\substack{b=p_1p_2 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9} \\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x; q, -2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$, *then* (ii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 1$, *then*

(iii) $$\sum_{\substack{b=p_1 \\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$, *then* (iii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, u, u) - \varepsilon$.

If $r = 0$ *and* $u = \frac{1}{500}$, *this simplifies as*

$$\sum_{\substack{q\leqslant x^{\frac{2497}{4000}} \\ q|P(x^{1/500}) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2) - \frac{\pi(x)}{\varphi(q)}\right) \ll \frac{x}{(\log x)^A}.$$

Lemma 2.4. *Let* $(D_1, \ldots, D_r) \in \mathbf{D}_r^{\mathrm{well}}(D)$ *and write* $D = x^{\boldsymbol{\vartheta}}, D_i = x^{t_i}$ *for* $i \leqslant r$. *Let* $\varepsilon > 0$ *and real numbers* $\varepsilon_1, \ldots, \varepsilon_k \geqslant \varepsilon$ *such that* $\sum_{i\leqslant k} \varepsilon_i = 1$, *and let* $\Delta = 1 + (\log x)^{-B}$. *If* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$, *then*

(i) $$\sum_{\substack{b=p_1'\cdots p_r' \\ D_i<p_i'\leqslant D_i^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(p_r') \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q) \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,2)=1 \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1\right) \ll \frac{x}{(\log x)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$ *and* $r \geqslant 3$, *then* (i) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, t_3) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 2$, *then*

(ii) $$\sum_{\substack{b=p_1'p_2' \\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9} \\ D_2<p_2'\leqslant D_2^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q) \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,2)=1 \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1\right) \ll \frac{x}{(\log x)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$, *then* (ii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1, t_2, u) - \varepsilon$.

If $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1) - \varepsilon$ *and* $r = 1$, *then*

(iii) $$\sum_{\substack{b=p_1' \\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}} \sum_{\substack{q=bc\leqslant D \\ c|P(x^u) \\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q) \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{(p_1\cdots p_k,2)=1 \\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1\right) \ll \frac{x}{(\log x)^A}.$$

Moreover, if $t_1 \leqslant \frac{1-\theta_0}{4-3\theta_0}$*, then* (iii) *holds if* $\boldsymbol{\vartheta} \leqslant \boldsymbol{\vartheta}_0(t_1,u,u)-\varepsilon$.

If $r=0$ *and* $u=\frac{1}{500}$*, this simplifies as*

$$\sum_{\substack{q\leqslant x^{\frac{2497}{4000}}\\ q\mid P(x^{1/500})\\ (q,2)=1}} \widetilde{\lambda}^{\pm}(q)\Bigg(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}} 1\Bigg) \ll \frac{x}{(\log x)^A}.$$

3. WEIGHTED SIEVE METHOD

Let $\mathcal{A}$ and $\mathcal{B}$ denote finite sets of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z\geqslant 2$. Put

$$\mathcal{A}=\{N-p: p\leqslant N\},\quad \mathcal{B}=\{p+2: p\leqslant x\},$$
$$\mathcal{P}=\{p:(p,2)=1\},\quad \mathcal{P}(q)=\{p: p\in\mathcal{P},\ (p,q)=1\},$$
$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}} p,\quad \mathcal{A}_d=\{a: a\in\mathcal{A},\ d\mid a\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}} 1.$$

Lemma 3.1. *We have*

$$\begin{aligned}
4D_{1,2}(N) \geqslant\ & 3S\big(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{11.49}}\big)+S\big(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{6.18}}\big)\\
&-2\sum_{\substack{N^{\frac{1}{11.49}}\leqslant p<N^{\frac{25}{128}}\\ (p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big)\\
&-\sum_{\substack{N^{\frac{25}{128}}\leqslant p<N^{\frac{1}{4}}\\ (p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big)\\
&-\sum_{\substack{N^{\frac{1}{4}}\leqslant p<N^{\frac{57}{224}}\\ (p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big)\\
&-\sum_{\substack{N^{\frac{57}{224}}\leqslant p<N^{\frac{1}{3}}\\ (p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big)\\
&-\sum_{\substack{N^{\frac{25}{128}}\leqslant p<N^{\frac{1}{2}-\frac{3}{11.49}}\\ (p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big)\\
&+\sum_{\substack{N^{\frac{1}{11.49}}\leqslant p_2<p_1<N^{\frac{1}{6.18}}\\ (p_1p_2,N)=1}} S\big(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{11.49}}\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_2 < N^{\frac{1}{6.18}} \leqslant p_1 < N^{\frac{25}{128}} \\ (p_1p_2, N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{11.49}}\big) \\
&+ \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_2 < N^{\frac{1}{6.18}} < N^{\frac{25}{128}} \leqslant p_1 < N^{\frac{1}{2}-\frac{3}{11.49}} \\ (p_1p_2, N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{11.49}}\big) \\
&- 2 \sum_{\substack{N^{\frac{1}{2}-\frac{3}{11.49}} \leqslant p_1 < p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2\big) \\
&- \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_1 < N^{\frac{1}{3}} \leqslant p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), p_2\big) \\
&- \sum_{\substack{N^{\frac{1}{6.18}} \leqslant p_1 < N^{\frac{1}{2}-\frac{3}{11.49}} \leqslant p_2 < (\frac{N}{p_1})^{\frac{1}{2}} \\ (p_1p_2, N)=1}} S\left(\mathcal{A}_{p_1p_2}; \mathcal{P}(Np_1), \left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\right) \\
&- \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_4 < p_3 < p_2 < p_1 < N^{\frac{1}{6.18}} \\ (p_1p_2p_3p_4, N)=1}} S\big(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_3\big) \\
&- \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_1 < p_2 < p_3 < N^{\frac{1}{6.18}} \leqslant p_4 < N^{\frac{1}{2}-\frac{2}{11.49}} p_3^{-1} \\ (p_1p_2p_3p_4, N)=1}} S\big(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_2\big) \\
&+ O(N^{\frac{10.49}{11.49}}) \\
={}& 3S_1 + S_2 - 2S_3 - S_4 - S_5 - S_6 - S_7 + S_8 + S_9 \\
&+ S_{10} - 2S_{11} - S_{12} - S_{13} - S_{14} - S_{15} + O(N^{\frac{10.49}{11.49}}).
\end{aligned}
$$

