ON THE ASYMPTOTICS OF HIGHER POWER MOMENTS OF HECKE EIGENVALUES OVER TWO SPARSE SEQUENCES OF POSITIVE INTEGERS

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Let f and g be two distinct normalized primitive holomorphic cusp forms of even integral weights κ_1 and κ_2 for the full modular group $\Gamma = SL(2,\mathbb{Z})$, and let $\lambda_f(n)$ and $\lambda_g(n)$ denote the n-th normalized Fourier coefficients of f and g, respectively. In this paper, we establish the asymptotic formulae for the summatory functions associated to $\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)$, with $\ell_1, \ell_2 \geq 2$ being any fixed integers, over two sparse sequences of positive integers, which generalizes the existing results in the literature in this direction.

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1. INTRODUCTION

The Fourier coefficients of automorphic forms are important and interesting objects in modern analytic number theory, which has been the main concern in the investigations of plenty of researches (see, e.g., [8,12,13,15,18,19,36,44]). In this paper, we consider the holomorphic cusp form for the full modular group $\Gamma = SL(2,\mathbb{Z})$, which are simultaneous eigenfunctions of all the Hecke operators T_n. Let H_κ^* denote the set of normalized primitive holomorphic Hecke cusp forms of even integral weight κ for $\Gamma = SL(2,\mathbb{Z})$. The $f \in H_\kappa^*$ admits Fourier expansion at the cusp ∞

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz), \qquad \Im(z) > 0,$$

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where $e(z) = e^{2\pi iz}$ and $\lambda_f(n)$ denote the n-th normalized Fourier coefficient (Hecke eigenvalue) of f. It is well known from the theory of Hecke operator that $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

where $m, n \geq 1$ are integers. In 1974, Deligne [8] proved the celebrated Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{2}$$

where $d(n)$ is the classical divisor function. By (2), Deligne's result is equivalent to the fact that there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$ satisfying

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{3}$$

More generally, for all positive integers $\ell \geq 1$ one has

$$\lambda_f(p^\ell) = \alpha_f(p)^\ell + \alpha_f(p)^{\ell-1}\beta_f(p) + \cdots + \alpha_f(p)\beta_f(p)^{\ell-1} + \beta_f(p)^\ell.$$

For more details, the interested reader can refer to Lau and Wu [33, Section 1].

It is of significant importance to consider the average behavior of Hecke eigenvalues of cusp forms in various aspects (see, e.g., [8, 13, 14, 16, 17, 19, 36]). In 2013, Zhai [60] considered the average behavior of the power sum

$$U_j(f;x) := \sum_{a^2+b^2 \leq x} \lambda_f(a^2+b^2)^j$$

for $x \geq 1, 2 \leq j \leq 8$ and $a, b, j \in \mathbb{Z}$. Indeed, he successfully proved that

$$U_j(f;x) = x\tilde{P}_j(\log x) + O_{f,\varepsilon}(x^{\alpha_j+\varepsilon}),$$

where $\tilde{P}_j$ with $j = 2, \ldots, 8$ are polynomials with degrees $\deg \tilde{P}_2 = 0$, $\deg \tilde{P}_4 = 1$, $\deg \tilde{P}_6 = 4$, $\deg \tilde{P}_8 = 13$, and $\deg \tilde{P}_j \equiv 0$ for $j = 3, 5, 7$. The exponents α_j are given by

$$\alpha_2 = \frac{8}{11}, \qquad \alpha_3 = \frac{17}{20}, \qquad \alpha_4 = \frac{43}{46}, \qquad \alpha_5 = \frac{83}{86},$$
$$\alpha_6 = \frac{184}{187}, \qquad \alpha_7 = \frac{355}{357}, \qquad \alpha_8 = \frac{752}{755}.$$

Afterwards, the results of Zhai were refined and generalized by Xu [59] and Liu [38], by using the recent breakthrough of Newton and Thorne [41, 42], along with some nice analytic properties of the associated L-functions.

Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the n-th normalized coefficient of the Dirichlet expansion of the j-th symmetric power L-function $L(\mathrm{sym}^j f, s)$. Fomenko [9] proved that

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^2 f}(n) \ll_f x^{\frac{1}{2}}(\log x)^2.$$

Later, this sum was studied by many authors (see, e.g., [29, 35, 52]). For more delightful results, the readers can refer to [36, 37, 53, 58] and the references therein.

In [54], Sharma and Sankaranarayanan considered the asymptotic behavior of the sum

$$\widetilde{S}_{f,r}(x) := \sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^r_{\mathrm{sym}^2 f}(n)$$

for $r = 2$ for $x \geq x_0$, where x_0 is sufficiently large. In fact, the authors established the formula

$$\widetilde{S}_{f,2}(x) = c_f x^2 + O_{f,\varepsilon}(x^{\frac{9}{5}+\varepsilon})$$

for any $\varepsilon > 0$, where $c_f > 0$ is some suitable constant depending on f. Very recently, Sharma and Sankaranarayanan [56] established the asymptotic formulae for $\widetilde{S}_{f,r}(x)$ with $r = 3, 4$. In fact, they proved that

$$\widetilde{S}_{f,3}(x) = c_1 x^2 + O_{f,\varepsilon}(x^{\frac{27}{14}+\varepsilon}),$$

and

$$\widetilde{S}_{f,4}(x) = c_2 x^2 \log x + O_{f,\varepsilon}(x^{\frac{160}{81}+\varepsilon}),$$

where c_1, c_2 are suitable effective constants depending on f. Afterwards, the author and his collaborators gave some refinements and generalizations concerning the above results of Sharma and Sankaranarayanan, the interested readers can refer to [14, 15, 21]. In particular, for $j \geq 2$ any fixed integer, in [15] the author established the asymptotic formula

$$\sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda^2_{\mathrm{sym}^j f}(n) = c_f(j)x^2 + O_{f,\varepsilon}(x^{2-\frac{60}{30(j+1)^2-13}+\varepsilon}), \tag{4}$$

where $c_f(j)$ is some suitable constant which can be evaluated explicitly. Very recently, Liu and Yang [39] gave further improvement with respect to the result from (4), reducing the exponent in the error term with $\delta_j := 2 - \frac{120}{60(j+1)^2-61}$.

Motivated by the above impressive results, in this paper the authors are interested in the higher power moments of Fourier coefficients associated to two distinct cusp forms. Let $f \in H^*_{\kappa_1}$ and $g \in H^*_{\kappa_2}$ be two distinct Hecke eigenforms, and let $\lambda_f(n)$ and $\lambda_g(n)$ denote the n-th normalized Fourier coefficients of f and g, respectively. Let

$$S_{f,g}(x;\ell_1,\ell_2) := \sum_{\substack{n=a^2+b^2+c^2+d^2\leq x\\ (a,b,c,d)\in\mathbb{Z}^4}} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n), \tag{5}$$

where $\ell_1, \ell_2 \geq 2$ are any given integers. The first purpose in this paper is to establish the following theorem.

THEOREM 1.1. *Let $S_{f,g}(x;\ell_1,\ell_2)$ be defined as in* (5)*, then*

$$S_{f,g}(x;\ell_1,\ell_2)=x^2P_{A_{\ell_1}A_{\ell_2}-1}(\log x)+O_{f,g,\varepsilon}(x^{2-\theta_{\ell_1,\ell_2}+\varepsilon}),$$

where $P_{A_{\ell_1}A_{\ell_2}-1}(t)$ is a polynomial in t of degree $A_{\ell_1}A_{\ell_2}-1$, and

$$\theta_{\ell_1,\ell_2}=\frac{2}{7\widetilde{\theta}_{\ell_1,\ell_2}}$$

with $\widetilde{\theta}_{\ell_1,\ell_2}$ defined as in (45)*. The highest-degree term of $P_{A_{\ell_1}A_{\ell_2}-1}(t)$ can be explicitly evaluated as*

$$\begin{aligned}
&P_{A_{\ell_1}A_{\ell_2}-1}(t)\\
&\quad=\frac{4(-1/2)^{A_{\ell_1}A_{\ell_2}}}{(A_{\ell_1}A_{\ell_2}-1)!}\zeta(2)^{A_{\ell_1}A_{\ell_2}}\big(L(\mathrm{sym}^{2\ell_1}f,2)L(\mathrm{sym}^{2\ell_1}f\otimes\widetilde{\chi}_0,1)\big)^{A_{\ell_2}}\\
&\qquad\cdot\big(L(\mathrm{sym}^{2\ell_2}g,2)L(\mathrm{sym}^{2\ell_2}g\otimes\widetilde{\chi}_0,1)\big)^{A_{\ell_1}}\\
&\qquad\cdot\prod_{1\le i\le\ell_1-1}\big(L(\mathrm{sym}^{2i}f,2)L(\mathrm{sym}^{2i}f\otimes\widetilde{\chi}_0,1)\big)^{A_{\ell_2}C_{\ell_1}(i)}\\
&\qquad\cdot\prod_{1\le j\le\ell_2-1}\big(L(\mathrm{sym}^{2j}g,2)L(\mathrm{sym}^{2j}g\otimes\widetilde{\chi}_0,1)\big)^{A_{\ell_1}C_{\ell_2}(j)}\\
&\qquad\cdot\prod_{1\le i\le\ell_1-1}\prod_{1\le j\le\ell_2-1}\big(L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g,2)\\
&\qquad\cdot L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g\otimes\widetilde{\chi}_0,1)\big)^{C_{\ell_1}(i)C_{\ell_2}(j)}\\
&\qquad\cdot\prod_{1\le j\le\ell_2-1}\big(L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g,2)L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g\otimes\widetilde{\chi}_0,1)\big)^{C_{\ell_2}(j)}\\
&\qquad\cdot\prod_{1\le i\le\ell_1-1}\big(L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g,2)L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g\otimes\widetilde{\chi}_0,1)\big)^{C_{\ell_1}(i)}\\
&\qquad\cdot\big(L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g,2)L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g\otimes\widetilde{\chi}_0,1)\big)H_{\ell_1,\ell_2}(2)\\
&\qquad\cdot t^{A_{\ell_1}A_{\ell_2}-1}+\cdots+C_{f,g},
\end{aligned}$$

here $\widetilde{\chi}_0$ is a character modulo 4 given by (17)*, and $H_{\ell_1,\ell_2}(2)\neq0$, and $C_{f,g}$ is a certain suitable constant depending on f,g and various associated L-functions.*

