ON THE LOWER BOUND FOR THE CARDINALITY OF SUMSET $A + 5 \cdot A$

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Communicated by Alexandru Zaharescu

Let A and B be two non-empty finite subsets of $\mathbb{Z}$. We consider the dilated sumset $A+k\cdot B = \{a+kb : a \in A, b \in B\}$, where $k \in \mathbb{Z}^+$. In 2010, Cilleruelo et al. proved that $|A+3\cdot A| \geq 4|A|-|A|$, where A is finite subset of $\mathbb{Z}$. They proposed the following conjecture. If A is a non-empty finite subset of $\mathbb{Z}$ with arbitrarily large cardinality and $k \in \mathbb{Z}^+$, then $|A+k\cdot A| \geq (k+1)|A| - \lceil (k^2+2k)/4 \rceil$. In 2009, Cilleruelo et al. proved this conjecture for $k = p$, with $|A| \geq 3(k-1)^2(k-1)!$. In this article, we prove that this conjecture is true for smaller as well as larger value of $|A|$ with $|A| \geq 6$ and $k = 5$.

AMS 2020 Subject Classification: 11B13, 11P70, 11B75.

Key words: sumset, additive combinatorics, direct problem.

1. INTRODUCTION

Let A and B be two non-empty finite subsets of $\mathbb{Z}$. For any $\alpha, \beta \in \mathbb{Z}^+$, the dilated sumset of A and B is defined as

$$\alpha \cdot A + \beta \cdot B = \{\alpha a + \beta b : a \in A, b \in B\}.$$

For $\alpha = \beta = 1$, this sumset is called the Minkowski Sumset. In 1996, Nathanson [6] proved that $|A + B| \geq |A| + |B| - 1$, and the equality holds if and only if A and B are in arithmetic progressions with the same common difference or if $|A| = 1$ and $|B| = 1$. In 2008, Bukh [1] proved that for any finite subset A of $\mathbb{Z}$ and $(\lambda_1, \lambda_2, \ldots, \lambda_h) \in \mathbb{Z}^h$, such that $(\lambda_1, \lambda_2, \ldots, \lambda_h) = 1$, then

$$|\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_h \cdot A| \geq (|\lambda_1| + |\lambda_2| + \cdots + |\lambda_h|)|A| - o(|A|),$$

where $o(|A|)$ is the error term. In 2010, Cilleruelo et al. [3] proved that

$$|A + 3 \cdot A| \geq 4|A| - 4$$

The research of S. Singh is supported by NBHM (Sanction No. 02011/49/2023/R&D-II/13983).

doi: 10.59277/mrar.2026.28.78.1.2.63

and equality holds for $A = \{0, 1, 3\}$ or $\{0, 1, 4\}$ or $3\cdot\{0, \ldots, n\} \cup (3\cdot\{0, \ldots, n\}+1)$ and all affine transforms of these sets. They also conjectured that if A is an arbitrarily large subset of $\mathbb{Z}$ and $k \in \mathbb{Z}^+$, then

$$|A + k \cdot A| \geq (k+1)|A| - \lceil (k^2 + 2k)/4 \rceil.$$

Cilleruelo et al. [2] settled the conjecture for $k = p$, (p is any prime) with $|A| \geq 3(k-1)^2(k-1)!$. In 2014, Du et al. [4] confirmed the conjecture for non-empty finite subset A of $\mathbb{Z}$ with $|A| \geq (k-1)^2 k!$, and for $k = p^n$ or $k = pq$ (p and q are distinct primes), $n \in \mathbb{Z}^+$. In 2024, Kaur et al. [5] proved the conjecture for k to be any positive integer except of the form $2^{n_1}p^{n_2}$ or $2^{n_1}3^{n_2}5^{n_3}$, where $n_i \geq 1$ for $i = 1, 2, 3$ and p be any odd prime, and some additional restrictions applied on A. In the most part of the literature, the conjecture is proved for large values of $|A|$. In 2014, Du et al. [4] found lower bound for cardinality of $A + k \cdot A$ for $|A| \geq 5$ and $k = 4$. But it is still pending to find a lower bound for the cardinality of $A + k \cdot A$ for $k(> 4) \in \mathbb{Z}^+$ and small cardinalities of A. In this article, we consider the case when $k = 5$ and smaller as well as larger values of $|A|$ with $|A| \geq 6$. We also observe that the conjecture is not true for $2 \leq |A| \leq 5$ and $k = 5$.