Proof. Taking $\kappa_1 = \frac{1}{11.49}$ and $\kappa_2 = \frac{1}{6.18}$ in [[22], Lemma 2.2], we get Lemma 3.1. □

LEMMA 3.2. *We have*

$$
\begin{aligned}
4\pi_{1,2}(x) \geqslant{}& 3S(\mathcal{B}; \mathcal{P}, x^{\frac{1}{12}}) + S(\mathcal{B}; \mathcal{P}, x^{\frac{1}{7.2}}) \\
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < p_1 < x^{\frac{1}{7.2}}} S(\mathcal{B}_{p_1p_2}; \mathcal{P}, x^{\frac{1}{12}}) \\
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < x^{\frac{1}{7.2}} \leqslant p_1 < x^{\frac{1}{4}}} S(\mathcal{B}_{p_1p_2}; \mathcal{P}, x^{\frac{1}{12}})
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{x^{\frac{1}{12}} \leqslant p_2 < x^{\frac{1}{7.2}} < x^{\frac{1}{4}} \leqslant p_1 < \min(x^{\frac{2}{7}}, x^{\frac{17}{42}} p_2^{-1})} S(\mathcal{B}_{p_1 p_2}; \mathcal{P}, x^{\frac{1}{12}}) \\
&- 2 \sum_{x^{\frac{1}{12}} \leqslant p < x^{\frac{1}{4}}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) - 2 \sum_{x^{\frac{1}{4}} \leqslant p < x^{\frac{2}{7}-\varepsilon}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) \\
&- \sum_{x^{\frac{2}{7}-\varepsilon} \leqslant p < x^{\frac{2}{7}}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) - \sum_{x^{\frac{2}{7}-\varepsilon} \leqslant p < x^{\frac{29}{100}}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) \\
&- \sum_{x^{\frac{29}{100}} \leqslant p < x^{\frac{1}{3}-\varepsilon}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) - \sum_{x^{\frac{1}{3}-\varepsilon} \leqslant p < x^{\frac{1}{3}}} S(\mathcal{B}_p; \mathcal{P}, x^{\frac{1}{12}}) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < x^{\frac{1}{3}} \leqslant p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\big(\mathcal{B}_{p_1 p_2}; \mathcal{P}(p_1), p_2\big) \\
&- \sum_{x^{\frac{1}{7.2}} \leqslant p_1 < x^{\frac{2}{7}} \leqslant p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1 p_2}; \mathcal{P}(p_1), \left(\frac{x}{p_1 p_2}\right)^{\frac{1}{2}}\right) \\
&- 2 \sum_{x^{\frac{2}{7}} \leqslant p_1 < p_2 < (\frac{x}{p_1})^{\frac{1}{2}}} S\big(\mathcal{B}_{p_1 p_2}; \mathcal{P}(p_1), p_2\big) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_4 < p_3 < p_2 < p_1 < x^{\frac{1}{7.2}}} S\big(\mathcal{B}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_3\big) \\
&- \sum_{x^{\frac{1}{12}} \leqslant p_1 < p_2 < p_3 < x^{\frac{1}{7.2}} < p_4 < \min(x^{\frac{2}{7}}, x^{\frac{17}{42}} p_3^{-1})} S\big(\mathcal{B}_{p_1 p_2 p_3 p_4}; \mathcal{P}(p_1), p_2\big) \\
&+ O(x^{\frac{11}{12}}) \\
= {} & 3S_1' + S_2' + S_3' + S_4' + S_5' - 2S_6' - 2S_7' - S_8' - S_9' \\
&- S_{10}' - S_{11}' - S_{12}' - S_{13}' - 2S_{14}' - S_{15}' - S_{16}' + O(x^{\frac{11}{12}}).
\end{aligned}
$$

Proof. This is [3, Lemma 3.2] and [13, Lemma 3.2]. □

4. PROOF OF THEOREM 1.1

In this section, sets $\mathcal{A}$ and $\mathcal{P}$ are defined respectively. Let γ denotes the Euler's constant, $F(s)$ and $f(s)$ are determined by the following differential-difference equation

$$
\begin{cases}
F(s) = \frac{2e^{\gamma}}{s}, \quad f(s) = 0, & 0 < s \leqslant 2, \\
(sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & s \geqslant 2,
\end{cases}
$$

and let $\omega(u)$ denotes the Buchstab function determined by the next differential-difference equation:

$$\begin{cases}\omega(u) = \frac{1}{u}, & 1 \leqslant u \leqslant 2,\\ (u\omega(u))' = \omega(u-1), & u \geqslant 2.\end{cases}$$