It is also interesting to investigate the cusp form coefficients over the sparse sequence of positive integers supported at sums of six squares. In [55], Sharma and Sankaranarayanan investigated another type of summatory function related to the coefficients of the symmetric power L-function

$$S_{f,j}(x)=\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\le x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}}\lambda^2_{\mathrm{sym}^jf}(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2),$$

with $j = 2$. In fact, they proved the asymptotic formula

$$S_{f,2}(x) = \tilde{c}_{f,2}x^3 + O_{f,\varepsilon}(x^{\frac{14}{5}+\varepsilon}),$$

where $\tilde{c}_{f,2}$ is an effective constant. Recently, Sharma and Sankaranarayanan [57] considered the asymptotic formulae for $S_{f,j}(x)$ for all $j \geq 2$, by using the celebrated work of Newton and Thorne [41, 42], along with some individual and averaged subconvexity bounds of associated L-functions. More precisely, for $j \geq 2$, they established that

$$S_{f,j}(x) = \tilde{c}_{f,j}x^3 + O_{f,\varepsilon}(x^{3-\frac{6}{3(j+1)^2+1}+\varepsilon}), \tag{6}$$

where $\tilde{c}_{f,j}$ is some effective constant depending on f and associated L-functions. In the same paper, Liu and Yang [39] also gave further refinement regarding result (6), with the exponent in the error term replaced by

$$\tilde{\delta}_j := 3 - \frac{210}{105(j+1)^2 - 103}.$$

More recently, the author [?] gave further generalizations of the above results.

As before, let $f \in H^*_{\kappa_1}$ and $g \in H^*_{\kappa_2}$ be two distinct Hecke eigenforms, with $\lambda_f(n)$ and $\lambda_g(n)$ the corresponding n-th normalized Fourier coefficients of f and g, respectively. Let

$$\widetilde{S}_{f,g}(x;\ell_1,\ell_2) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leq x\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n), \tag{7}$$

where $\ell_1, \ell_2 \geq 2$ are any given integers.

Theorem 1.2. *Let $\widetilde{S}_{f,g}(x;\ell_1,\ell_2)$ be defined as in (7), then*

$$\widetilde{S}_{f,g}(x;\ell_1,\ell_2) = x^3 P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x) + O_{f,g,\varepsilon}(x^{3-\vartheta_{\ell_1,\ell_2}+\varepsilon}),$$

where $P^*_{A_{\ell_1}A_{\ell_2}-1}(t)$ is a polynomial in t of degree $A_{\ell_1}A_{\ell_2} - 1$, and

$$\theta_{\ell_1,\ell_2} = \frac{2}{7\widetilde{\theta}_{\ell_1,\ell_2}}$$

with $\widetilde{\theta}_{\ell_1,\ell_2}$ defined as in (45). The polynomial $P^*_{A_{\ell_1}A_{\ell_2}-1}(t)$ can be explicitly evaluated as

$$\begin{aligned}
&P^*_{A_{\ell_1}A_{\ell_2}-1}(t)\\
&\quad= \frac{16}{3\cdot(A_{\ell_1}A_{\ell_2}-1)!}L(3,\chi)^{A_{\ell_1}A_{\ell_2}}\big(L(\mathrm{sym}^{2\ell_1}f,1)L(\mathrm{sym}^{2\ell_1}f\otimes\chi,3)\big)^{A_{\ell_2}}\\
&\qquad\cdot\big(L(\mathrm{sym}^{2\ell_2}g,1)L(\mathrm{sym}^{2\ell_2}g\otimes\chi,3)\big)^{A_{\ell_1}}
\end{aligned}$$

$$\cdot \prod_{1\le i\le \ell_1-1} \big(L(\mathrm{sym}^{2i} f, 1) L(\mathrm{sym}^{2i} f\otimes\chi, 3)\big)^{A_{\ell_2} C_{\ell_1}(i)}$$
$$\cdot \prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2j} g, 1) L(\mathrm{sym}^{2j} g\otimes\chi, 3)\big)^{A_{\ell_1} C_{\ell_2}(j)}$$
$$\cdot \prod_{1\le i\le \ell_1-1}\prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2j} g, 1)$$
$$\cdot L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2j} g\otimes\chi, 3)\big)^{C_{\ell_1}(i)C_{\ell_2}(j)}$$
$$\cdot \prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2j} g, 1) L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2j} g\otimes\chi, 3)\big)^{C_{\ell_2}(j)}$$
$$\cdot \prod_{1\le i\le \ell_1-1} \big(L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2\ell_2} g, 1) L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, 3)\big)^{C_{\ell_1}(i)}$$
$$\cdot \big(L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g, 1) L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, 3)\big) H^{\dagger}_{\ell_1,\ell_2}(3)$$
$$\cdot t^{A_{\ell_1}A_{\ell_2}-1} + \cdots + \widetilde{C}_{f,g},$$

where χ is the non-principal Dirichlet character modulo 4, and $H^{\dagger}_{\ell_1,\ell_2}(3)\neq 0$, and $\widetilde{C}_{f,g}$ is some suitable constant depending on f, g and various associated L-functions.

Remark 1.3. For simplicity of exposition, in this paper, we are dealing with the setting for the normalized Fourier coefficients of primitive holomorphic cusp forms of level 1, but our method can be extended to the general case of cuspidal Hecke newform for the Hecke group $\Gamma_0(N)$ with a bit extra effort.

The investigations of the average behavior of higher power moments of cusp forms coefficients is an intriguing and significant topic in modern number theory, which occupies the central status in automorphic forms. The research concerning the asymptotic behavior of cusp forms over certain sparse sequences may lead to the fascinating and hidden structure of automorphic forms, which may unveil some unexpected properties. In Theorems 1.1 and 1.2, we established the asymptotic estimates for the Fourier coefficients associated to two distinct cuspidal Hecke eigenforms on certain quadratic forms of arithmetic interests, which further generalizes the previous results to broad framework. Furthermore, the author also provides an explicit expression for the main terms of Theorems 1.1 and 1.2, which gives a clear and accessible description of the asymptotic relations.

Let

$$r_k(n) := \#\{(a_1, a_2, \ldots, a_k)\in\mathbb{Z}^k : n = a_1^2 + a_2^2 + \ldots + a_k^2\}, \tag{8}$$

allowing zeros, distinguishing signs, and orders. The proofs of Theorems 1.1

and 1.2 are mainly built upon the truncated Perron's formula, with the applications of the Cauchy residue theorem, along with the individual and average convexity/subconvexity bounds for the associated L-functions. The intrinsic structure of $r_k(n)$ with $k = 4, 6$ are also closely related to the demonstration of the proofs for Theorems 1.1 and 1.2. The milestones of Newton and Thorne [41, 42] also play a crucial role in the proofs.

The organization of this paper is as follows. In Section 2, we review the background of automorphic forms and relevant L-functions, and also give some preliminary lemmas. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

Notations. Throughout the paper, we always assume that $f \in H^*_{\kappa_1}$ and $g \in H^*_{\kappa_2}$ are two Hecke eigenforms.We also assume that $x > 0$ is sufficiently large. The letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each fixed $\varepsilon > 0$, and any implied constant in such a statement is allowed to depend on ε. We use $\ll$ and O to denote the Vinogradov's and Landau's well-known notations, respectively, and the implied constants depend at most on f, g and ε. The symbol p always denotes a prime number.

2. PRELIMINARIES

In this section, we collect some facts concerning automorphic forms and the associated L-functions, along with some useful lemmas, which play a prominent role in the proof of the main results in this paper.

2.1. Automorphic forms and L-functions

We recall some fundamental facts concerning cuspidal forms; the interested reader is invited to consult the monograph of Iwaniec and Kowalski [25] for more systematic descriptions. Let $k \geq 2$ be an even integer and $N > 0$ be an integer. Let χ be a primitive character to modulus q such that $N|q$, satisfying $\chi(-1) = (-1)^k$. Denote by $\mathcal{S}_k(N, \chi)$ the vector space of holomorphic cuspidal forms on $\Gamma_0(N)$ with nebentypus χ and weight k. For any $f \in \mathcal{S}_k(N, \chi)$, it admits the Fourier expansion at the cusp ∞

$$f(z) = \sum_{n\geq 1} \psi_f(n) n^{\frac{k-1}{2}} e(nz), \quad z \in \mathbb{H},$$

where $\mathbb{H}$ denotes the upper half-plane. The space $\mathcal{S}_k(N, \chi)$ is a finite dimensional Hilbert space which can be equipped with the Petersson inner products

$$\langle f_1, f_2 \rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)} y^{k-2} dxdy,$$

for any $f_1, f_2 \in \mathcal{S}_k(N, \chi)$. Recall that the Hecke operators T_n with $(n, N) = 1$, which satisfy the multiplicative relation

$$T_n T_m = \sum_{d|(n,m)} \chi(d) T_{\frac{nm}{d^2}}. \tag{9}$$

Furthermore, for any $f_1, f_2 \in \mathcal{S}_k(N, \chi)$, one has

$$\langle T_n f_1, f_2 \rangle = \overline{\chi(n)} \langle f_1, T_n f_2 \rangle.$$

One might find an orthogonal basis $\mathcal{B}_k(N, \chi)$ of $\mathcal{S}_k(N, \chi)$ consisting of common eigenfunctions of all the Hecke operators T_n with $(n, N) = 1$. For each cusp form $f \in \mathcal{B}_k(N, \chi)$, denote by $\lambda_f(n)$ the n-th Hecke eigenvalues which satisfies

$$T_n f(z) = \lambda_f(n) f(z)$$

for all $(n, N) = 1$. From (9), it can be shown that

$$\psi_f(m) \lambda_f(n) = \sum_{d|(n,m)} \chi(d) \psi_f\left(\frac{mn}{d^2}\right)$$

for any $m, n \geq 1$ with $(n, N) = 1$. In particular, $\psi_f(1)\lambda_f(n) = \psi_f(n)$ for $(n, N) = 1$. Therefore,

$$\overline{\lambda_f(n)\chi(n)}\lambda_f(n), \qquad \lambda_f(m)\lambda_f(n) = \sum_{d|(n,m)} \chi(d) \lambda_f\left(\frac{mn}{d^2}\right),$$

provided that $(mn, N) = 1$.