In Section 2, we discuss some important assumptions and already proved results that are useful to prove our main theorem. In Section 3, we give our main results.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout the paper, we use the following notations and assumptions. Let A be a finite subset of $\mathbb{Z}$ with $|A| \geq 2$. For $k \in \mathbb{Z}^+$, define

$$\hat{A} = \{\bar{a} = a + k\mathbb{Z} : a \in A\}.$$

Let $A_1, A_2, \ldots, A_t$ be distinct residue classes of A modulo k such that

$$A = \bigcup_{i=1}^{t} A_i = \bigcup_{i=1}^{t} (k \cdot X_i + r_i),$$

where $A_i = k \cdot X_i + r_i$ such that $0 \leq r_i < k$ for $i = 1, 2, \ldots, t$. Assume $\Delta_{ri} = (A_r + k \cdot A) \setminus (A_r + k \cdot A_i)$. Without loss of generality, we can make the following assumptions:

1. $r_1 = 0$ and $|A_1| \geq |A_2| \geq \cdots \geq |A_t|$,

2. $\gcd(A) = 1$,

3. $2 \leq |\hat{A}| \leq k$.

Next, we have some key results that are frequently used to prove our main theorem.

Lemma 2.1 ([3]). *For the following two arbitrary non-empty sets B and $A = \bigcup_{i=1}^{t} A_i = \bigcup_{i=1}^{t}(k \cdot X_i + r_i)$,*

(i) $|A + k \cdot B| = \sum_{i=1}^{t} |X_i + B|$.

(ii) $|A + k \cdot B| \geq |A| + t(|B| - 1)$.

(iii) *Furthermore, if equality holds in* (ii)*, then either $|B| = 1$ or $|X_i| = 1$ for all $i = 1, 2, \ldots, t$ or B and all the sets X_i with more than one element are arithmetic progressions with the same common difference.*

Lemma 2.2 ([6]). *For $n \geq 2$, let A and B be non-empty subsets of $\mathbb{Z}/n\mathbb{Z}$. If $0 \in B$ and $(b, n) = 1$ for all $b \in B \setminus \{0\}$, then*

$$|A + B| \geq \min\{n, |A| + |B| - 1\}.$$

Lemma 2.3 ([2]). *For each subset $I \subseteq \{1, 2, \ldots, t\}$, we have*

$$\sum_{i \in I} |\Delta_{ii}| \geq |I|(|I| - 1).$$

Lemma 2.4 ([2]). *Let k be a prime and $E = \{i : 0 < |\hat{X}_i| < k\}$ and $F = \{i : |\hat{X}_i| = k\}$. Then*

(i) *If $i \in E$, then $|\Delta_{ii}| \geq |A_s|$ for any $s \neq i$.*

(ii) *If $\sum_{i \in E} |\Delta_{ii}| \geq (|E| - 1)|A_1| + |A_2|$.*

3. MAIN RESULTS

To prove our main theorem, we prove the following important lemmas.

Lemma 3.1. *Let A be a non-empty finite subset of $\mathbb{Z}$. Then the following hold:*

(I) *If $|A| = 2$, then $|A + 5 \cdot A| = 4$.*

(II) *If $|A| = 3$, then $8 \leq |A + 5 \cdot A| \leq 9$.*

(III) *If $|A| = 4$, then $12 \leq |A + 5 \cdot A| \leq 16$.*

(IV) *If $|A| = 5$, then $19 \leq |A + 5 \cdot A| \leq 25$.*

Proof. (I) Let $A = \{x, y\}$, where $x \neq y$. Then

$$A + 5 \cdot A = \{6x, x + 5y, y + 5x, 6y\}.$$

It is easy to verify that all these elements are different. Hence $|A + 5.A| = 4$.

(II) Let $A = \{x, y, z\}$, $x \neq y \neq z$. Then

$$A + 5 \cdot A = \{6x, x + 5y, x + 5z, y + 5x, 6y, y + 5z, z + 5x, z + 5y, 6z\}.$$

This data can be written as

$$\begin{bmatrix} 6x & x + 5y & x + 5z \\ y + 5x & 6y & y + 5z \\ z + 5x & z + 5y & 6z \end{bmatrix}. \tag{1}$$

Observe that elements of each row(column) cannot be equal to any other element of respective row (column) because $x \neq y \neq z$. By the same fact, one can see that diagonal entries of the matrix are not equal to each other. Next, we consider other possible cases.

Case 1. Assume that $6x = y+5z$. Then $x = \frac{y+5z}{6}$. Therefore, matrix (1) can be written as:

$$\begin{bmatrix} y + 5z & \frac{31y+5z}{6} & \frac{y+35z}{6} \\ \frac{11y+25z}{6} & 6y & y + 5z \\ \frac{5y+31z}{6} & z + 5y & 6z \end{bmatrix}.$$

If possible, let $y + 5z = z + 5y$, then $4z = 4y$, which is a contradiction. Thus, $|A + 5 \cdot A| \geq 8$. Suppose that $\frac{31y+5z}{6} = \frac{11y+25z}{6}$. Then $31y + 5z = 11y + 25z$, which is again a contradiction. Next, assume that $\frac{31y+5z}{6} = \frac{5y+31z}{6}$, again a contradiction. In similar manner, one can check that the different elements of the matrix is 8. Thus, $|A + 5 \cdot A| = 8$.