We first consider S_1 and S_2. By Buchstab's identity, we have

$$\begin{aligned} S_1 &= S\big(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{11.49}}\big) \\ &= S\big(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad - \sum_{\substack{N^{\frac{1}{500}} \leqslant p < N^{\frac{1}{11.49}} \\ (p,N)=1}} S\big(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad + \sum_{\substack{N^{\frac{1}{500}} \leqslant p_2 < p_1 < N^{\frac{1}{11.49}} \\ (p_1p_2,N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad - \sum_{\substack{N^{\frac{1}{500}} \leqslant p_3 < p_2 < p_1 < N^{\frac{1}{11.49}} \\ (p_1p_2p_3,N)=1}} S\big(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(N), p_3\big) \end{aligned} \tag{9}$$

and

$$\begin{aligned} S_2 &= S\big(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{6.18}}\big) \\ &= S\big(\mathcal{A}; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad - \sum_{\substack{N^{\frac{1}{500}} \leqslant p < N^{\frac{1}{6.18}} \\ (p,N)=1}} S\big(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad + \sum_{\substack{N^{\frac{1}{500}} \leqslant p_2 < p_1 < N^{\frac{1}{6.18}} \\ (p_1p_2,N)=1}} S\big(\mathcal{A}_{p_1p_2}; \mathcal{P}(N), N^{\frac{1}{500}}\big) \\ &\quad - \sum_{\substack{N^{\frac{1}{500}} \leqslant p_3 < p_2 < p_1 < N^{\frac{1}{6.18}} \\ (p_1p_2p_3,N)=1}} S\big(\mathcal{A}_{p_1p_2p_3}; \mathcal{P}(N), p_3\big). \end{aligned} \tag{10}$$

By Lemma 2.1, Iwaniec's linear sieve method and arguments in [15, 16] and [13] we have

$$S_1 \geqslant (1+o(1))\frac{2}{e^{\gamma}}\Bigg(500 f(500\boldsymbol{\vartheta}_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{1}{11.49}} \frac{F(500(\boldsymbol{\vartheta}_1(t, \frac{1}{500}, \frac{1}{500}) - t))}{t} dt$$

$$+ 500\int_{\frac{1}{500}}^{\frac{1}{11.49}} \int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_1(t_1, t_2, \frac{1}{500}) - t_1 - t_2))}{t_1 t_2} dt_2 dt_1$$

$$-\int_{\frac{1}{500}}^{\frac{1}{11.49}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \tag{11}$$

and

$$S_2 \geqslant (1+o(1))\frac{2}{e^\gamma}\Bigg(500f(500\boldsymbol{\vartheta}_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{1}{6.18}}\frac{F(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$

$$+500\int_{\frac{1}{500}}^{\frac{1}{6.18}}\int_{\frac{1}{500}}^{t_1}\frac{f(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$

$$-\int_{\frac{1}{500}}^{\frac{1}{6.18}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}, \tag{12}$$

where $\boldsymbol{\vartheta}_{\frac{1}{500}} = \frac{19101}{32000}$. By numerical calculations we get that

$$S_1 \geqslant 12.902021\frac{C(N)N}{(\log N)^2} \tag{13}$$

and

$$S_2 \geqslant 6.533916\frac{C(N)N}{(\log N)^2}. \tag{14}$$

For S_3, we can either use Buchstab's identity and Lichtman's method to estimate S_3 with a better distribution level as in [15] or use Chen's double sieve technique as in [22]. The first option leads to

$$\begin{aligned}\sum_{\substack{p\\(p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{11.49}}\big) = &\sum_{\substack{p\\(p,N)=1}} S\big(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{k}}\big)\\ &-\sum_{\substack{p_1\\N^{\frac{1}{k}}\leqslant p_2<N^{\frac{1}{11.49}}\\(p_1p_2,N)=1}} S\big(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{k}}\big)\\ &+\sum_{\substack{p_1\\N^{\frac{1}{k}}\leqslant p_3<p_2<N^{\frac{1}{11.49}}\\(p_1p_2p_3,N)=1}} S\big(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_3\big)\end{aligned} \tag{15}$$

for some $k \geqslant 11.49$, while the second option creates a small saving on S_3 itself. We can also use Chen's double sieve on the first two sums on the right-hand side of (15) after applying Buchstab's identity. We do not know which of these options gives a smaller value, hence we take a minimum. By Lemma 2.1, Iwaniec's linear sieve method and arguments in [15, 16] and [13] we have

$$
\begin{aligned}
S_3 \leqslant& \big(1+o(1)\big)\frac{2}{e^\gamma}\Bigg(\int_{\frac{1}{11.49}}^{\frac{25}{128}} \min\Bigg(11.49\frac{F(11.49(\boldsymbol{\vartheta}_1(t_1,\frac{1}{11.49},\frac{1}{11.49})-t_1))}{t_1}\\
&-\frac{22.98e^\gamma H(11.49(\frac{1}{2}-t_1))}{(11.49(\frac{1}{2}-t_1))t_1}, \min_{11.49\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1}\\
&-\frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1} - k\int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{f(k(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}dt_2\\
&-2ke^\gamma\int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2\\
&+\int_{\frac{1}{k}}^{\frac{1}{11.49}}\int_{\frac{1}{k}}^{t_2} \frac{F(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C(N)N}{(\log N)^2}\\
\leqslant& 10.436523\frac{C(N)N}{(\log N)^2},
\end{aligned}
\tag{16}
$$

where we choose $k = 12.3$ and $H(s) = H_{1/2}(s)$ and $h(s) = h_{1/2}(s)$ are defined as the same in [22]. We have used the following lower bounds of $H(s)$ and $h(s)$ for $2.0 \leqslant s \leqslant 4.9$. These values can be found at Tables 1 and 2 of [22]. We remark that we have $H_{\boldsymbol{\vartheta}}(s) \geqslant H_{1/2}(s)$ and $h_{\boldsymbol{\vartheta}}(s) \geqslant h_{1/2}(s)$ for $\boldsymbol{\vartheta} > \frac{1}{2}$.