Note that the Hecke eigenbasis $\mathcal{B}_k(N, \chi)$ also contains a subset of newforms $\mathcal{B}_k^*(N, \chi)$, forms which are simultaneous eigenfunctions of all Hecke operators T_n for any $n \geq 1$, and normalized to have first Fourier coefficient $\psi_f(1) = 1$. The elements of $\mathcal{B}_k^*(N, \chi)$ are commonly called primitive forms in the sense of Atkin–Lehner [1], namely, each one is orthogonal to all oldforms, and is an eigenfunction of all the Hecke and Atkin–Lehner operators. In this paper, for the sake of brevity, we only consider the Hecke newforms with trivial nebentypus and of level $N = 1$.

Let $f \in H_k^*$ be a Hecke eigenform. The j-th symmetric power L-function associated to f is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s}\right)^{-1}, \quad \Re(s) > 1, \tag{10}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are the local parameters defined as in (3). We may expand it into a Dirichlet series

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \tag{11}$$
$$= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}} + \cdots\right)$$

for $\Re(s) > 1$. Obviously, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. In particular, for $j = 1$, we have $L(\mathrm{sym}^1 f, s) = L(f, s)$.

Let χ be a Dirichlet character modulo q. Similarly, for $j \geq 1$, the j-th twisted symmetric L-function is defined as

$$L(\mathrm{sym}^j f \otimes \chi, s) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m} \beta_f(p)^m \chi(p) p^{-s}\right)^{-1}$$
$$= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}, \qquad \Re(s) > 1.$$

Let $f \in H_{k_1}^*$ and $g \in H_{k_2}^*$ be two distinct Hecke eigenforms. The Rankin–Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$$
$$= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \alpha_f(p)^{i-2m} \alpha_g(p)^{j-2m'} p^{-s}\right)^{-1}, \ \Re(s) > 1. \tag{12}$$

For $\Re(s) > 1$, we may expand it into a Dirichlet series

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$$
$$= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} = \prod_p \left(1 + \sum_{r \geq 1} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^r)}{p^{rs}}\right), \tag{13}$$

where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is a real and multiplicative function of n. In particular, for $i = j = 1$, one has the next equalities: $L(\mathrm{sym}^1 f \times \mathrm{sym}^1 g, s) = L(f \times g, s)$ and $L(\mathrm{sym}^1 f \times \mathrm{sym}^j g, s) = L(f \times \mathrm{sym}^j g, s)$.

In a similar manner, for χ being a Dirichlet character with modulus q, we can also define the twisted Rankin–Selberg L-function

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \chi, s)$$
$$= \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \alpha_f(p)^{i-m} \beta_f(p)^m \alpha_g(p)^{j-m'} \beta_g(p)^{m'} \chi(p) p^{-s}\right)^{-1}$$
$$= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)\chi(n)}{n^s}, \qquad \Re(s) > 1.$$

It is standard to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m, \tag{14}$$

which can also be written as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j\big(\lambda_f(p)/2\big), \tag{15}$$

where $U_j(x)$ is the j-th Chebyshev polynomial of the second kind. It is also easily seen that

$$\begin{aligned} \lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(p) &= \sum_{m=0}^{i}\sum_{m'=0}^{j} \alpha_f(p)^{i-2m}\alpha_g(p)^{j-2m'} \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p). \end{aligned} \tag{16}$$

From (3), (10)–(13), it is not hard to find that

$$|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n), \quad |\lambda_{\mathrm{sym}^i f\times \mathrm{sym}^j g}(n)| \le d_{(i+1)(j+1)}(n)$$

for any $i, j \ge 1$, where $d_\nu(n)$ denotes the ν-dimensional divisor function, which is defined as the number of ordered representations $n = n_1 \dots n_\nu$ with integers $n_1, \dots, n_\nu \ge 1$.

As it is well known, to a primitive form f is associated an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_\mathbb{Q})$ and hence an automorphic L-function $L(\pi_f, s)$ which coincides with $L(f, s)$. For $1 \le j \le 8$, the special Langlands functoriality conjecture which states that $\mathrm{sym}^j f$ is automorphic cuspidal has been established in a series of important works of Gelbart and Jacquet [10], Kim [30], Kim and Shahidi [31, 32], Shahidi [50], Clozel and Thorne [3–5]. Very recently, Newton and Thorne [41, 42] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_\mathbb{Q})$ for all $j \ge 1$ (with f being a holomorphic cusp form). Furthermore, by combining the works of Newton–Thorne and Cogdell–Michel [6], together with the Rankin–Selberg theory associated with two cuspidal automorphic representations developed by Jacquet, Piatetski-Shapiro and Shalika [26], Jacquet and Shalika [27, 28], Shahidi [47–49,51], and the reformulation of Rudnick and Sarnak [46], we know that the L-functions $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ have an analytic continuation as an entire function (except for the case $\mathrm{sym}^j f \cong \mathrm{sym}^j g$ with simple poles at $s = 0, 1$) in the whole complex plane and satisfies a certain functional equation of Riemann zeta-type. For a more comprehensive investigation, the interested readers can refer to [25, Chapter 5].

2.2. Preliminary lemmas

Now, we need to interpret the two summatory functions $S_{f,g}(x;\ell_1,\ell_2)$ and $U_{f,g}(x;\ell_1,\ell_2)$ in a more feasible manner. We tackle the sum $S_{f,g}(x;\ell_1,\ell_2)$ in the first place. Now, we specify (8) with $k=4$. It is well known that $r_4(n)=8r(n)$ (see, e.g., [15, Section 2]), where

$$r(n)=\sum_{d|n}\widetilde{\chi}_0(d)d$$

is multiplicative, and $\widetilde{\chi}_0$ is a character modulo 4 given by

$$\widetilde{\chi}_0(p^\nu)=\begin{cases}\chi_0(p^\nu), & \text{if } p>2,\\ 3, & \text{if } p=2,\end{cases} \tag{17}$$

and χ_0 is a principal character modulo 4. In particular, for any prime p, we have

$$r(p)=\sum_{d|p}\widetilde{\chi}_0(d)d=1+p\widetilde{\chi}_0(p).$$

It is well known that $r(n)\ll n^{1+\varepsilon}$ for any $\varepsilon>0$ (cf. [11, (1.1)]).

Hence, for $\ell_1,\ell_2\geq 2$ being any fixed integers, we can rephrase the summation $S_{f,g}(x;\ell_1,\ell_2)$ as

$$\begin{aligned}S_{f,g}(x;\ell_1,\ell_2)&=\sum_{n\leq x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)\sum_{\substack{n=a^2+b^2+c^2+d^2\\(a,b,c,d)\in\mathbb{Z}^4}}1\\&=\sum_{n\leq x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)r_4(n)=8\sum_{n\leq x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)r(n),\end{aligned} \tag{18}$$

where $S_{f,g}(x;\ell_1,\ell_2)$ is defined as in (5).

In order to obtain the asymptotic behavior for the summatory function $S_{f,g}(x;\ell_1,\ell_2)$, we define the associated L-series as follows

$$\mathfrak{F}_{\ell_1,\ell_2}(s):=\sum_{n=1}^{\infty}\frac{\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)r(n)}{n^s},\quad \Re(s)>1,$$

where $\ell_1,\ell_2\geq 1$ are any fixed integers. From the multiplicative property of $\lambda_f(n),\lambda_g(n)$ and $r(n)$, along with the upper bound $r(n)\ll n^{1+\varepsilon}$, for $\Re(s)>2$, we have the Euler product identity

$$\mathfrak{F}_{\ell_1,\ell_2}(s):=\prod_p\left(1+\sum_{j\geq 1}\frac{\lambda_f^{2\ell_1}(p^j)\lambda_g^{2\ell_2}(p^j)r(p^j)}{p^{js}}\right). \tag{19}$$

For simplicity, for $i,j\geq 0$ being any fixed integers, we set

$$\prod_{\widetilde{\chi}_0}^{*}L(\mathrm{sym}^i f\times\mathrm{sym}^j g,s):=L(\mathrm{sym}^i f\times\mathrm{sym}^j g,s)L(\mathrm{sym}^i f\times\mathrm{sym}^j g\otimes\widetilde{\chi}_0,s-1),$$

where the symbol $\prod^*_{\widetilde{\chi}_0}$ denotes that the L-functions $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ and $L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \widetilde{\chi}_0, s-1)$ occur in pairs.

For notational convenience, we also define

$$\begin{aligned} D_{\ell_1,\ell_2}(s) := \zeta(s)^{A_{\ell_1}A_{\ell_2}} & L(\mathrm{sym}^{2\ell_1} f, s)^{A_{\ell_2}} L(\mathrm{sym}^{2\ell_2} g, s)^{A_{\ell_1}} \\ & \cdot \Big(\prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i} f, s)^{A_{\ell_2}C_{\ell_1}(i)} \Big) \\ & \cdot \Big(\prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2j} g, s)^{A_{\ell_1}C_{\ell_2}(j)} \Big) \\ & \cdot \prod_{1\le i\le \ell_1-1} \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2j} g, s)^{C_{\ell_1}(i)C_{\ell_2}(j)} \\ & \cdot \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2j} g, s)^{C_{\ell_2}(j)} \\ & \cdot \prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2\ell_2} g, s)^{C_{\ell_1}(i)} \\ & \cdot L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2\ell_2} g, s), \end{aligned} \tag{20}$$

and the constants $A_j, C_j(r)$ $(1 \le r \le j-1)$ are given by

$$A_j = \frac{(2j)!}{j!(j+1)!}, \qquad C_j(r) = \frac{(2j)!(2r+1)}{(j-r)!(j+r+1)!}. \tag{21}$$

Now, we are ready to give the lemma concerning the decomposition of $\mathfrak{F}_{\ell_1,\ell_2}(s)$.

