

ON EIGENFUNCTIONS AND NODAL SETS OF THE WITTEN-LAPLACIAN

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Communicated by Lucian Beznea

In this paper, we successfully establish a Courant-type nodal domain theorem for both the Dirichlet eigenvalue problem and the closed eigenvalue problem of the Witten-Laplacian. Moreover, we also characterize the properties of the nodal lines of the eigenfunctions of the Witten-Laplacian on smooth Riemannian 2-manifolds. Besides, for a Riemann surface with genus g , an upper bound for the multiplicity of closed eigenvalues of the Witten-Laplacian can be provided.

AMS 2020 Subject Classification: 35P15, 49J35, 35J15.

Key words: Witten-Laplacian, Dirichlet eigenvalues, Laplacian, the fixed membrane problem, isoperimetric inequalities.

1. INTRODUCTION

Let $(M^n, \langle \cdot, \cdot \rangle)$ be an n -dimensional ($n \geq 2$) complete Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$. Let $\Omega \subseteq M^n$ be a domain in M^n , and $\phi \in C^\infty(M^n)$ be a smooth¹ real-valued function defined on M^n . In this setting, on Ω , the following elliptic operator

$$\Delta_\phi := \Delta - \langle \nabla \phi, \nabla \cdot \rangle$$

can be well defined, where ∇ , Δ are the gradient and the Laplace operators on M^n , respectively. The operator Δ_ϕ with respect to the metric $\langle \cdot, \cdot \rangle$ is called the *Witten-Laplacian* (also called the *drifting Laplacian* or the *weighted Laplacian*). The K -dimensional Bakry–Émery Ricci curvature Ric_ϕ^K on M^n can be defined as follows

$$\text{Ric}_\phi^K := \text{Ric} + \text{Hess}\phi - \frac{d\phi \otimes d\phi}{K - n},$$

where Ric denotes the Ricci curvature tensor on M^n , and Hess is the Hessian operator on M^n associated to the metric $\langle \cdot, \cdot \rangle$. Here, $K > n$ or $K = n$ if ϕ is a

¹ In fact, one might see that $\phi \in C^2$ is suitable to derive our main conclusions in this paper. However, in order to avoid putting too much attention on discussion for the regularity of ϕ and follow the assumption on conformal factor $e^{-\phi}$ for the notion of *smooth metric measure spaces* in many literatures (including, of course, those cited in this paper), without specification, we prefer to assume that ϕ is smooth on the domain Ω .

constant function. When $K = \infty$, the so-called ∞ -dimensional Bakry–Émery Ricci curvature Ric_ϕ (simply, *Bakry–Émery Ricci curvature* or *weighted Ricci curvature*), can be defined as follows:

$$\text{Ric}_\phi := \text{Ric} + \text{Hess}\phi.$$

These notions were introduced by D. Bakry and M. Émery in [2]. Many interesting results (under suitable assumptions on the Bakry–Émery Ricci curvature) have been obtained; see, e.g., [9, 11–13]. Besides, we refer readers to [7, Section 1] for a nice introduction about the motivation, the meaning, and the reason why people considered geometric problems under the assumption of Bakry–Émery Ricci curvature bounded.

For a bounded domain Ω (with boundary $\partial\Omega$) in a given complete n -manifold M^n , $n \geq 2$, consider the Dirichlet eigenvalue problem of the Witten-Laplacian as follows

$$(1) \quad \begin{cases} \Delta_\phi u + \lambda u = 0 & \text{in } \Omega \subset M^n, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and it is not hard to check that the operator Δ_ϕ in (1) is *self-adjoint* with respect to the following inner product

$$(2) \quad (\widehat{h_1}, \widehat{h_2}) := \int_{\Omega} h_1 h_2 d\eta = \int_{\Omega} h_1 h_2 e^{-\phi} dv,$$

$h_1, h_2 \in \widehat{W}_0^{1,2}(\Omega)$, where $d\eta := e^{-\phi} dv$ with dv the Riemannian volume element of M^n , and $\widehat{W}_0^{1,2}(\Omega)$ stands for a Sobolev space, which is the completion of the set of smooth functions (with compact support) $C_0^\infty(\Omega)$ under the following Sobolev norm:

$$(3) \quad \|\widehat{f}\|_{1,2} := \left(\int_{\Omega} f^2 d\eta + \int_{\Omega} |\nabla f|^2 d\eta \right)^{1/2}.$$

Clearly, if $\phi = \text{const}$ is a constant function, then Δ_ϕ degenerates into the Laplacian Δ , and (1) becomes the classical fixed membrane problem (i.e., the Dirichlet eigenvalue problem of the Laplacian). Since Δ_ϕ in (1) is self-adjoint with respect to the inner product $(\widehat{\cdot}, \widehat{\cdot})$, it is natural to ask:

- *Whether or not Δ_ϕ in (1) has some similar spectral properties as the Dirichlet Laplacian defined in $\Omega \subset M^n$?*

The answer is affirmative. For instance, as we have pointed out in [7, Section 1] or [10], using similar arguments² to those of the classical fixed membrane problem of the Laplacian (i.e., the discussions about the existence of

² In fact, as explained in [7, 10], if one wants to get similar spectral properties for the

discrete spectrum, Rayleigh's theorem, Max-min theorem, Domain monotonicity of eigenvalues with vanishing Dirichlet data, etc. Those discussions are standard, and for details, see, for instance, [5]), it is not hard to know:

- The self-adjoint elliptic operator $-\Delta_\phi$ in (1) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows

$$(4) \quad 0 < \lambda_{1,\phi}(\Omega) < \lambda_{2,\phi}(\Omega) \leq \lambda_{3,\phi}(\Omega) \leq \cdots \uparrow +\infty.$$