$$
H(s) \geqslant \begin{cases}
0.0223939, & 2.0 < s \leqslant 2.2,\\
0.0217196, & 2.2 < s \leqslant 2.3,\\
0.0202876, & 2.3 < s \leqslant 2.4,\\
0.0181433, & 2.4 < s \leqslant 2.5,\\
0.0158644, & 2.5 < s \leqslant 2.6,\\
0.0129923, & 2.6 < s \leqslant 2.7,\\
0.0100686, & 2.7 < s \leqslant 2.8,\\
0.0078162, & 2.8 < s \leqslant 2.9,\\
0.0072943, & 2.9 < s \leqslant 3.0,\\
0.0061642, & 3.0 < s \leqslant 3.1,\\
0.0052233, & 3.1 < s \leqslant 3.2,\\
0.0044073, & 3.2 < s \leqslant 3.3,\\
0.0036995, & 3.3 < s \leqslant 3.4,\\
0.0030860, & 3.4 < s \leqslant 3.5,
\end{cases}
\begin{cases}
0.0025551, & 3.5 < s \leqslant 3.6,\\
0.0020972, & 3.6 < s \leqslant 3.7,\\
0.0017038, & 3.7 < s \leqslant 3.8,\\
0.0013680, & 3.8 < s \leqslant 3.9,\\
0.0010835, & 3.9 < s \leqslant 4.0,\\
0.0008451, & 4.0 < s \leqslant 4.1,\\
0.0006482, & 4.1 < s \leqslant 4.2,\\
0.0004882, & 4.2 < s \leqslant 4.3,\\
0.0003602, & 4.3 < s \leqslant 4.4,\\
0.0002592, & 4.4 < s \leqslant 4.5,\\
0.0001803, & 4.5 < s \leqslant 4.6,\\
0.0001187, & 4.6 < s \leqslant 4.7,\\
0.0000702, & 4.7 < s \leqslant 4.8,\\
0.0000313, & 4.8 < s \leqslant 4.9,
\end{cases}
\tag{17}
$$

$$h(s) \geqslant \begin{cases} 0.0232385, & s = 2.0, \\ 0.0211041, & 2.0 < s \leqslant 2.1, \\ 0.0191556, & 2.1 < s \leqslant 2.2, \\ 0.0173631, & 2.2 < s \leqslant 2.3, \\ 0.0157035, & 2.3 < s \leqslant 2.4, \\ 0.0141585, & 2.4 < s \leqslant 2.5, \\ 0.0127132, & 2.5 < s \leqslant 2.6, \\ 0.0113556, & 2.6 < s \leqslant 2.7, \\ 0.0100756, & 2.7 < s \leqslant 2.8, \\ 0.0088648, & 2.8 < s \leqslant 2.9, \\ 0.0077612, & 2.9 < s \leqslant 3.0, \\ 0.0066236, & 3.0 < s \leqslant 3.1, \\ 0.0055818, & 3.1 < s \leqslant 3.2, \\ 0.0046164, & 3.2 < s \leqslant 3.3, \\ 0.0037529, & 3.3 < s \leqslant 3.4, \end{cases} \begin{cases} 0.0030123, & 3.4 < s \leqslant 3.5, \\ 0.0023901, & 3.5 < s \leqslant 3.6, \\ 0.0018997, & 3.6 < s \leqslant 3.7, \\ 0.0015336, & 3.7 < s \leqslant 3.8, \\ 0.0012593, & 3.8 < s \leqslant 3.9, \\ 0.0010120, & 3.9 < s \leqslant 4.0, \\ 0.0008099, & 4.0 < s \leqslant 4.1, \\ 0.0006440, & 4.1 < s \leqslant 4.2, \\ 0.0005084, & 4.2 < s \leqslant 4.3, \\ 0.0003980, & 4.3 < s \leqslant 4.4, \\ 0.0003085, & 4.4 < s \leqslant 4.5, \\ 0.0002365, & 4.5 < s \leqslant 4.6, \\ 0.0001791, & 4.6 < s \leqslant 4.7, \\ 0.0001396, & 4.7 < s \leqslant 4.8, \\ 0.0000981, & 4.8 < s \leqslant 4.9. \end{cases} \tag{18}$$

Similarly, for S_4, S_5 and S_7, we have

$$\begin{aligned} S_4 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{25}{128}}^{\frac{1}{4}} \min\Bigg(11.49\frac{F(11.49(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} \\ & - \frac{22.98e^\gamma H(11.49(\frac{1}{2}-t_1))}{(11.49(\frac{1}{2}-t_1))t_1}, \\ & \min_{11.49\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1} \\ & - \int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{f(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2})}{t_1t_2^2}dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \\ \leqslant{} & 3.311305\frac{C(N)N}{(\log N)^2}, \end{aligned} \tag{19}$$

$$\begin{aligned} S_5 \leqslant{} & (1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{1}{4}}^{\frac{57}{224}} \min\Bigg(11.49\frac{F(11.49(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1}, \\ & \min_{11.49\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} \\ & - \int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{f(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2})}{t_1t_2^2}dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \end{aligned}$$

$$\leqslant 0.272301\frac{C(N)N}{(\log N)^2}, \tag{20}$$

$$S_7 \leqslant \big(1+o(1)\big)\frac{2}{e^\gamma}\Bigg(\int_{\frac{25}{128}}^{\frac{1}{2}-\frac{3}{11.49}} \min\Bigg(11.49\frac{F(11.49(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{22.98e^\gamma H(11.49(\frac{1}{2}-t_1))}{(11.49(\frac{1}{2}-t_1))t_1},$$