Lemma 2.1. *Let $f \in H^*_{\kappa_1}$ and $g \in H^*_{\kappa_2}$ be distinct Hecke eigenforms. For $\Re(s) > 2$, we have*

$$\mathfrak{F}_{\ell_1,\ell_2}(s) = F_{\ell_1,\ell_2}(s) H_{\ell_1,\ell_2}(s),$$

where $F_{\ell_1,\ell_2}(s) = \prod^*_{\widetilde{\chi}_0} D_{\ell_1,\ell_2}(s)$ with $D_{\ell_1,\ell_2}(s)$ defined as in (20), and the function $H_{\ell_1,\ell_2}(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \ge \frac{3}{2} + \varepsilon$ for any $\varepsilon > 0$, and $H_{\ell_1,\ell_2}(s) \ne 0$ for $\Re(s) = 2$.

Proof. By comparing the p-th coefficient in [36, Lemma 7.1], for $\ell = 2j$,

$$\lambda_f^{\ell}(p) = A_j + \sum_{1\le r\le j-1} C_j(r)\lambda_{\mathrm{sym}^{2r} f}(p) + \lambda_{\mathrm{sym}^{2j} f}(p),$$

where $A_j, C_j(r)$ are defined as in (21). Hence, by standard manipulations with little extra efforts, we can derive that

$$\lambda_f^{2\ell_1}(p)\lambda_g^{2\ell_2}(p) = \big(A_{\ell_1} + \sum_{1\le i\le \ell_1-1} C_{\ell_1}(i)\lambda_{\mathrm{sym}^{2i} f}(p) + \lambda_{\mathrm{sym}^{2\ell_1} f}(p)\big)$$

$$\cdot \big(A_{\ell_2} + \sum_{1\le j\le \ell_2-1} C_{\ell_2}(j)\lambda_{\mathrm{sym}^{2j}g}(p) + \lambda_{\mathrm{sym}^{2\ell_2}g}(p)\big)$$

$$\begin{aligned}
&= A_{\ell_1}A_{\ell_2} + A_{\ell_2}\lambda_{\mathrm{sym}^{2\ell_1}f}(p) + A_{\ell_1}\lambda_{\mathrm{sym}^{2\ell_2}g}(p)\\
&+ \sum_{1\le i\le \ell_1-1} A_{\ell_2}C_{\ell_1}(i)\lambda_{\mathrm{sym}^{2i}f}(p)\\
&+ \sum_{1\le j\le \ell_2-1} A_{\ell_1}C_{\ell_2}(j)\lambda_{\mathrm{sym}^{2j}g}(p)\\
&+ \sum_{1\le i\le \ell_1-1}\sum_{1\le j\le \ell_2-1} C_{\ell_1}(i)C_{\ell_2}(j)\lambda_{\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g}(p)\\
&+ \sum_{1\le j\le \ell_2-1} C_{\ell_2}(j)\lambda_{\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g}(p)\\
&+ \sum_{1\le i\le \ell_1-1} C_{\ell_1}(i)\lambda_{\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g}(p) + \lambda_{\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g}(p).
\end{aligned} \tag{22}$$

In the decomposition of the given L-function, the p-th coefficient of the L-function determines the analytic properties of that L-function. To obtain the required result for the decomposition of $\mathfrak{F}_{\ell_1,\ell_2}(s)$, it is a key component to determine the p-th coefficient in the decomposition. For $\Re(s) > 2$, the L-function

$$\begin{aligned}
F_{\ell_1,\ell_2}(s) = \prod_{\widetilde{\chi}_0}^{*} \big(&\zeta(s)^{A_{\ell_1}A_{\ell_2}} L(\mathrm{sym}^{2\ell_1}f,s)^{A_{\ell_2}} L(\mathrm{sym}^{2\ell_2}g,s)^{A_{\ell_1}}\\
&\cdot \Big(\prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i}f,s)^{A_{\ell_2}C_{\ell_1}(i)}\Big)\\
&\cdot \Big(\prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2j}g,s)^{A_{\ell_1}C_{\ell_2}(j)}\Big)\\
&\cdot \prod_{1\le i\le \ell_1-1}\prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g,s)^{C_{\ell_1}(i)C_{\ell_2}(j)}\\
&\cdot \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g,s)^{C_{\ell_2}(j)}\\
&\cdot \prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g,s)^{C_{\ell_1}(i)}\\
&\cdot L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g,s)\big)
\end{aligned} \tag{23}$$

can be represented as

$$F_{\ell_1,\ell_2}(s) = \prod_p \Big(1 + \sum_{j\ge1}\frac{b(p^j)}{p^{js}}\Big). \tag{24}$$

By comparing the p-th coefficient of (23) and (24), noting the relation (22), it is apparent that

$$b(p) = \lambda_f^{2\ell_1}(p)\lambda_g^{2\ell_2}(p)\big(1 + p\widetilde{\chi}_0(p)\big) = \lambda_f^{2\ell_1}(p)\lambda_g^{2\ell_2}(p)r(p). \tag{25}$$

Putting all the equations (19), and (23)–(25) together, by standard argument (see, e.g., [23]), we are lead to

$$\begin{aligned}\mathfrak{F}_{\ell_1,\ell_2}(s) &= \prod_p \left(1 + \frac{\lambda_f^{2\ell_1}(p)\lambda_g^{2\ell_2}(p)r(p)}{p^s} + \frac{\lambda_f^{2\ell_1}(p^2)\lambda_g^{2\ell_2}(p^2)r(p^2)}{p^{2s}} + \cdots\right)\\ &= F_{\ell_1,\ell_2}(s)\cdot\prod_p\left(1 + \frac{\lambda_f^{2\ell_1}(p^2)\lambda_g^{2\ell_2}(p^2)r(p^2) - b(p^2)}{p^{2s}} + \cdots\right)\\ &:= F_{\ell_1,\ell_2}(s)H_{\ell_1,\ell_2}(s),\end{aligned}$$

where $F_{\ell_1,\ell_2}(s)$ is defined by (23), and $H_{\ell_1,\ell_2}(s)$ is a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{3}{2} + \varepsilon$. □

Now, we turn to the summatory function $U_{f,g}(x;\ell_1,\ell_2)$ using a different approach. For $k = 6$ in (8), we learn from [57, Lemma 2.1] that

$$r_6(n) = 16\sum_{d|n}\chi(d')d^2 - 4\sum_{d|n}\chi(d)d^2,$$

where $n = dd'$, where χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases}1, & \text{if } n \equiv 1(\mathrm{mod}4),\\ -1, & \text{if } n \equiv -1(\mathrm{mod}4),\\ 0, & \text{if } n \equiv 0(\mathrm{mod}2).\end{cases} \tag{26}$$

We can also rewrite $r_6(n)$ as

$$\begin{aligned}r_6(n) &= 16\sum_{d|n}\chi(d)\frac{n^2}{d^2} - 4\sum_{d|n}\chi(d)d^2\\ &:= 16l(n) - 4v(n) := l_1(n) - v_1(n).\end{aligned}$$

Clearly, the functions $l(n)$ and $v(n)$ are both multiplicative, due to the fact that the non-principal character χ is multiplicative.

It is clear that

$$\begin{aligned}l(p) &= p^2 + \chi(p),\\ l(p^2) &= p^4 + p^2\chi(p) + \chi(p^2),\end{aligned}$$

and

$$v(p) = 1 + p^2\chi(p),$$

$$v(p^2) = 1 + p^2\chi(p) + p^4\chi(p^2),$$

respectively. By employing the similar argument as with $S_{f,g}(x;\ell_1,\ell_2)$, we can deduce that

$$\begin{aligned}
\widetilde{S}_{f,g}(x;\ell_1,\ell_2) &= \sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\\(a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1\\
&= \sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) r_6(n)\\
&= \sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)(l_1(n) - v_1(n))\\
&= 16\sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) l(n) - 4\sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) v(n). \qquad (27)
\end{aligned}$$

Similarly, in order to get the asymptotic behavior of $U_{f,g}(x;\ell_1,\ell_2)$, for $\Re(s) > 3$, we also need to define two associated L-series

$$\mathfrak{G}_{\ell_1,\ell_2}(s) := \sum_{n=1}^{\infty} \frac{\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) l(n)}{n^s},$$

and

$$\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s) := \sum_{n=1}^{\infty} \frac{\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n) v(n)}{n^s}.$$

For simplicity, for $i, j \ge 0$ being any fixed integers, we set

$$\begin{aligned}
&\prod_{\chi}{}' L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)\\
&\qquad := L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s-2) L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \chi, s),
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{\chi}{}^{\flat} L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)\\
&\qquad := L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \chi, s-2).
\end{aligned}$$

In a similar manner, we can also prove the following decompositions of $\mathfrak{G}_{\ell_1,\ell_2}(s)$ and $\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)$, respectively.

LEMMA 2.2. *Let $f \in H_{\kappa_1}^*$ and $g \in H_{\kappa_2}^*$ be distinct Hecke eigenforms. For $\Re(s) > 3$, we have*

$$\mathfrak{G}_{\ell_1,\ell_2}(s) = G_{\ell_1,\ell_2}(s) H_{\ell_1,\ell_2}^{\dagger}(s),$$

where $G_{\ell_1,\ell_2}(s) = \prod_{\chi}' D_{\ell_1,\ell_2}(s)$ with $D_{\ell_1,\ell_2}(s)$ defined as in (20) and the constants $A_j, C_j(r)$ $(1 \le r \le j-1)$ are defined as in (21), and the function $H_{\ell_1,\ell_2}^{\dagger}(s)$

admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{5}{2} + \varepsilon$ *for any* $\varepsilon > 0$, *and* $H^{\dagger}_{\ell_1,\ell_2}(s) \neq 0$ *for* $\Re(s) = 3$.