Case 2. Let $x + 5y = y + 5z$. Then the original matrix (1) becomes

$$\begin{bmatrix} 30z - 24y & 5z + y & 10z - 4y \\ 25z - 19y & 6y & y + 5z \\ 26z - 20y & z + 5y & 6z \end{bmatrix}.$$

By the fact that $x \neq y \neq z$, we have $|A + 5 \cdot A| = 8$. Similarly, the result holds for the remaining cases.

(III) Let $A = \{x, y, z, w\}$, where $x \neq y \neq z \neq w$. Then

$$A + 5 \cdot A = \{6x, x + 5y, x + 5z, x + 5w, y + 5x, 6y, y + 5z, y + 5w,$$
$$z + 5x, z + 5y, 6z, z + 5w, w + 5x, w + 5y, w + 5z, 6w\}.$$

This data can be written as

$$(2) \qquad \begin{bmatrix} 6x & x+5y & x+5z & x+5w \\ y+5x & 6y & y+5z & y+5w \\ z+5x & z+5y & 6z & z+5w \\ w+5x & w+5y & w+5z & 6w \end{bmatrix}.$$

Note that each row (column) elements cannot be equal to elements of that row (column) since $x \neq y \neq z \neq w$. This fact also indicates that the diagonal entries of the matrix are not equal to one another. Further, we consider the other possibilities of equality of elements of matrix (2).

Case 1. Let $x + 5y = z + 5x$. Therefore $z = 5y - 4z$. Thus, matrix (2) transforms as:

$$\begin{bmatrix} 6x & x+5y & 25y-19x & x+5w \\ y+5x & 6y & 26y-20x & y+5w \\ 5y+x & 10y-4x & 30y-24x & 5y-4x+5w \\ w+5x & w+5y & w+25y-20x & 6w \end{bmatrix}.$$

Suppose that $x+5w = 30y-24x$. Therefore, $w = \frac{30y-25x}{5}$ and the transformed matrix can be written as

$$\begin{bmatrix} 6x & x+5y & 25y-19x & 30y-24x \\ y+5x & 6y & 26y-20x & \frac{155y-125x}{5} \\ 5y+x & 10y-4x & 30y-24x & \frac{160y-145x}{5} \\ 6y & \frac{40y-25x}{5} & \frac{155y-125x}{5} & \frac{180y-150x}{5} \end{bmatrix}.$$

Consider $25y - 19x = y + 5x$, which is not possible. Observe that any other two elements of the above matrix cannot be equal. In all the cases, we can see that $12 \leq |A + 5 \cdot A| \leq 16$. Hence, the result.

Case 2. Assume that $x+5y = y+5z$. Hence, $x = 5z-4y$ and matrix (2) becomes

$$\begin{bmatrix} 30z-24y & 5z+y & 10z-4y & 5z-4y+5w \\ 25z-19y & 6y & y+5z & y+5w \\ 26z-20y & z+5y & 6z & z+5w \\ 25z-20y+w & w+5y & w+5z & 6w \end{bmatrix}.$$

Let $5z + y = z + 5w$. Then the above matrix becomes

$$\begin{bmatrix} 126z-120w & z+5w & 26z-20w & 21z-15w \\ 101z-95w & 30w-24z & z+5w & 10w-4z \\ 106z-100w & -19z+25w & 6z & z+5w \\ 105z-99w & 26w-20z & w+5z & 6w \end{bmatrix}.$$

By the fact that $z \neq w$, clearly $12 \leq |A + 5 \cdot A| \leq 16$.

(IV) Let $A = \{x, y, z, w, t\}$, where $x \neq y \neq z \neq w \neq t$. Then

$$\begin{aligned} A + 5 \cdot A &= \{x, y, z, w, t\} + \{5x, 5y, 5z, 5w, 5t\} \\ &= \{6x, x + 5y, x + 5z, x + 5w, x + 5t, y + 5x, 6y, y + 5z, y + 5w, \\ &\qquad y + 5t, z + 5x, z + 5y, 6z, z + 5w, z + 5t, w + 5x, w + 5y, \\ &\qquad w + 5z, 6w, w + 5t, t + 5x, t + 5y, t + 5z, t + 5w, 6t\}. \end{aligned}$$

Also, $A + 5 \cdot A$ can be written as

$$\begin{bmatrix} 6x & x + 5y & x + 5z & x + 5w & x + 5t \\ y + 5x & 6y & y + 5z & y + 5w & y + 5t \\ z + 5x & z + 5y & 6z & z + 5w & z + 5t \\ w + 5x & w + 5y & w + 5z & 6w & w + 5t \\ t + 5x & t + 5y & t + 5z & t + 5w & 6t \end{bmatrix}. \tag{3}$$

Since $x \neq y \neq z \neq w \neq t$, therefore the elements in each row (column) cannot be equivalent to any other element of respective row (column). This fact also shows that the diagonal entries in the matrix are not equal to each other. Next, we discuss the possibilities of equality of other elements of matrix (3).