$$\min_{11.49\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1}$$

$$- \int_{\frac{1}{k}}^{\frac{1}{11.49}} \frac{f(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2})}{t_1t_2^2}dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C(N)N}{(\log N)^2}$$

$$\leqslant 2.659313\frac{C(N)N}{(\log N)^2}. \tag{21}$$

By the classical linear sieve, for S_6 we have

$$S_6 \leqslant \big(1+o(1)\big)\frac{2}{e^\gamma}\Bigg(11.49\int_{\frac{57}{224}}^{\frac{1}{3}}\frac{F(11.49(\frac{1}{2}-t))}{t}dt\Bigg)\frac{C(N)N}{(\log N)^2}$$

$$\leqslant 5.259433\frac{C(N)N}{(\log N)^2}. \tag{22}$$

For S_8–S_{10} we can also use Chen's double sieve to gain some savings. Using similar methods as above together with [[22], Propositions 4.2 and 4.3], we have

$$S_8 \geqslant$$

$$\big(1+o(1)\big)\frac{2}{e^\gamma}\Bigg(11.49\int_{\frac{1}{11.49}}^{\frac{1}{6.18}}\int_{\frac{1}{11.49}}^{t_1}\frac{f(11.49(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{11.49})-t_1-t_2))}{t_1t_2}dt_2dt_1$$

$$+\ 11.49\int_{\frac{1}{11.49}}^{\frac{1}{6.18}}\int_{\frac{1}{11.49}}^{t_1}\frac{2e^\gamma h(11.49(\frac{1}{2}-t_1-t_2))}{(11.49(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}$$

(23)

$$\geqslant 2.421452\frac{C(N)N}{(\log N)^2},$$

$$S_9 \geqslant$$

$$\big(1+o(1)\big)\frac{2}{e^\gamma}\Bigg(11.49\int_{\frac{1}{6.18}}^{\frac{25}{128}}\int_{\frac{1}{11.49}}^{\frac{1}{6.18}}\frac{f(11.49(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{11.49})-t_1-t_2))}{t_1t_2}dt_2dt_1$$

$$+\,11.49\int_{\frac{1}{6.18}}^{\frac{1}{2}-\frac{2}{6.18}}\int_{\frac{1}{11.49}}^{\frac{1}{6.18}}\frac{2e^\gamma h(11.49(\frac{1}{2}-t_1-t_2))}{(11.49(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1$$

$$+\ 11.49\int_{\frac{1}{2}-\frac{2}{6.18}}^{\frac{25}{128}}\int_{\frac{1}{11.49}}^{\frac{39}{256}}\frac{2e^{\gamma}h(11.49(\frac{1}{2}-t_1-t_2))}{(11.49(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}$$

(24)

$$\geqslant 1.382532\frac{C(N)N}{(\log N)^2},$$

$$S_{10}\geqslant$$

$$(1+o(1))\frac{2}{e^{\gamma}}\Bigg(11.49\int_{\frac{25}{128}}^{\frac{1}{2}-\frac{3}{11.49}}\int_{\frac{1}{11.49}}^{\frac{1}{6.18}}\frac{f(11.49(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{11.49})-t_1-t_2))}{t_1t_2}dt_2dt_1$$

$$+\ 11.49\int_{\frac{25}{128}}^{\frac{1}{2}-\frac{3}{11.49}}\int_{\frac{1}{11.49}}^{\frac{1.5}{11.49}}\frac{2e^{\gamma}h(11.49(\frac{1}{2}-t_1-t_2))}{(11.49(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}$$

(25)

$$\geqslant 0.960457\frac{C(N)N}{(\log N)^2}.$$

For the remaining terms, we can use Chen's switching principle together with Lemma 2.2 to estimate them. Namely, for S_{11} we have

$$(26)\quad S_{11}=\sum_{\substack{N^{\frac{1}{2}-\frac{3}{11.49}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}}S\big(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2\big)=S\big(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\big),$$

where the set $\mathcal{A}'$ is defined as

$$\mathcal{A}'=\big\{N-p_1p_2m:N^{\frac{1}{2}-\frac{3}{11.49}}\leqslant p_1<p_2<(N/p_1)^{\frac{1}{2}},\ p'\mid m\Rightarrow p'>p_2\text{ or }p'=p_1\big\}.$$

We note that each m above must be a prime number or a P_2 since $\frac{1}{2}-\frac{3}{11.49}>\frac{1}{5}$. By Buchstab's identity, we have

$$\begin{aligned}S_{11}&=S\big(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\big)\\&\leqslant S\big(\mathcal{A}';\mathcal{P}(N),N^{\frac{25}{128}}\big)\\&=S\big(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{500}}\big)-\sum_{\substack{N^{\frac{1}{500}}\leqslant p'<N^{\frac{25}{128}}\\(p',N)=1}}S\big(\mathcal{A}'_{p'};\mathcal{P}(N),N^{\frac{1}{500}}\big)\\&\quad+\sum_{\substack{N^{\frac{1}{500}}\leqslant p'_2<p'_1<N^{\frac{25}{128}}\\(p'_1p'_2,N)=1}}S\big(\mathcal{A}'_{p'_1p'_2};\mathcal{P}(N),N^{\frac{1}{500}}\big)\\&\quad-\sum_{\substack{N^{\frac{1}{500}}\leqslant p'_3<p'_2<p'_1<N^{\frac{25}{128}}\\(p'_1p'_2p'_3,N)=1}}S\big(\mathcal{A}'_{p'_1p'_2p'_3};\mathcal{P}(N),p'_3\big).\end{aligned}\tag{27}$$