Proof. This follows the similar argument as that of Lemma 2.1, and the interested readers are also invited to refer to [57, Lemma 2.2]. □

Lemma 2.3. *Let* $f \in H^*_{\kappa_1}$ *and* $g \in H^*_{\kappa_2}$ *be distinct Hecke eigenforms. For* $\Re(s) > 3$, *we have*

$$\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s) = G^*_{\ell_1,\ell_2}(s) H^*_{\ell_1,\ell_2}(s), \tag{28}$$

where $G^*_{\ell_1,\ell_2}(s) = \prod^{\flat}_{\chi} D_{\ell_1,\ell_2}(s)$ *with* $D_{\ell_1,\ell_2}(s)$ *defined as in* (20), *and the constants* $A_j, C_j(r)$ $(1 \leq r \leq j-1)$ *are defined as in* (21), *and the function* $H^*_{\ell_1,\ell_2}(s)$ *admits a Dirichlet series which converges absolutely and uniformly in the half-plane* $\Re(s) \geq \frac{5}{2} + \varepsilon$ *for any* $\varepsilon > 0$, *and* $H^*_{\ell_1,\ell_2}(s) \neq 0$ *for* $\Re(s) = 3$.

Proof. This can be proved by following the similar argument as that of Lemma 2.2. □

To prove the main results, we also need the following individual or average subconvexity bounds for the associated automorphic L-functions.

Lemma 2.4. *For any* $\varepsilon > 0$, *one has*

$$\int_1^T \left|\zeta\left(\frac{5}{7} + it\right)\right|^{12} dt \ll T^{1+\varepsilon} \tag{29}$$

uniformly for $T \geq 1$, *and*

$$\zeta(\sigma + it) \ll (1 + |t|)^{\max\{\frac{13}{42}(1-\sigma), 0\}+\varepsilon} \tag{30}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ *and* $|t| \geq 1$.

Proof. The first result is established by Ivić [22], and the second result is the recent breakthrough of Bourgain [2]. □

Lemma 2.5 ([45, Lemma 1]). *For* $\frac{1}{2} \leqslant \sigma \leqslant 2$, *and* T *sufficiently large, there exists a* $T^* \in [T, T + T^{1/3}]$ *such that the bound*

$$\log \zeta(\sigma + iT^*) \ll (\log\log T^*)^2 \ll (\log\log T)^2$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$, *and hence*

$$\left|\zeta(\sigma + iT^*)\right| \ll \exp\left((\log\log T^*)^2\right) \ll T^{\varepsilon} \tag{31}$$

on the horizontal line with $t = T^*$ *for* $\frac{1}{2} \leqslant \sigma \leqslant 2$.

Remark 2.6. Let $\chi' = \chi$ or $\widetilde{\chi}_0$ as defined in (26) and (17), respectively. By employing a similar argument as that of [39, Lemma 2.6], we know that (31) also holds for Dirichlet L-functions $L(\sigma + it, \chi')$ with respect to the variable t (other parameters are fixed).

LEMMA 2.7. *For any $\varepsilon > 0$, we have*

$$L(\mathrm{sym}^2 f, \sigma + iT) \ll \big(1 + |T|\big)^{\max\{\frac{8}{7}(1-\sigma),0\}+\varepsilon} \tag{32}$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|T| \ge 1$.

Proof. The estimate (32) follows from the recent remarkable work of Dasgupta, Leung, and Young [7, Corollary 1.4] and Phragmén–Lindelöf convexity principle for a strip. □

LEMMA 2.8. *Let $\chi' = \chi$ or $\widetilde{\chi}_0$ be defined in* (26) *and* (17), *respectively. For $\varepsilon > 0$, we have*

$$L(\sigma + it, \chi') \ll (1 + |t|)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon},$$
$$L(\mathrm{sym}^2 f \otimes \chi', \sigma + it) \ll (1 + |t|)^{\max\{\frac{8}{7}(1-\sigma),0\}+\varepsilon},$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|t| \ge 1$, and

$$\int_1^T \left| L\left(\frac{5}{7} + it, \chi'\right)\right|^{12} dt \ll T^{1+\varepsilon},$$

uniformly for $T \ge 1$.

Proof. By adopting a similar argument as that of [39, Lemma 2.6], it can be seen that L-functions twisting by character $\chi' = \chi$ or $\widetilde{\chi}_0$ does not affect the subconvexity bounds, convexity bounds and integral mean estimates of the corresponding L-functions in the t-aspect. □

Let $\boldsymbol{d} := (d_1, \ldots, d_J)$, $\boldsymbol{m} = (m_1, \ldots, m_J)$ and $\boldsymbol{n} = (n_1, \ldots, n_J)$ with $d_j, m_j, n_j \in (\mathbb{N} \cup \{0\})$, and set

$$A(\boldsymbol{d}, \boldsymbol{m}, \boldsymbol{n}) = \frac{1}{2} \sum_{j=1}^{J} d_j (m_j + 1)(n_j + 1).$$

Let χ be a primitive character modulo q, and define

$$\mathfrak{L}^{\boldsymbol{d}}_{\boldsymbol{m},\boldsymbol{n}}(f, g, \chi, s) := \prod_{j=1}^{J} L(\mathrm{sym}^{m_j} f \times \mathrm{sym}^{n_j} g \otimes \chi, s)^{d_j}. \tag{33}$$

This general L-function is in the sense of Perelli [43] due to the recent deep works of Newton and Thorne [41, 42]. The following lemma follows plainly from Perelli [43, Theorem 4].

LEMMA 2.9. *Let $\mathfrak{L}_{\boldsymbol{m}}^{\boldsymbol{d}}(f,g,\chi,s)$ be defined as in (33), for any $\varepsilon>0$, we have*

$$\int_T^{2T}|\mathfrak{L}_{\boldsymbol{m},\boldsymbol{n}}^{\boldsymbol{d}}(f,g,\chi,\sigma+it)|^2dt\ll_{f,g,\varepsilon,\boldsymbol{d},\boldsymbol{m},\boldsymbol{n}}\big(q(1+T)\big)^{2A(\boldsymbol{d},\boldsymbol{m},\boldsymbol{n})(1-\sigma)+\varepsilon}\tag{34}$$

uniformly for $\frac{1}{2}\le\sigma\le 1+\varepsilon$ and $T\ge 1$, and

$$\mathfrak{L}_{\boldsymbol{m},\boldsymbol{n}}^{\boldsymbol{d}}(f,g,\chi,\sigma+it)\ll_{f,g,\varepsilon,\boldsymbol{d},\boldsymbol{m},\boldsymbol{n}}\big(q(1+|t|)\big)^{A(\boldsymbol{d},\boldsymbol{m},\boldsymbol{n})(1-\sigma)+\varepsilon}\tag{35}$$

uniformly for $\frac{1}{2}\le\sigma\le 1+\varepsilon$ and $|t|\ge 1$.

Proof. In light of the recent groundbreaking works of Newton and Thorne, see [41,42], we learn that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j\ge 1$. Using the Rankin–Selberg convolution theory associated to two automorphic cuspidal representations, we infer that $L(\mathrm{sym}^i f\times\mathrm{sym}^j g,s),(i,j\geqslant 0)$ can be regarded as the general L-function in the sense of Perelli [43]. Henceforth, the results follow from [43, Theorem 4] and [40, Proposition 1] in a straightforward nature. □

3. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1, by following the similar argument exhibited in [15, Section 3] with suitable modifications. Applying Perron's formula (see, e.g., [34, Theorem 2.1]) and invoking Lemma 2.1, along with the reinterpretation for $S_{f,g}(x;\ell_1,\ell_2)$ in (18), we have

$$\begin{aligned}S_{f,g}(x;\ell_1,\ell_2)&=\sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)r_4(n)\\&=8\sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)r(n)\\&=\frac{8}{2\pi i}\int_{\eta-iT}^{\eta+iT}\mathfrak{F}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds+O_{f,g,\varepsilon}\left(\frac{x^{2+\varepsilon}}{T}\right),\end{aligned}\tag{36}$$

where $\eta=2+\varepsilon$, and we make the special choice $T=T^*$ which satisfies (31), here $2\le T\le x$ is some suitable parameter to be specified later.

By shifting the line of integration in equality (36) to the parallel line with $\Re(s)=\alpha:=\frac{12}{7}$, together with Cauchy's residue theorem, we get

$$\begin{aligned}S_{f,g}(x;\ell_1,\ell_2)=8\operatorname*{Res}_{s=2}&\left\{\mathfrak{F}_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\}\\&+\frac{8}{2\pi i}\left\{\int_{\alpha-iT}^{\alpha+iT}+\int_{\alpha+iT}^{\eta+iT}+\int_{\eta-iT}^{\alpha-iT}\right\}\mathfrak{F}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds\end{aligned}$$

$$+ O_{f,g,\varepsilon}\left(\frac{x^{2+\varepsilon}}{T}\right)$$

$$:= x^2 P_{A_{\ell_1}A_{\ell_2}-1}(\log x) + J_1 + J_2 + J_3 + O_{f,g,\varepsilon}\left(\frac{x^{2+\varepsilon}}{T}\right), \tag{37}$$

where $P_j(t)$ denotes the polynomial in t with degree j. In the region

$$\alpha \le \Re(s) \le \eta, -T \le \Im(s) \le T,$$

we note that the L-series $\mathfrak{F}_{\ell_1,\ell_2}(s)$ is a meromorphic function having a pole at $s = 2$ of order $A_{\ell_1}A_{\ell_2}$ coming from the factor $L(s-1, \widetilde{\chi}_0)^{A_{\ell_1}A_{\ell_2}}$.