Case 1. Let $x + 5y = z + 5x$, then $z = 5y - 4x$. Then the original matrix (3) becomes

$$\begin{bmatrix} 6x & 5y + x & 25y - 19x & x + 5w & x + 5t \\ y + 5x & 6y & 26y - 20x & y + 5w & y + 5t \\ 5y + x & 10y - 4x & 30y - 24x & 5y - 4x + 5w & 5y - 4x + 5t \\ w + 5x & w + 5y & w + 25y - 20x & 6w & w + 5t \\ t + 5x & t + 5y & t + 25y - 20x & t + 5w & 6t \end{bmatrix}.$$

Next, assume that $x + 5w = 30y - 24x$, then $y = \frac{5w+25x}{30}$. The above matrix transforms into

$$\begin{bmatrix} 6x & \frac{155x+25w}{30} & \frac{55x+125w}{30} & x + 5w & x + 5t \\ \frac{5w+175x}{30} & \frac{30w+150x}{30} & \frac{125w+50x}{30} & \frac{155w+25x}{30} & \frac{5w+25x+150t}{30} \\ \frac{155x+25w}{30} & \frac{50w+130x}{30} & \frac{150w+30x}{30} & \frac{5x+175w}{30} & \frac{25w+5x+150t}{30} \\ w + 5x & \frac{55w+125x}{30} & \frac{155w+25x}{30} & 6w & w + 5t \\ t + 5x & \frac{30t+25w+125x}{30} & \frac{30t+125w+25x}{30} & t + 5w & 6t \end{bmatrix}.$$

Again, let $\frac{50w+130x}{30} = t + 5x$, then $t = \frac{50w-20x}{30}$. Then the transformed matrix is:

$$\begin{bmatrix} 6x & \frac{155x+25w}{30} & \frac{55x+125w}{30} & x + 5w & \frac{250w-70x}{30} \\ \frac{5w+175x}{30} & w + 5x & \frac{125w+50x}{30} & \frac{155w+25x}{30} & \frac{255w-75x}{30} \\ \frac{155x+25w}{30} & \frac{50w+130x}{30} & x + 5w & \frac{5x+175w}{30} & \frac{275w-75x}{30} \\ w + 5x & \frac{55w+125x}{30} & \frac{155w+25x}{30} & 6w & \frac{55w-100x}{30} \\ \frac{20w+130x}{30} & \frac{75w+106x}{30} & \frac{175w+5x}{30} & \frac{200w-20x}{30} & \frac{10w-4x}{30} \end{bmatrix}.$$

If possible, $\frac{155w+25x}{30} = \frac{5w+175x}{30}$, then $w = t$, which is a contradiction, and we have $|A + 5 \cdot A| = 19$. On similar arguments, we can check for the other possibilities of equality of the elements in matrix (3). Observe that in all the cases, we have $19 \leq |A + 5 \cdot A| \leq 25$. □

Observe that Lemma 3.1 holds for any value of $k \in \mathbb{Z}^+$.

LEMMA 3.2. *If $|A| \geq 5$ and $|A_i| \leq 5$ for some i, then the following hold:*

(I) *If $|\hat{A}| = 4$, then $|A_i + 5 \cdot A| \geq |A| + 2|A_i| - 2$.*

(II) *If $|\hat{A}| = 3$, then $|A_i + 5 \cdot A| \geq |A| + 3|A_i| - 3$.*

(III) *If $|\hat{A}| = 2$, then $|A_i + 5 \cdot A| \geq |A| + 4|A_i| - 4$.*

Proof. (I) We divide the proof into the following cases:

Case 1. Let $|\hat{X}_i| = 1$. Then by Lemma 2.1,

$$\begin{aligned}|X_i + A| &= \sum_{j=1}^{4} |X_i + A_j| \\ &\geq |X_i| + |A_1| - 1 + |X_i| + |A_2| - 1 + |X_i| + |A_3| - 1 + |X_i| + |A_4| - 1 \\ &\geq 4|X_i| + |A| - 4.\end{aligned}$$

Case 2. Assume that $|\hat{X}_i| = 2$. Then

$$\begin{aligned}|X_i + A| &= |X_i + A_1| + |(X_i + A) \setminus (X_i + A_1)| \\ &\geq |A_i| + 2|A_1| - 2 + |A_2| + |A_3| + |A_4| \\ &\geq 2|X_i| + |A| - 2.\end{aligned}$$

Case 3. Suppose that $|\hat{X}_i| = 3$. Then by Lemma 2.1,

1. If $|X_i| = 3$, then $|X_i + A_1| \geq 3|A_1|$,
2. If $|X_i| = 4$, then $|X_i + A_1| \geq 3|A_1| + 1$,
3. If $|X_i| = 5$, then $|X_i + A_1| \geq 3|A_1| + 2$.