Then by Lemma 2.2, Iwaniec's linear sieve method and arguments in [15, 16] and [13] we have

$$\begin{aligned}
S_{11} \leqslant & \big(1+o(1)\big)\frac{2C(N)|\mathcal{A}'|}{e^{\gamma}\log N}\Bigg(500F(500\boldsymbol{\vartheta}_{\frac{1}{500}}) \\
& - 500\int_{\frac{1}{500}}^{\frac{25}{128}} \frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt \\
& + 500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1} \frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1 \\
& - \int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{f(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg) \\
\leqslant & (1+o(1))\frac{2G_1}{e^{\gamma}}\Bigg(\int_{\frac{1}{2}-\frac{3}{11.49}}^{\frac{1}{3}}\int_{t_1}^{\frac{1}{2}(1-t_1)} \frac{\omega(\frac{1-t_1-t_2}{t_2})}{t_1t_2^2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \\
\leqslant & 1.30656\frac{C(N)N}{(\log N)^2}, \qquad (28)
\end{aligned}$$

where

$$\begin{aligned}
G_1 = & 500F(500\boldsymbol{\vartheta}_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{25}{128}} \frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt \\
& + 500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1} \frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1 \\
& - \int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{f(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2dt_1 \\
< & 6.06932. \qquad (29)
\end{aligned}$$

Similarly, for S_{12} and S_{13} we have

$$\begin{aligned}
S_{12} \leqslant & \big(1+o(1)\big)\frac{2G_1}{e^{\gamma}}\Bigg(\int_{\frac{1}{11.49}}^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{1}{2}(1-t_1)} \frac{\omega(\frac{1-t_1-t_2}{t_2})}{t_1t_2^2}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \\
\leqslant & 3.912436\frac{C(N)N}{(\log N)^2}, \qquad (30)\\
S_{13} \leqslant & (1+o(1))\frac{2G_1}{e^{\gamma}}\Bigg(\int_{\frac{1}{6.18}}^{\frac{1}{2}-\frac{3}{11.49}}\int_{\frac{1}{2}-\frac{3}{11.49}}^{\frac{1}{2}(1-t_1)} \frac{1}{t_1t_2(1-t_1-t_2)}dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \\
\leqslant & 2.835087\frac{C(N)N}{(\log N)^2}. \qquad (31)
\end{aligned}$$

For S_{14} and S_{15}, we use a device that has been used a lot in Harman's sieve. Since $p_3 > p_4$, we have

$$\sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_4 < p_3 < p_2 < p_1 < N^{\frac{1}{6.18}} \\ (p_1p_2p_3p_4, N)=1}} S\big(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_3\big)$$

$$\leqslant \sum_{\substack{N^{\frac{1}{11.49}} \leqslant p_4 < p_3 < p_2 < p_1 < N^{\frac{1}{6.18}} \\ (p_1p_2p_3p_4, N)=1}} S\big(\mathcal{A}_{p_1p_2p_3p_4}; \mathcal{P}(N), p_4\big). \tag{32}$$

Here we can apply Lemma 2.1 with $r = 4$ to handle part of the sum on the right-hand side of (32) if $(D_1, \ldots, D_4) \in \mathbf{D}_4^{\text{well}}(D)$. We use the similar arguments as above to deal with other parts. Thus, we have

$$\begin{aligned} S_{14} \leqslant \big(1 + o(1)\big) \frac{C(N)N}{(\log N)^2} \Bigg(\int_{\frac{1}{11.49}}^{\frac{1}{6.18}} \int_{\frac{1}{11.49}}^{t_1} \int_{\frac{1}{11.49}}^{t_2} \int_{\frac{1}{11.49}}^{t_3} \\ \Bigg(\mathbf{1}_{(D_1,\ldots,D_4)\in \mathbf{D}_4^{\text{well}}(D)} \min\Bigg(\frac{2}{e^\gamma} \frac{F\big(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3-t_4)}{t_4}\big)}{t_1t_2t_3t_4^2}, \\ \frac{2G_1}{e^\gamma} \frac{\omega\big(\frac{1-t_1-t_2-t_3-t_4}{t_3}\big)}{t_1t_2t_3^2t_4} \Bigg) \\ + \mathbf{1}_{(D_1,\ldots,D_4)\notin \mathbf{D}_4^{\text{well}}(D)} \frac{2G_1}{e^\gamma} \frac{\omega\big(\frac{1-t_1-t_2-t_3-t_4}{t_3}\big)}{t_1t_2t_3^2t_4} \Bigg) dt_4dt_3dt_2dt_1 \Bigg) \end{aligned}$$

$$\leqslant 0.193502 \frac{C(N)N}{(\log N)^2}. \tag{33}$$

Similarly, for S_{15} we have

$$\begin{aligned} S_{15} \leqslant \big(1 + o(1)\big) \frac{C(N)N}{(\log N)^2} \Bigg(\int_{\frac{1}{11.49}}^{\frac{1}{6.18}} \int_{t_1}^{\frac{1}{6.18}} \int_{t_2}^{\frac{1}{6.18}} \int_{\frac{1}{6.18}}^{\frac{1}{2}-\frac{2}{11.49}-t_3} \\ \Bigg(\mathbf{1}_{(D_4,\ldots,D_1)\in \mathbf{D}_4^{\text{well}}(D)} \min\Bigg(\frac{2}{e^\gamma} \frac{F\big(\frac{(\boldsymbol{\vartheta}_1(t_4,t_3,t_2)-t_1-t_2-t_3-t_4)}{t_1}\big)}{t_1^2t_2t_3t_4}, \\ \frac{2G_1}{e^\gamma} \frac{\omega\big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\big)}{t_1t_2^2t_3t_4} \Bigg) \\ + \mathbf{1}_{(D_4,\ldots,D_1)\notin \mathbf{D}_4^{\text{well}}(D)} \frac{2G_1}{e^\gamma} \frac{\omega\big(\frac{1-t_1-t_2-t_3-t_4}{t_2}\big)}{t_1t_2^2t_3t_4} \Bigg) dt_4dt_3dt_2dt_1 \Bigg) \end{aligned}$$