From [15, Section 3], we learn that

$$L(s-1, \widetilde{\chi}_0) = \left(1 - \frac{3}{2^{s-1}}\right)^{-1}\left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta(s-1), \tag{38}$$

and, for any $j \ge 1$,

$$\begin{aligned} &L(\mathrm{sym}^j f \otimes \widetilde{\chi}_0, s-1) \\ &= \left(1 - \frac{3\lambda_{\mathrm{sym}^j f}(2)}{2^{s-1}}\right)^{-1}\left(1 - \frac{\lambda_{\mathrm{sym}^j f}(2)}{2^{s-1}}\right)^2 L(\mathrm{sym}^j f, s-1). \end{aligned} \tag{39}$$

By following the similar approach as in [15, Section 3], for $i, j \ge 1$ being any fixed integers, we can also derive that

$$\begin{aligned} &L(\mathrm{sym}^i f \times \mathrm{sym}^j g \otimes \widetilde{\chi}_0, s-1) \\ &= \left(1 - \frac{3\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j g}(2)}{2^{s-1}}\right)^{-1}\left(1 - \frac{\lambda_{\mathrm{sym}^i f\times\mathrm{sym}^j g}(2)}{2^{s-1}}\right)^2 \\ &\quad \cdot L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s-1). \end{aligned} \tag{40}$$

Indeed, we can evaluate the main term explicitly as follows. From [24, (1.11)], we learn that $\zeta(s)$ has a Laurent expansion at the simple pole $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \sum_{n=1}^{\infty} \gamma_j (s-1)^j,$$

where $\gamma_j, j \ge 0$ are suitable constants. In particular, γ_0 is the Euler–Mascheroni constant. By the Leibniz's rule and the Cauchy residue theorem, along with (38), we obtain

$$\begin{aligned} x^2 P_{A_{\ell_1}A_{\ell_2}-1}(\log x) &= 8 \operatorname*{Res}_{s=2}\left\{\mathfrak{F}_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\} \\ &= 8 \operatorname*{Res}_{s=2}\left\{F_{\ell_1,\ell_2}(s) H_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\} \\ &= \frac{4(-1/2)^{A_{\ell_1}A_{\ell_2}}}{(A_{\ell_1}A_{\ell_2}-1)!}\zeta(2)^{A_{\ell_1}A_{\ell_2}}\left(L(\mathrm{sym}^{2\ell_1} f, 2) L(\mathrm{sym}^{2\ell_1} f \otimes \widetilde{\chi}_0, 1)\right)^{A_{\ell_2}} \end{aligned}$$

$$
\begin{aligned}
&\cdot \big(L(\mathrm{sym}^{2\ell_2} g, 2) L(\mathrm{sym}^{2\ell_2} g \otimes \widetilde{\chi}_0, 1)\big)^{A_{\ell_1}}\\
&\cdot \prod_{1\le i\le \ell_1-1} \big(L(\mathrm{sym}^{2i} f, 2) L(\mathrm{sym}^{2i} f \otimes \widetilde{\chi}_0, 1)\big)^{A_{\ell_2} C_{\ell_1}(i)}\\
&\cdot \prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2j} g, 2) L(\mathrm{sym}^{2j} g \otimes \widetilde{\chi}_0, 1)\big)^{A_{\ell_1} C_{\ell_2}(j)}\\
&\cdot \prod_{1\le i\le \ell_1-1}\ \prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2j} g, 2)\\
&\cdot L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2j} g \otimes \widetilde{\chi}_0, 1)\big)^{C_{\ell_1}(i) C_{\ell_2}(j)}\\
&\cdot \prod_{1\le j\le \ell_2-1} \big(L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2j} g, 2) L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2j} g \otimes \widetilde{\chi}_0, 1)\big)^{C_{\ell_2}(j)}\\
&\cdot \prod_{1\le i\le \ell_1-1} \big(L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2\ell_2} g, 2) L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2\ell_2} g \otimes \widetilde{\chi}_0, 1)\big)^{C_{\ell_1}(i)}\\
&\cdot \big(L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2\ell_2} g, 2) L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2\ell_2} g \otimes \widetilde{\chi}_0, 1)\big) H_{\ell_1,\ell_2}(2)\\
&\cdot x^2 (\log x)^{A_{\ell_1} A_{\ell_2} - 1} + \cdots + C_{f,g} x^2,
\end{aligned}
$$

and $C_{f,g}$ is some suitable constant depending on f, g and various associated L-functions.

Now, we need to estimate the integrals J_1, J_2 and J_3. For simplicity, we rephrase $D_{\ell_1,\ell_2}(s)$ defined by (20) as

$$
\begin{aligned}
D_{\ell_1,\ell_2}(s) = \zeta(s)^{A_{\ell_1} A_{\ell_2}} L(\mathrm{sym}^2 f, s)^{A_{\ell_2} C_{\ell_1}(1)} L(\mathrm{sym}^2 g, s)^{A_{\ell_1} C_{\ell_2}(1)}\\
\cdot L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2\ell_2} g, s) \mathcal{F}_{\ell_1,\ell_2}(s),
\end{aligned}
\tag{41}
$$

where $\mathcal{F}_{\ell_1,\ell_2}(s)$ is explicitly defined by

$$
\begin{aligned}
\mathcal{F}_{\ell_1,\ell_2}(s) := L(\mathrm{sym}^{2\ell_1} f, s)^{A_{\ell_2}} L(\mathrm{sym}^{2\ell_2} g, s)^{A_{\ell_1}}&\\
\cdot \Big(\prod_{2\le i\le \ell_1-1} L(\mathrm{sym}^{2i} f, s)^{A_{\ell_2} C_{\ell_1}(i)}\Big)&\\
\cdot \Big(\prod_{2\le j\le \ell_2-1} L(\mathrm{sym}^{2j} g, s)^{A_{\ell_1} C_{\ell_2}(j)}\Big)&\\
\cdot \prod_{1\le i\le \ell_1-1}\ \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2j} g, s)^{C_{\ell_1}(i) C_{\ell_2}(j)}&\\
\cdot \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2j} g, s)^{C_{\ell_2}(j)}&\\
\cdot \prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i} f \times \mathrm{sym}^{2\ell_2} g, s)^{C_{\ell_1}(i)}.&
\end{aligned}
$$

It is not hard to find that $\mathcal{F}_{\ell_1,\ell_2}(s)$ is an L-function of degree

$$g_{\ell_1,\ell_2} := 2^{\ell_1+\ell_2} - A_{\ell_1}A_{\ell_2} - 3\big(A_{\ell_2}C_{\ell_1}(1) + A_{\ell_1}C_{\ell_2}(1)\big) - (2\ell_1+1)(2\ell_2+1), \tag{42}$$

where $A_j, C_j(r)$ $(1 \le r \le j-1)$ are constants defined as in (21).

For the integral over the vertical segment J_1, by Lemma 2.1 and Hölder's inequality, on noting (38)–(40), we have

$$\begin{aligned}
J_1 &= \frac{8}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \mathfrak{F}_{\ell_1,\ell_2}(s)\frac{x^s}{s}dt \\
&\ll x^{\frac{12}{7}+\varepsilon}\int_1^T \left|D_{\ell_1,\ell_2}\left(\frac{5}{7}+it\right)t^{-1}\right|dt + x^{\frac{12}{7}+\varepsilon} \\
&\ll x^{\frac{12}{7}+\varepsilon}\sup_{1\le T_1\le T/2}\sup_{T_1\le t\le 2T_1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{A_{\ell_1}A_{\ell_2}-4} \\
&\quad\cdot\left|L\left(\mathrm{sym}^2 f, \frac{5}{7}+it\right)\right|^{A_{\ell_2}C_{\ell_1}(1)}\left|L\left(\mathrm{sym}^2 g, \frac{5}{7}+it\right)\right|^{A_{\ell_1}C_{\ell_2}(1)} \\
&\quad\cdot I_1(T_1)^{\frac{1}{3}}I_2(T_1)^{\frac{1}{2}}I_3(T_1)^{\frac{1}{6}}T_1^{-1} + x^{\frac{12}{7}+\varepsilon},
\end{aligned}$$

where

$$\begin{aligned}
I_1(T_1) &:= \int_{T_1}^{2T_1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{12}dt, \\
I_2(T_1) &:= \int_{T_1}^{2T_1}\left|\mathcal{F}_{\ell_1,\ell_2}\left(\frac{5}{7}+it\right)\right|^{2}dt, \\
I_3(T_1) &:= \int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g, \frac{5}{7}+it\right)\right|^{6}dt. \qquad (43)
\end{aligned}$$

By virtue of (29) and (34), we have

$$\begin{aligned}
I_1(T_1) &\ll_\varepsilon T_1^{1+\varepsilon}, \\
I_2(T_2) &\ll_{f,g,\varepsilon} T_1^{\frac{2}{7}g_{\ell_1,\ell_2}+\varepsilon},
\end{aligned}$$

and

$$\begin{aligned}
I_3(T_1) &= \int_{T_1}^{2T_1}\left|L\left(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g, \frac{5}{7}+it\right)^3\right|^{2}dt \\
&\ll T_1^{(2\ell_1+1)(2\ell_2+1)\times 3\times\frac{2}{7}+\varepsilon} \\
&\ll T_1^{\frac{6}{7}(2\ell_1+1)(2\ell_2+1)+\varepsilon},
\end{aligned}$$

where g_{ℓ_1,ℓ_2} is defined as in (42). Therefore, by invoking (30), (32) and the above estimates, we obtain

$$\begin{aligned} J_1 \ll x^{\frac{12}{7}+\varepsilon} \sup_{1\le T_1\le T/2} \sup_{T_1\le t\le 2T_1} \left|\zeta\left(\frac{5}{7}+it\right)\right|^{A_{\ell_1}A_{\ell_2}-4} \\ \cdot \left|L\left(\mathrm{sym}^2 f, \frac{5}{7}+it\right)\right|^{A_{\ell_2}C_{\ell_1}(1)} \left|L\left(\mathrm{sym}^2 g, \frac{5}{7}+it\right)\right|^{A_{\ell_1}C_{\ell_2}(1)} \\ \cdot I_1(T_1)^{\frac{1}{3}} I_2(T_1)^{\frac{1}{2}} I_3(T_1)^{\frac{1}{6}} T_1^{-1} + x^{\frac{12}{7}+\varepsilon} \\ \ll x^{\frac{12}{7}+\varepsilon} T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}. \end{aligned} \tag{44}$$

Here, $\widetilde{\theta}_{\ell_1,\ell_2}$ is the exponent defined as

$$\begin{aligned} \widetilde{\theta}_{\ell_1,\ell_2} := \frac{13}{147} A_{\ell_1}A_{\ell_2} + \frac{16}{49}\big(A_{\ell_2}C_{\ell_1}(1) + A_{\ell_1}C_{\ell_2}(1)\big) + \frac{1}{7} g_{\ell_1,\ell_2} \\ + \frac{1}{7}(2\ell_1+1)(2\ell_2+1) - \frac{1}{49}. \end{aligned} \tag{45}$$