Also, for $|A_1| \geq |A_4|$, then $|X_i + A| \geq |X_i + A_1| + |A_2| + |A_3| \geq 2|X_i| + |A| - 2$. Hence the result.

Case 4. Let $|\hat{X}_i| = 4$. Then $4 \leq |X_i| \leq 5$. For the set $|X_i| = 4$, $|X_i + A_1| \geq 4|A_1|$. Also, for $|X_i| = 5$, then $|X_i + A_1| \geq 4|A_1| + 1$. Thus, $|X_i + A| \geq 2|X_i| + |A| - 2$.

Case 5. Finally, assume that $|\hat{X}_i| = 5$. Suppose that $x_i = \min(X_i)$ and $z = \max(A)$. Therefore

$$|X_i + A| \geq |x_i + A| + |z + X_i| - 1 + c$$

and

$$c = |(X_i + (A_i \setminus z) \setminus ((x_i + A) \cup (z + X_i))| \geq 4.$$

Hence $|X_i + A| \geq 2|X_i| + |A| - 2$.

(II) We prove the result by considering different cardinalities of $\hat{X}_i$.

Case 1. Let $|\hat{X}_i| = 1$. By Lemma 3.1, we have

$$\begin{aligned} |X_i + A| &= |X_i + A_i| + |(X_i + A) \setminus (X_i + A_i)| \\ &\geq |X_i + 4 \cdot X_i| + |(X_i + A) \setminus (X_i + A_i)| \\ &\geq |X_i + 4 \cdot X_i| + |A| - |A_i| + |A_i| - 1 \\ &\geq 3|X_i| + |A| - 3. \end{aligned}$$

Case 2. Let $|\hat{X}_i| = 2$. Then

$$\begin{aligned} |X_i + A| &= |X_i + A_i| + |(X_i + A) \setminus (X_i + A_i)| \\ &\geq |X_i + 4 \cdot X_i| + |(X_i + A) \setminus (X_i + A_i)| \\ &\geq |X_i + 4 \cdot X_i| + 2(|A| - |A_i|) \\ &\geq 3|X_i| + |A| - 3. \end{aligned}$$

Case 3. Let $|\hat{X}_i| = 3$. Then $3 \leq |X_i| \leq 5$. If the set $|X_i| = 3$, then $|X_i + A_1| \geq 3|A_1|$. Also if $|X_i| = 4$, then we have $|X_i + A_1| \geq 3|A_1| + 2$. Hence $|X_i + A| \geq 3|X_i| + |A| - 3$.

Case 4. Let $|\hat{X}_i| = 4$. By Lemma 2.2, $4 \leq |X_i| \leq 5$. If $|X_i| = 4$, then $|X_i + A_1| \geq 4|A_1|$. Again if $|X_i| = 5$, then $|X_i + A_1| \geq 4|A_1| + 1$ and hence $|X_i + A| \geq 3|X_i| + |A| - 3$.

Case 5. Suppose that $|\hat{X}_i| = 5$. By Lemma 3.2, we have the next inequality $|X_i + A| \geq 3|X_i| + |A| - 3$.

(III) The proof of this case follows the same approach as cases (I) and (II). □

THEOREM 3.3. *Let A be a non-empty finite subset of $\mathbb{Z}$ with $|A| \geq 6$ and $k = 5$. Then*

$$|A + 5 \cdot A| \geq 6|A| - 9.$$

Proof. Assume that $|\hat{A}| = 5$. Then by Lemma 2.1, $|A + 5 \cdot A| \geq 6|A| - 9$. Next, assume that $|\hat{A}| \leq 4$. We prove our result by mathematical induction on $|A|$. By Lemma 3.2, the result holds for $|A| = 6$. Suppose that the result holds for all subsets of $\mathbb{Z}$ having cardinality greater than or equal to 6 and less than the cardinality of A. We prove the final step of the result by considering the following cases.

Case 1. Let $|\hat{A}| = 4$. Then, we have the following two subcases.

Subcase 1. Suppose that $|A_4| \leq 5$. Let $|\hat{X}_i| = 5$ for all i. Consider the following subparts:

(i) Let $|A_i| = 5$, $i = 1, 2, 3$ or all i. Then by Lemma 3.2,

$$|A + 5 \cdot A| = \sum_{i=1}^{4} |X_i + A| \geq 6|A| - 6.$$

(ii) Consider $|A_i| = 5$ for $i = 2, 3$ and $|A_1| > 5$. Then by Lemmas 2.1 and 3.2,

$$\begin{aligned} |A + 5 \cdot A| &= |A_1 + 5 \cdot A| + \sum_{i \neq 1} |A_i + 5 \cdot A| \\ &\geq |6A_1| - 5 + |\Delta_{11}| + \sum_{i \neq 1} |A_i + 5 \cdot A| \\ &\geq 6|A| - 9. \end{aligned}$$