$$\leqslant 0.183611 \frac{C(N)N}{(\log N)^2}. \tag{34}$$

Finally, by Lemma 3.1 and (9)–(34) we get

$$4D_{1,2}(N) \geqslant (3S_1 + S_2 + S_8 + S_9 + S_{10})$$

$$
\begin{aligned}
&-(2S_3+S_4+S_5+S_6+S_7+2S_{11}+S_{12}+S_{13}+S_{14}+S_{15})\\
&\geqslant 7.8912\frac{C(N)N}{(\log N)^2},
\end{aligned}
$$

hence

$$D_{1,2}(N)\geqslant 1.9728\frac{C(N)N}{(\log N)^2}.$$

Theorem 1.1 is proved. Since the detail of the proof of Theorem 1.2 is similar to those of Theorem 1.1 and Theorem 1.1 in [14], we omit it in this paper.

5. PROOF OF THEOREM 1.3

In this section, sets $\mathcal{B}$ and $\mathcal{P}$ are defined respectively. For S_1' and S_2', by Buchstab's identity, we have

$$
\begin{aligned}
S_1' = S(\mathcal{B};\mathcal{P},x^{\frac{1}{12}}) &= S(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}) - \sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{12}}} S(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}})\\
&+ \sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{12}}} S(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}})\\
&- \sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{12}}} S(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3) \qquad (35)
\end{aligned}
$$

and

$$
\begin{aligned}
S_2' = S(\mathcal{B};\mathcal{P},x^{\frac{1}{7.2}}) &= S(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}) - \sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{7.2}}} S(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}})\\
&+ \sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{7.2}}} S(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}})\\
&- \sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{7.2}}} S(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3). \qquad (36)
\end{aligned}
$$

By Lemma 2.3, Iwaniec's linear sieve method and arguments in [16], [15] and [13] we have

$$
\begin{aligned}
S_1' \geqslant (1+o(1))\frac{1}{e^\gamma}\Bigg(&500f(500\boldsymbol{\vartheta}'_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{1}{12}}\frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\\
&+500\int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1}\frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\
&-\int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}
\end{aligned}
$$

$$\geqslant 6.737439\frac{C_2x}{(\log x)^2} \tag{37}$$

and

$$S_2' \geqslant (1+o(1))\frac{1}{e^\gamma}\Bigg(500f(500\boldsymbol{\vartheta}'_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{1}{7.2}}\frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$
$$+500\int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1}\frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$
$$-\int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$
$$\geqslant 4.011646\frac{C_2x}{(\log x)^2}, \tag{38}$$

where $\boldsymbol{\vartheta}'_{\frac{1}{500}} = \frac{2497}{4000}$. For S_3'–S_7', by Lemma 2.3, Iwaniec's linear sieve method and the above discussion, we have

$$S_3' \geqslant \big(1+o(1)\big)\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{12}}^{t_1}\max\Bigg(12\frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1t_2},$$
$$\max_{12\leqslant k\leqslant 500}\Bigg(k\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}$$
$$-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3\Bigg)\Bigg)dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$
$$\geqslant 0.875194\frac{C_2x}{(\log x)^2}, \tag{39}$$

$$S_4' \geqslant \big(1+o(1)\big)\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{7.2}}^{\frac{1}{4}}\int_{\frac{1}{12}}^{\frac{1}{7.2}}\max\Bigg(12\frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1t_2},$$
$$\max_{12\leqslant k\leqslant 500}\Bigg(k\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}$$
$$-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3\Bigg)\Bigg)dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$
$$\geqslant 1.917212\frac{C_2x}{(\log x)^2}, \tag{40}$$

$$S_5' \geqslant \big(1+o(1)\big)\frac{1}{e^\gamma}$$
$$\Bigg(12\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{4}}^{\min(\frac{2}{7},\frac{17}{42}-t_1)}\frac{f(12(\boldsymbol{\vartheta}_0(t_2)-t_1-t_2))}{t_1t_2}dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$

$$\geqslant 0.282826\frac{C_2x}{(\log x)^2}, \tag{41}$$

$$S_6' \leqslant \big(1+o(1)\big)\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{4}} \min\Bigg(12\frac{F(12(\boldsymbol{\vartheta}_0(t_1,\frac{1}{12},\frac{1}{12})-t_1))}{t_1},$$

$$\min_{12\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_0(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1}$$

$$-k\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}dt_2$$

$$+\int_{\frac{1}{k}}^{\frac{1}{12}}\int_{\frac{1}{k}}^{t_2}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$

$$\leqslant 7.410929\frac{C_2x}{(\log x)^2}, \tag{42}$$

$$S_7' \leqslant (1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{4}}^{\frac{2}{7}} \min\Bigg(12\frac{F(12(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1},$$

$$\min_{12\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1}$$

$$-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{f(\frac{(\boldsymbol{\vartheta}_0(t_1)-t_1-t_2)}{t_2})}{t_1t_2^2}dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$

$$\leqslant 0.925271\frac{C_2x}{(\log x)^2}. \tag{43}$$

For S_{12}'–S_{16}', by Chen's switching principle, Lemma 2.4 and above arguments on estimating S_{11}–S_{15}, we have

$$S_{12}' \leqslant (1+o(1))\frac{G_2}{e^\gamma}\left(\int_2^{11}\frac{\log(2-\frac{3}{t+1})}{t}dt\right)\frac{C_2x}{(\log x)^2}$$