For the integrals J_2 and J_3, by using Lemma 2.1, (31), (32) and (35), along with (38)–(40), we have

$$\begin{aligned} J_2 + J_3 \ll \int_{\frac{12}{7}}^{2+\varepsilon} \left|F_{\ell_1,\ell_2}(\sigma+iT) x^{\sigma} T^{-1}\right| d\sigma \\ \ll \sup_{\frac{5}{7}\le\sigma\le 1+\varepsilon} x^{1+\sigma} \left|\zeta(\sigma+iT)\right|^{A_{\ell_1}A_{\ell_2}} \\ \cdot \left|L(\mathrm{sym}^2 f, \sigma+iT)\right|^{A_{\ell_2}C_{\ell_1}(1)} \left|L(\mathrm{sym}^2 g, \sigma+iT)\right|^{A_{\ell_1}C_{\ell_2}(1)} \\ \cdot \left|L(\mathrm{sym}^{2\ell_1} f \times \mathrm{sym}^{2\ell_2} g, \sigma+iT) \mathcal{F}_{\ell_1,\ell_2}(\sigma+iT)\right| T^{-1} \\ \ll \frac{x^{2+\varepsilon}}{T} + x^{\frac{12}{7}+\varepsilon} T^{\rho_{\ell_1,\ell_2}-1+\varepsilon}, \end{aligned} \tag{46}$$

where ρ_{ℓ_1,ℓ_2} is given by

$$\begin{aligned} \rho_{\ell_1,\ell_2} := \frac{16}{49}\big(A_{\ell_2}C_{\ell_1}(1) + A_{\ell_1}C_{\ell_2}(1)\big) + \frac{1}{7} g_{\ell_1,\ell_2} \\ + \frac{1}{7}(2\ell_1+1)(2\ell_2+1). \end{aligned} \tag{47}$$

Putting all equations (37), (44) and (46) together, we obtain

$$\begin{aligned} S_{f,g}(x;\ell_1,\ell_2) = x^2 P_{A_{\ell_1}A_{\ell_2}-1}(\log x) \\ + O_{f,g,\varepsilon}\left(\frac{x^{2+\varepsilon}}{T} + x^{\frac{12}{7}+\varepsilon} T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}\right). \end{aligned} \tag{48}$$

Taking $T = x^{\frac{2}{7\widetilde{\theta}_{\ell_1,\ell_2}}}$ in (48), it leads to

$$S_{f,g}(x;\ell_1,\ell_2) = x^2 P_{A_{\ell_1}A_{\ell_2}-1}(\log x) + O_{f,g,\varepsilon}(x^{2-\theta_{\ell_1,\ell_2}+\varepsilon}),$$

where $\theta_{\ell_1,\ell_2} = \frac{2}{7\widetilde{\theta}_{\ell_1,\ell_2}}$.

4. PROOF OF THEOREM 1.2

From (27), we know that

$$\begin{aligned}\widetilde{S}_{f,g}(x;\ell_1,\ell_2) &= \sum_{n\le x} \lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)(l_1(n)-v_1(n))\\ &= 16\sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)l(n) - 4\sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)v(n)\\ &:= \mathcal{S}_1+\mathcal{S}_2. \qquad (49)\end{aligned}$$

First, we treat the sum $\mathcal{S}_1$. By applying Perron's formula and Lemma 2.2, we get

$$\begin{aligned}\mathcal{S}_1 &= \sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)l_1(n)\\ &= \frac{16}{2\pi i}\int_{\widetilde{\eta}-iT}^{\widetilde{\eta}+iT}\mathfrak{G}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds + O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right), \qquad (50)\end{aligned}$$

where $\widetilde{\eta} = 3+\varepsilon$, and $6 \le T = T^* \le x$ is some parameter satisfying (31) to be determined later. We note that the utilization of Lemma 2.5 in handling the integrals over the horizontal segments can be ensured by the choice of the parameter T, which is taken as a positive power of sufficiently large x.

By shifting the line of integration in equation (50) to the parallel line with $\Re(s) = \beta := \frac{19}{7}$, and appealing to Cauchy's residue theorem, we obtain

$$\begin{aligned}\mathcal{S}_1 &= 16\operatorname*{Res}_{s=3}\left\{\mathfrak{G}_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\}\\ &\quad + \frac{16}{2\pi i}\left\{\int_{\beta-iT}^{\beta+iT}+\int_{\beta+iT}^{\widetilde{\eta}+iT}+\int_{\widetilde{\eta}-iT}^{\beta-iT}\right\}\mathfrak{G}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds\\ &\quad + O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right)\\ &:= x^3P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x) + I_1+I_2+I_3+O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right), \qquad (51)\end{aligned}$$

where $P_j^*(t)$ denotes the polynomial in t with degree j. Here, the only pole at $s=3$ of order $A_{\ell_1}A_{\ell_2}$ coming from the factor $\zeta(s-3)^{A_{\ell_1}A_{\ell_2}}$, contributes the main term $x^3P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x)$. To be more precise,

$$x^3P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x) = 16\operatorname*{Res}_{s=3}\left\{\mathfrak{G}_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\} = 16\operatorname*{Res}_{s=3}\left\{G_{\ell_1,\ell_2}(s)H^{\dagger}_{\ell_1,\ell_2}(s)\frac{x^s}{s}\right\}$$

$$
\begin{aligned}
&= \frac{16}{3\cdot(A_{\ell_1}A_{\ell_2}-1)!}L(3,\chi)^{A_{\ell_1}A_{\ell_2}}\big(L(\mathrm{sym}^{2\ell_1}f,1)L(\mathrm{sym}^{2\ell_1}f\otimes\chi,3)\big)^{A_{\ell_2}}\\
&\quad\cdot\big(L(\mathrm{sym}^{2\ell_2}g,1)L(\mathrm{sym}^{2\ell_2}g\otimes\chi,3)\big)^{A_{\ell_1}}\\
&\quad\cdot\prod_{1\le i\le \ell_1-1}\big(L(\mathrm{sym}^{2i}f,1)L(\mathrm{sym}^{2i}f\otimes\chi,3)\big)^{A_{\ell_2}C_{\ell_1}(i)}\\
&\quad\cdot\prod_{1\le j\le \ell_2-1}\big(L(\mathrm{sym}^{2j}g,1)L(\mathrm{sym}^{2j}g\otimes\chi,3)\big)^{A_{\ell_1}C_{\ell_2}(j)}\\
&\quad\cdot\prod_{1\le i\le \ell_1-1}\ \prod_{1\le j\le \ell_2-1}\big(L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g,1)\\
&\quad\cdot L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2j}g\otimes\chi,3)\big)^{C_{\ell_1}(i)C_{\ell_2}(j)}\\
&\quad\cdot\prod_{1\le j\le \ell_2-1}\big(L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g,1)L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2j}g\otimes\chi,3)\big)^{C_{\ell_2}(j)}\\
&\quad\cdot\prod_{1\le i\le \ell_1-1}\big(L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g,1)L(\mathrm{sym}^{2i}f\times\mathrm{sym}^{2\ell_2}g\otimes\chi,3)\big)^{C_{\ell_1}(i)}\\
&\quad\cdot\big(L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g,1)L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g\otimes\chi,3)\big)H^{\dagger}_{\ell_1,\ell_2}(3)\\
&\quad\cdot x^3(\log x)^{A_{\ell_1}A_{\ell_2}-1}+\cdots+\widetilde{C}_{f,g}x^3,
\end{aligned}
$$

and $\widetilde{C}_{f,g}$ is some suitable constant depending on f,g and various associated L-functions.

For the contribution of vertical line integral I_1, by Lemma 2.2 and Hölder's inequality, together with (30), (32), with $I_1(T_1), I_2(T_1)$ and $I_3(T_1)$ defined the same as in (43), we have

$$
\begin{aligned}
I_1 &= \frac{16}{2\pi i}\int_{\beta-iT}^{\beta+iT}\mathfrak{G}_{\ell_1,\ell_2}(s)\frac{x^s}{s}dt\\
&\ll x^{\frac{19}{7}+\varepsilon}\int_1^T\left|D_{\ell_1,\ell_2}\left(\frac{5}{7}+it\right)t^{-1}\right|dt+x^{\frac{19}{7}+\varepsilon}\\
&\ll x^{\frac{19}{7}+\varepsilon}\sup_{1\le T_1\le T/2}\ \sup_{T_1\le t\le 2T_1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{A_{\ell_1}A_{\ell_2}-4}\\
&\qquad\cdot\left|L\left(\mathrm{sym}^2f,\frac{5}{7}+it\right)\right|^{A_{\ell_2}C_{\ell_1}(1)}\left|L\left(\mathrm{sym}^2g,\frac{5}{7}+it\right)\right|^{A_{\ell_1}C_{\ell_2}(1)}\\
&\qquad\cdot I_1(T_1)^{\frac13}I_2(T_1)^{\frac12}I_3(T_1)^{\frac16}T_1^{-1}+x^{\frac{19}{7}+\varepsilon}\\
&\ll x^{\frac{19}{7}+\varepsilon}T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon},
\end{aligned}
\tag{52}
$$

where $\widetilde{\theta}_{\ell_1,\ell_2}$ is the exponent defined as (45).

For the integrals over the horizontal segments I_2 and I_3, using Lemma 2.2, (31), (32) and (35), with $\mathcal{F}_{\ell_1,\ell_2}(s)$ defined as in (41), we have

$$
\begin{aligned}
I_2+I_3 &\ll \int_{\frac{19}{7}}^{3+\varepsilon} \left|G_{\ell_1,\ell_2}(\sigma+iT)x^{\sigma}T^{-1}\right|d\sigma \\
&\ll \sup_{\frac{5}{7}\le\sigma\le 1+\varepsilon} x^{2+\sigma}\left|\zeta(\sigma+iT)\right|^{A_{\ell_1}A_{\ell_2}}\left|L(\mathrm{sym}^2 f,\sigma+iT)\right|^{A_{\ell_2}C_{\ell_1}(1)} \\
&\qquad \cdot\left|L(\mathrm{sym}^2 g,\sigma+iT)\right|^{A_{\ell_1}C_{\ell_2}(1)} \\
&\qquad \cdot\left|L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g,\sigma+iT)\mathcal{F}_{\ell_1,\ell_2}(\sigma+iT)\right|T^{-1} \\
&\ll \frac{x^{3+\varepsilon}}{T}+x^{\frac{19}{7}+\varepsilon}T^{\rho_{\ell_1,\ell_2}-1+\varepsilon}, \qquad (53)
\end{aligned}
$$

where ρ_{ℓ_1,ℓ_2} is determined by (47).