(iii) Let $|A_i| = 5$ for $i = 3$ and $|A_1|, |A_2| > 5$. Then, by using Lemmas 2.1 and 3.2,

$$\begin{aligned} |A + 5 \cdot A| &= |X_1 + A| + |X_2 + A| + \sum_{i \neq 1,2} |A_i + 5 \cdot A| \\ &\geq |X_1 + 5 \cdot X_1| + |\Delta_{11}| + |X_2 + 5 \cdot X_2| + |\Delta_{22}| + \sum_{i \neq 1,2} |A_i + 5 \cdot A| \\ &\geq 6|A_1| - 5 + 6|A_2| - 5 + \sum_{i \neq 1,2} |A| + 2|A_i| - 2 \\ &\geq 6|A| - 9. \end{aligned}$$

Similarly for $|A_i| = 5$ for $i = 2$ or $i = 2, 4$.

(iv) Let $|A_i| > 5$ for $i = 1, 2, 3$, by Lemmas 2.1, 2.3 and 3.2. Then

$$|A + 5 \cdot A| = \sum_{i=1}^{4} |X_i + A| \geq \sum_{i \neq 4} |X_i + A_i| + |\Delta_{ii}| + |X_4 + 5 \cdot A| \geq 6|A| - 9.$$

Further, assume that $|\hat{X}_i| < 5$ for all i. Then

(i) Let $|A_i| \leq 5$ for all i. By Lemmas 2.1 and 3.2,

$$|A + 5 \cdot A| \geq 4|A| + 2(|A_1| + |A_2| + |A_3| + |A_4|) - 6 \geq 6|A| - 9.$$

(ii) Consider $|A_i| \leq 5$ for $i = 2, 3$ and $|A_1| > 5$. Then by Lemmas 2.4 and 3.2,

$$\begin{aligned}|A + 5 \cdot A| &= |X_1 + A| + \sum_{i \neq 1} |X_i + 5 \cdot A| \\ &\geq 6|A_1| - 9 + |\Delta_{11}| + \sum_{i \neq 1} |X_i + 5 \cdot A| \\ &\geq 6|A| - 9.\end{aligned}$$

(iii) Let $|A_i| \leq 5$ for $i = 3$ and $|A_1|, |A_2| > 5$. We use Lemmas 2.4 and 3.2,

$$\begin{aligned}|A + 5 \cdot A| &= |X_1 + A| + |X_2 + A| + \sum_{i \neq 1,2} |X_i + 5 \cdot A| \\ &\geq 6|A_1| - 9 + |\Delta_{11}| + 6|A_2| - 9 + |\Delta_{22}| + \sum_{i \neq 1,2} |X_i + 5 \cdot A| \\ &\geq 6|A| - 9.\end{aligned}$$

(iv) Let $|A_i| > 5$ for $i = 1, 2, 3$. Then by the induction hypothesis, and Lemmas 2.4 and 3.2,

$$\begin{aligned}|A + 5 \cdot A| &= \sum_{i \neq 4} |X_i + 5 \cdot A| + |X_4 + 5 \cdot A| \\ &\geq \sum_{i \neq 4} |X_i + 5 \cdot X_i| + |\Delta_{ii}| + |X_4 + 5 \cdot A| \\ &\geq \sum_{i \neq 4} 6|A_i| - 9 + |\Delta_{ii}| + 2|X_4| + |A| - 2 \\ &\geq 6|A| - 9.\end{aligned}$$

Suppose that $|\hat{X}_i| < 5$ for $i = 1, 2$.

(i) Let $|A_i| > 5$ for $i = 1, 2, 3$. By Lemma 2.4 and the induction hypothesis,

$$|A + 5 \cdot A| = \sum_{i \neq 4} |X_i + A| + |X_4 + 5 \cdot A| \geq 6|A| + 2|A_2| + 3|A_1| + |A_3| - 29 \geq 6|A| - 9.$$

(ii) Let $|A_i|>5$ for $i=1,2$ and $|A_3|\le 5$. Then

$$|A+5\cdot A|=\sum_{i\neq 3,4}|X_i+A|+\sum_{i\neq 1,2}|X_i+5\cdot A|\ge 6|A|-9.$$

(iii) Let $|A_i|>5$ for $i=1$ and $|A_2|,|A_3|\le 5$. Then

$$|A+5\cdot A|=|X_1+A|+\sum_{i\neq 1}|X_i+5\cdot A|\ge 6|A|-9.$$

Similarly for $|\{1\le i\le 4:|\hat{X}_i|<5\}|=2$.

Now, assume that $|\hat{X}_i|<5$ for $i=1$.