$$\leqslant 1.960955\frac{C_2x}{(\log x)^2}, \tag{44}$$

$$S_{13}' \leqslant (1+o(1))\frac{G_2}{e^\gamma}\left(\int_{2.5}^{6.2}\frac{\log(2.5-\frac{3.5}{t+1})}{t}dt\right)\frac{C_2x}{(\log x)^2}$$

$$\leqslant 1.699112\frac{C_2x}{(\log x)^2}, \tag{45}$$

$$S_{14}' \leqslant (1+o(1))\frac{G_2}{e^\gamma}\left(\int_2^{2.5}\frac{\log(t-1)}{t}dt\right)\frac{C_2x}{(\log x)^2}$$

$$\leqslant 0.152213\frac{C_2x}{(\log x)^2}, \tag{46}$$

$$S'_{15} \leqslant (1+o(1))\frac{C_2 x}{(\log x)^2}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{12}}^{t_1}\int_{\frac{1}{12}}^{t_2}\int_{\frac{1}{12}}^{t_3}$$
$$\Bigg(\mathbf{1}_{(D_1,\ldots,D_4)\in\mathbf{D}_4^{\text{well}}(D)}\min\Bigg(\frac{1}{e^\gamma}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3-t_4)}{t_4})}{t_1t_2t_3t_4^2},$$
$$\frac{G_2}{e^\gamma}\frac{\omega(\frac{1-t_1-t_2-t_3-t_4}{t_3})}{t_1t_2t_3^2t_4}\Bigg)$$
$$+\ \mathbf{1}_{(D_1,\ldots,D_4)\notin\mathbf{D}_4^{\text{well}}(D)}\frac{G_2}{e^\gamma}\frac{\omega(\frac{1-t_1-t_2-t_3-t_4}{t_3})}{t_1t_2t_3^2t_4}\Bigg)dt_4dt_3dt_2dt_1\Bigg)$$
$$\leqslant 0.031709\frac{C_2 x}{(\log x)^2}, \tag{47}$$

$$S'_{16} \leqslant (1+o(1))\frac{C_2 x}{(\log x)^2}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\min(\frac{2}{7},\frac{17}{42}-t_3)}$$
$$\Bigg(\mathbf{1}_{(D_4,\ldots,D_1)\in\mathbf{D}_4^{\text{well}}(D)}\min\Bigg(\frac{1}{e^\gamma}\frac{F(\frac{(\boldsymbol{\vartheta}_0(t_4,t_3,t_2)-t_1-t_2-t_3-t_4)}{t_1})}{t_1^2t_2t_3t_4},$$
$$\frac{G_2}{e^\gamma}\frac{\omega(\frac{1-t_1-t_2-t_3-t_4}{t_2})}{t_1t_2^2t_3t_4}\Bigg)$$
$$+\ \mathbf{1}_{(D_4,\ldots,D_1)\notin\mathbf{D}_4^{\text{well}}(D)}\frac{G_2}{e^\gamma}\frac{\omega(\frac{1-t_1-t_2-t_3-t_4}{t_2})}{t_1t_2^2t_3t_4}\Bigg)dt_4dt_3dt_2dt_1\Bigg)$$
$$\leqslant 0.245969\frac{C_2 x}{(\log x)^2}, \tag{48}$$

where

$$G_2 = 500F(500\boldsymbol{\vartheta}'_{\frac{1}{500}}) - 500\int_{\frac{1}{500}}^{\frac{1}{5}}\frac{f(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt$$
$$+500\int_{\frac{1}{500}}^{\frac{1}{5}}\int_{\frac{1}{500}}^{t_1}\frac{F(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1$$
$$-\int_{\frac{1}{500}}^{\frac{1}{5}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{f(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3})}{t_1t_2t_3^2}dt_3dt_2dt_1.$$
$$< 5.81637. \tag{49}$$

For the remaining terms, by the arguments in [3] and [22], we have

$$S'_8 \ll \frac{\varepsilon C_2 x}{(\log x)^2}, \tag{50}$$

$$S'_9 \leqslant (1+o(1))\frac{12}{e^\gamma}\left(\int_{(\frac{11}{20}-\frac{29}{100})12}^{(\frac{4}{7}-\frac{2}{7})12}\frac{F(t)}{2\times 12-t}dt\right) \leqslant 0.111039\frac{C_2 x}{(\log x)^2}, \tag{51}$$

$$(52)\qquad S'_{10} \leqslant (1+o(1))\frac{12}{e^{\gamma}}\left(\int_{(\frac{11}{20}-\frac{1}{3})12}^{(\frac{11}{20}-\frac{29}{100})12} \frac{F(t)}{\frac{11}{20}\times 12 - t}dt\right) \leqslant 1.169696\frac{C_2 x}{(\log x)^2},$$

$$(53)\qquad S'_{11} \ll \frac{\varepsilon C_2 x}{(\log x)^2}.$$

Finally, by Lemma 3.2 and (35)–(53) we get

$$\begin{aligned} 4\pi_{1,2}(x) \geqslant{}& (3S'_1 + S'_2 + S'_3 + S'_4 + S'_5) \\ &- (2S'_6 + 2S'_7 + S'_8 + S'_9 + S'_{10} + S'_{11} + S'_{12} + S'_{13} \\ &\quad + 2S'_{14} + S'_{15} + S'_{16} + S'_{17} + S'_{18} + S'_{19}) \\ \geqslant{}& 5.1036\frac{C_2 x}{(\log x)^2}, \end{aligned}$$

hence

$$\pi_{1,2}(x) \geqslant 1.2759\frac{C_2 x}{(\log x)^2}.$$

Theorem 1.3 is proved.

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