Therefore, inserting the estimates (52) and (53) into (50), we obtain

$$
\mathcal{S}_1 = x^3 P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x)+O_{f,g,\varepsilon}\left(x^{\frac{19}{7}+\varepsilon}T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}+\frac{x^{3+\varepsilon}}{T}\right), \qquad (54)
$$

where $\widetilde{\theta}_{\ell_1,\ell_2}$ is defined as in (45).

Similarly, we can also handle the sum $\mathcal{S}_2$. By applying Perron's formula to $\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)$ given by (28), shifting the line of integration to the parallel line with $\Re(s)=\beta:=\frac{19}{7}$, and invoking Cauchy's residue theorem, we are led to

$$
\begin{aligned}
\mathcal{S}_2 &= \sum_{n\le x}\lambda_f^{2\ell_1}(n)\lambda_g^{2\ell_2}(n)v_1(n) \\
&= \frac{4}{2\pi i}\int_{\widetilde{\eta}-iT}^{\widetilde{\eta}+iT}\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds+O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right) \\
&= \frac{4}{2\pi i}\left\{\int_{\beta-iT}^{\beta+iT}+\int_{\beta+iT}^{\widetilde{\eta}+iT}+\int_{\widetilde{\eta}-iT}^{\beta-iT}\right\}\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)\frac{x^s}{s}ds+O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right) \\
&:= E_1+E_2+E_3+O_{f,g,\varepsilon}\left(\frac{x^{3+\varepsilon}}{T}\right), \qquad (55)
\end{aligned}
$$

where $\widetilde{\eta}=3+\varepsilon$, and $6\le T=T^*\le x$ is some parameter satisfying (31) to be chosen later. In this case, the integrand $\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)\frac{x^s}{s}$ has no singularity in the region $-\beta\le\sigma\le\widetilde{\eta}, -T\le t\le T$, since $\widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)$ is analytic in this occurrence.

Now, we are ready to handle the three integrals E_1, E_2 and E_3. For simplicity, we define

$$
\begin{aligned}
D^*_{\ell_1,\ell_2}(s,\chi) := L(s,\chi)^{A_{\ell_1}A_{\ell_2}}&L(\mathrm{sym}^{2\ell_1}f\otimes\chi,s)^{A_{\ell_2}}L(\mathrm{sym}^{2\ell_2}g\otimes\chi,s)^{A_{\ell_1}} \\
&\cdot\prod_{1\le i\le\ell_1-1}L(\mathrm{sym}^{2i}f\otimes\chi,s)^{A_{\ell_2}C_{\ell_1}(i)} \\
&\cdot\prod_{1\le j\le\ell_2-1}L(\mathrm{sym}^{2j}g\otimes\chi,s)^{A_{\ell_1}C_{\ell_2}(j)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \prod_{1\le i\le \ell_1-1}\ \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2j} g\otimes\chi, s)^{C_{\ell_1}(i)C_{\ell_2}(j)}\\
&\cdot \prod_{1\le j\le \ell_2-1} L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2j} g\otimes\chi, s)^{C_{\ell_2}(j)}\\
&\cdot \prod_{1\le i\le \ell_1-1} L(\mathrm{sym}^{2i} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, s)^{C_{\ell_1}(i)}\\
&\cdot L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, s)\\
:= {}& L(s,\chi)^{A_{\ell_1}A_{\ell_2}} L(\mathrm{sym}^2 f\otimes\chi, s)^{A_{\ell_2}C_{\ell_1}(1)}\\
&\cdot L(\mathrm{sym}^2 g\otimes\chi, s)^{A_{\ell_1}C_{\ell_2}(1)} L(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, s)\\
&\cdot \mathcal{G}_{\ell_1,\ell_2}(s,\chi).
\end{aligned}
$$

where χ is the non-principal Dirichlet character modulo 4, and the constants $A_j, C_j(r)$ $(1\le r\le j-1)$ are given by (21). Clearly, the L-function $\mathcal{G}_{\ell_1,\ell_2}(s,\chi)$ is of degree g_{ℓ_1,ℓ_2} defined as in (42).

For simplicity, we define

$$
\begin{aligned}
E_1(T_1) &:= \int_{T_1}^{2T_1} \left| L\left(\frac{5}{7}+it,\chi\right)\right|^{12} dt,\\
E_2(T_1) &:= \int_{T_1}^{2T_1} \left| \mathcal{G}_{\ell_1,\ell_2}\left(\frac{5}{7}+it,\chi\right)\right|^{2} dt,\\
E_3(T_1) &:= \int_{T_1}^{2T_1} \left| L\left(\mathrm{sym}^{2\ell_1} f\times \mathrm{sym}^{2\ell_2} g\otimes\chi, \frac{5}{7}+it\right)\right|^{6} dt,
\end{aligned}
$$

and in view of Lemma 2.8 and (34), we get

$$
\begin{aligned}
E_1(T_1) &\ll T_1^{1+\varepsilon},\\
E_2(T_1) &\ll T_1^{\frac{2}{7}g_{\ell_1,\ell_2}+\varepsilon},\\
E_3(T_1) &\ll T_1^{\frac{6}{7}(2\ell_1+1)(2\ell_2+1)+\varepsilon},
\end{aligned}
$$

From Lemma 2.3, applying Hölder's inequality, together with Lemma 2.8, the contribution of the vertical line integral E_1 can be bounded by

$$
\begin{aligned}
E_1 &= \frac{4}{2\pi i}\int_{\beta-iT}^{\beta+iT} \widetilde{\mathfrak{G}}_{\ell_1,\ell_2}(s)\frac{x^s}{s}dt\\
&\ll x^{\frac{19}{7}+\varepsilon}\int_1^T \left| D^*_{\ell_1,\ell_2}\left(\frac{5}{7}+it,\chi\right)t^{-1}\right| dt + x^{\frac{19}{7}+\varepsilon}\\
&\ll x^{\frac{19}{7}+\varepsilon} \sup_{1\le T_1\le T/2}\ \sup_{T_1\le t\le 2T_1} \left| L\left(\frac{5}{7}+it,\chi\right)\right|^{A_{\ell_1}A_{\ell_2}-4}
\end{aligned}
$$

$$\cdot\left|L\left(\mathrm{sym}^2 f\otimes\chi,\frac{5}{7}+it\right)\right|^{A_{\ell_2}C_{\ell_1}(1)}\left|L\left(\mathrm{sym}^2 g\otimes\chi,\frac{5}{7}+it\right)\right|^{A_{\ell_1}C_{\ell_2}(1)}$$
$$\cdot E_1(T_1)^{\frac{1}{3}}E_2(T_1)^{\frac{1}{2}}E_3(T_1)^{\frac{1}{6}}T_1^{-1}+x^{\frac{19}{7}+\varepsilon}$$
$$\ll x^{\frac{19}{7}+\varepsilon}T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}, \tag{56}$$

where $\widetilde{\theta}_{\ell_1,\ell_2}$ is defined by (45).

From Lemma 2.3, by appealing to Lemma 2.8 and (35), the integrals over the horizontal segments I_2 and I_3 can be dominated by

$$E_2+E_3\ll\int_{\frac{19}{7}}^{3+\varepsilon}\left|G^*_{\ell_1,\ell_2}(\sigma+iT)x^{\sigma}T^{-1}\right|d\sigma$$
$$\ll\sup_{\frac{5}{7}\le\sigma\le1+\varepsilon}x^{2+\sigma}\left|L(\sigma+iT,\chi)\right|^{A_{\ell_1}A_{\ell_2}}$$
$$\cdot\left|L(\mathrm{sym}^2 f\otimes\chi,\sigma+iT)\right|^{A_{\ell_2}C_{\ell_1}(1)}\left|L(\mathrm{sym}^2 g\otimes\chi,\sigma+iT)\right|^{A_{\ell_1}C_{\ell_2}(1)}$$
$$\cdot\left|L(\mathrm{sym}^{2\ell_1}f\times\mathrm{sym}^{2\ell_2}g\otimes\chi,\sigma+iT)\mathcal{G}_{\ell_1,\ell_2}(\sigma+iT,\chi)\right|T^{-1}$$
$$\ll\frac{x^{3+\varepsilon}}{T}+x^{\frac{19}{7}+\varepsilon}T^{\rho_{\ell_1,\ell_2}-1+\varepsilon}, \tag{57}$$

where ρ_{ℓ_1,ℓ_2} is exactly the same as given by (47). Hence, by inserting (56) and (57) into (55), we obtain

$$\mathcal{S}_2=O_{f,g,\varepsilon}\left(x^{\frac{19}{7}+\varepsilon}T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}+\frac{x^{3+\varepsilon}}{T}\right), \tag{58}$$

where $\widetilde{\theta}_{\ell_1,\ell_2}$ is defined as in (45).

On combining (49), (54) and (58), we can get the asymptotic formula

$$\begin{aligned}\widetilde{S}_{f,g}(x;\ell_1,\ell_2)&=x^3P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x)\\&\quad+O_{f,g,\varepsilon}\left(x^{\frac{19}{7}+\varepsilon}T^{\widetilde{\theta}_{\ell_1,\ell_2}-1+\varepsilon}+\frac{x^{3+\varepsilon}}{T}\right),\end{aligned} \tag{59}$$

where $\widetilde{\theta}_{\ell_1,\ell_2}$ is given by (45). On taking $T=x^{\vartheta_{\ell_1,\ell_2}}$ with $\vartheta_{\ell_1,\ell_2}=\frac{2}{7\widetilde{\theta}_{\ell_1,\ell_2}}$ in (59), we can obtain the desirable asymptotic formula

$$\widetilde{S}_{f,g}(x;\ell_1,\ell_2)=x^3P^*_{A_{\ell_1}A_{\ell_2}-1}(\log x)+O_{f,g,\varepsilon}(x^{3-\vartheta_{\ell_1,\ell_2}+\varepsilon}).$$

This completes the proof of Theorem 1.2.

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