(i) Suppose $|A_i|>5$ for $i=1,2,3$. Then by the induction hypothesis, and Lemmas 2.4 and 3.2,

$$\begin{aligned}|A+5\cdot A|&=\sum_{i\neq 4}|X_i+A|+|X_4+A|\\&\ge 6(|A_1|+|A_2|+|A_3|)-27+2|A_4|+|A|-2+|\Delta_{11}|+|\Delta_{22}|+|\Delta_{33}|\\&\ge 6|A|-9.\end{aligned}$$

(ii) Let $|A_i|>5$ for $i=1,2$. Then

$$\begin{aligned}|A+5\cdot A|&=|X_1+A|+|X_2+A|+\sum_{i\neq 1,2}|X_i+A|\\&\ge |X_1+5\cdot X_1|+|\Delta_{11}|+|X_2+5\cdot X_2|+|\Delta_{22}|+\sum_{i\neq 1,2}|A|+3|A_i|-3\\&\ge 6|A|-9.\end{aligned}$$

(iii) Let $|A_i|>5$ for $i=1$. Then

$$\begin{aligned}|A+5\cdot A|&=|X_1+A|+\sum_{i\neq 1}|X_i+A|\\&\ge |X_1+5\cdot X_1|+|\Delta_{11}|+\sum_{i\neq 1}|A|+3|A_i|-3\\&\ge 6|A|-9.\end{aligned}$$

Similarly for $|\{1\le i\le 4:|\hat{X}_i|<5\}|=1$.

Subcase 2. Suppose that $|A_4|>5$. Consider that $|\hat{X}_i|=5$ for all i. Then by Lemma 2.1,

$$|A+5\cdot A|=\sum_{i=1}^{4}|X_i+A|\ge\sum_{i=1}^{4}|X_i+5\cdot X_i|+|\Delta_{ii}|\ge 6|A|-9.$$

Assume that $|\hat{X}_i| < 5$ for all i. If $|A_1| = |A_2|$, then the result holds easily by the direct analysis. If $|A_1| > |A_2|$, by induction and Lemma 2.4,

$$\begin{aligned}
|A + 5 \cdot A| &= \sum_{i=1}^{4} |X_i + A| \\
&\geq \sum_{i=1}^{4} |X_i + 5 \cdot X_i| + |\Delta_{ii}| \\
&\geq 6|A_1| - 9 + 6|A_2| - 9 + 6|A_3| - 9 + 6|A_4| - 9 + |A_2| + 3|A_1| \\
&\geq 6|A| - 9.
\end{aligned}$$

Suppose that $|\hat{X}_i| < 5$ for $i = 1, 2, 3$. The induction hypothesis and Lemma 2.4 imply

$$\begin{aligned}
|A + 5 \cdot A| &= \sum_{i \neq 4} |X_i + A| + |X_4 + A| \\
&\geq \sum_{i \neq 4} |X_i + 5 \cdot X_i| + |\Delta_{ii}| + |X_4 + 5 \cdot X_1| \\
&\geq 6|A| + |A_2| + 2|A_1| + 5(|A_1| - |A_4|) - 32 \\
&\geq 6|A| - 9.
\end{aligned}$$

Similarly for $|\{1 \leq i \leq 4 : |\hat{X}_i| < 5\}| = 3$.

Next, suppose that $|\hat{X}_i| < 5$ for $i = 1, 2$. Then

$$\begin{aligned}
|A + 5 \cdot A| &= \sum_{i \neq 4} |X_i + A| + |X_4 + A| \\
&\geq \sum_{i \neq 4} |X_i + 5 \cdot X_i| + |\Delta_{ii}| + |X_4 + 5 \cdot X_1| \\
&\geq 6|A| - 9.
\end{aligned}$$

Similarly for $|\{1 \leq i \leq 4 : |\hat{X}_i| < 5\}| = 2$.

Further, assume that $|\hat{X}_i| < 5$ for $i = 1$. By Lemmas 2.1 and 2.4,

$$\begin{aligned}
|A + 5 \cdot A| &= \sum_{i=1}^{4} |X_i + A| \\
&\geq \sum_{i=1,2} |X_i + 5 \cdot X_i| + |\Delta_{ii}| + |X_3 + 5 \cdot A| + |X_4 + 5 \cdot A| \\
&\geq 6|A| + 5(|A_1| - |A_3|) + 5(|A_1| - |A_4|) - 19. \\
&\geq 6|A| - 9.
\end{aligned}$$

Similarly for $|\{1 \leq i \leq 4 : |\hat{X}_i| < 5\}| = 2$.

Case 2. Consider $|\hat{A}| = 3$. We discuss the following two subcases depending upon $|A_3|$.

Subcase 1. Let $|A_3| \leq 5$. Observe that if $|A_i| \leq 5$ for all i, then the result holds trivially by Lemma 3.2.

Firstly, assume that $|X_i| = 5$ for all i. Then we have the following subparts:

(i) Let $|A_i| \leq 5$ for $i = 3, 4$. Then by Lemmas 2.1 and 3.2 , we have

$$\begin{aligned}|A+5\cdot A| &= |X_1+5\cdot A| + \sum_{i\neq 1}|X_i+A|\\ &\geq |X_1+5\cdot X_1| + |\Delta_{11}| + \sum_{i\neq 1}|A|+3|A_i|-3 \geq 6|A|-9.\end{aligned}$$

For the rest of the possibilities, we can see the result with the help of Lemmas 2.1, 2.3 and 3.2.

Assume that $|\hat{X}_i| < 5$ for all i or $i = 1, 2$.

(ii) Let $|A_i| > 5$ for $i = 1, 2$. Then we use the induction hypothesis and Lemma 3.2 to obtain

$$\begin{aligned}|A+5\cdot A| &= \sum_{i=1}^{3}|X_i+A|\\ &\geq \sum_{i\neq 3}|X_i+5\cdot X_i| + |\Delta_{ii}| + |X_3+5\cdot A|\\ &\geq 6|A_1|-5+6|A_2|-5+|A_2|+|A_1|+|A|+3|A_3|-3\\ &\geq 6|A|-9.\end{aligned}$$

(iii) Let $|A_i| > 5$ for $i = 1$. Then, in light of Lemma 3.2 and the induction hypothesis, we have

$$\begin{aligned}|A+5\cdot A| &\geq |X_1+5\cdot X_1| + |\Delta_{11}| + \sum_{i\neq 1}|X_3+5\cdot A|\\ &\geq 6|A_1|-5+|A_2| + \sum_{i\neq 1}|A|+3|A_i|-3\\ &\geq 6|A|-9.\end{aligned}$$

Similarly, the result holds for $i = 2$.

Consider that $|\hat{X}_i| < 5$ for $i = 1$ or $i = 2$.

(i) For $|A_i| > 5$ for $i = 1, 2$. Then by Lemmas 2.4, 3.2 and the induction hypothesis,

$$|A+5\cdot A| \geq \sum_{i\neq 3}|X_i+5\cdot X_i| + |\Delta_{ii}| + |X_3+5\cdot A|$$

$$\geq 6|A_1| - 9 + 6|A_2| - 5 + |A_2| + |A| + 3|A_3| - 3$$
$$\geq 6|A| - 9.$$

(ii) For $|A_i| > 5$ for $i = 1$. Then

$$\begin{aligned}
|A + 5 \cdot A| &= |X_1 + A| + \sum_{i \neq 1} |X_i + A| \\
&\geq |X_1 + 5 \cdot X_1| + |\Delta_{11}| + \sum_{i \neq 1} |A| + 3|A_3| - 3 \\
&\geq 6|A_1| - 9 + |A_2| + |A| + 3|A_3| - 3 \\
&\geq 6|A| - 27 + 18 \\
&\geq 6|A| - 9.
\end{aligned}$$

Similarly for $i = 2$.

Subcase 2. Assume that $|A_3| > 5$. Now, we have the following subparts:

(i) If $|\hat{X}_i| = 5$ for all i, then the result holds trivially by Lemmas 2.1 and 2.3.

(ii) Assume that $|\hat{X}_i| < 5$ for all i. Then by the induction hypothesis and Lemma 2.4,

$$\begin{aligned}
|A + 5 \cdot A| &\geq \sum_{i=1}^{3} |X_i + A_i| + |\Delta_{ii}| \\
&\geq \sum_{i=1}^{3} 6|A_i| - 9 + |\Delta_{ii}| \\
&\geq 6|A| - 9.
\end{aligned}$$

(iii) Let $|\hat{X}_i| = 5$ for $i = 1, 2$. Then

$$|A + 5 \cdot A| \geq \sum_{i=1}^{3} |X_i + 5 \cdot X_i| + |\Delta_{ii}| \geq 6|A| - 9.$$

(iv) Let $|\hat{X}_i| = 5$ for $i = 1$, then $|\Delta_{11}| \geq 2$ in case of $|A_1| = |A_2| = |A_3|$ by Lemma 2.1. Then

$$\begin{aligned}
|A + 5 \cdot A| &\geq \sum_{i=1}^{3} |X_i + 5 \cdot X_i| + |\Delta_{ii}| \\
&\geq 6(|A_1| + |A_2| + |A_3|) - 23 + 2|A_1| \\
&\geq 6|A| - 9.
\end{aligned}$$

Similarly for $|\hat{X}_i| = 5$ for $i = 2$ or $i = 3$.

Case 3. Finally, let $|\hat{A}| = 2$.

The proof of this case follows the same pattern as for Cases 1 and 2. This completes the proof of theorem. □

Acknowledgments. The first author gratefully acknowledges the financial support received from Akal University, Talwandi Sabo, for carrying out this research.

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Received 4 February 2025

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