Each eigenvalue $\lambda_{i,\phi}$, $i = 1, 2, \dots$, in sequence (4) was repeated according to its multiplicity (which is finite and equals to the dimension of the eigenspace of $\lambda_{i,\phi}$). By applying the standard variational principles, one can obtain that the k -th Dirichlet eigenvalue $\lambda_{k,\phi}(\Omega)$ can be characterized as follows³

$$(5) \quad \lambda_{k,\phi}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2 e^{-\phi} dv}{\int_\Omega f^2 e^{-\phi} dv} \middle| f \in \widehat{W}_0^{1,2}(\Omega), f \neq 0, \int_\Omega f f_i e^{-\phi} dv = 0 \right\},$$

where f_i , $i = 1, 2, \dots, k-1$, denotes an eigenfunction of $\lambda_{i,\phi}(\Omega)$. Moreover, the first Dirichlet eigenvalue $\lambda_{1,\phi}(\Omega)$ of the eigenvalue problem (1) satisfies

$$(6) \quad \lambda_{1,\phi}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2 d\eta}{\int_\Omega f^2 d\eta} \middle| f \in \widehat{W}_0^{1,2}(\Omega), f \neq 0 \right\}.$$

If the bounded domain $\Omega \subset M^n$ does not have boundary, then the closed eigenvalue problem of the Witten-Laplacian

$$(7) \quad \Delta_\phi u + \lambda u = 0 \quad \text{in } \Omega$$

can be considered. Similar to the Dirichlet case, it is not hard to know:

- The self-adjoint elliptic operator $-\Delta_\phi$ in (7) *only* has discrete spectrum, and all the eigenvalues in this discrete spectrum can be listed non-decreasingly as follows

$$(8) \quad 0 = \lambda_{0,\phi}^c < \lambda_{1,\phi}^c(\Omega) \leq \lambda_{2,\phi}^c(\Omega) \leq \lambda_{3,\phi}^c(\Omega) \leq \cdots \uparrow +\infty.$$

It is easy to see that the first closed eigenvalue $\lambda_{0,\phi}^c = 0$ is simple and its eigenfunctions are nonzero constant functions. Each nonzero eigenvalue $\lambda_{i,\phi}^c$, $i = 1, 2, \dots$, in sequence (8) was repeated according to its multiplicity (which is finite and equals to the dimension of the eigenspace of $\lambda_{i,\phi}^c$). Applying the

Dirichlet Witten-Laplacian, the key point is using the weighted Riemannian volume element $d\eta = e^{-\phi} dv$ to replace the usual one dv in proofs of those spectral properties of the eigenvalue problem of the Dirichlet Laplacian. Of course, one needs to check carefully whether or not every step still works for the weighted case.

³ For simplification and without specification, in the sequel, we prefer to write $\lambda_{i,\phi}(\Omega)$ as $\lambda_{i,\phi}$. Similar things are done for the closed eigenvalues $\lambda_{i,\phi}^c(\Omega)$ too.

standard variational principles, one can obtain that the k -th nonzero closed eigenvalue $\lambda_{k,\phi}^c(\Omega)$ can be characterized as follows:

$$(9) \quad \lambda_{k,\phi}^c(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla f|^2 e^{-\phi} dv}{\int_{\Omega} f^2 e^{-\phi} dv} \mid f \in \widehat{W}^{1,2}(\Omega), f \neq 0, \int_{\Omega} f f_i e^{-\phi} dv = 0 \right\},$$

where f_i , $i = 1, 2, \dots, k-1$, denotes an eigenfunction of $\lambda_{i,\phi}^c(\Omega)$. Moreover, the first nonzero closed eigenvalue $\lambda_{1,\phi}^c(\Omega)$ of the eigenvalue problem (7) satisfies

$$(10) \quad \lambda_{1,\phi}^c(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla f|^2 d\eta}{\int_{\Omega} f^2 d\eta} \mid f \in \widehat{W}^{1,2}(\Omega), f \neq 0, \int_{\Omega} f e^{-\phi} dv = 0 \right\}.$$

Here, $\widehat{W}^{1,2}(\Omega)$ is the admissible space of the eigenvalue problem (7), and is actually the completion of the set of smooth functions $C^\infty(\Omega)$ under the Sobolev norm (3).

Of course, for a bounded domain with boundary, similar to the Laplacian case, one can also propose the Neumann eigenvalue problem and the mixed eigenvalue problem for the Witten-Laplacian. One can see, e.g., [6, 7], for some interesting spectral properties of the Neumann eigenvalue problem of the Witten-Laplacian.

Maybe one might find previous contents (i.e., the existence conclusion on discrete spectrum and also the characterizations of eigenvalues of the eigenvalue problems (1) and (7) of the Witten-Laplacian) in other literatures (not ours), but for the completion of the structure of this paper and also for the convenience of readers, here we prefer to write them down.

In order to state our main conclusions clearly, we need the following concept, which can be found, e.g., in [5].

Definition 1.1. Let $f : M^n \rightarrow \mathbb{R} \in C^0$. Then the *nodal set* is the set $f^{-1}[0]$, and a *nodal domain* of f is a component on $\overline{M^n} \setminus f^{-1}[0]$.

For the eigenvalue problems (1) and (7), first we can get the following Courant-type nodal domain theorem.

THEOREM 1.2 (Nodal domain theorem for the Witten-Laplacian). *Assume that Ω is a regular domain on a given Riemannian n -manifold M^n . For the Dirichlet eigenvalue problem (1), its eigenvalues consist of a non-decreasing sequence (4). Denote by f_i an eigenfunction of the i -th eigenvalue $\lambda_{i,\phi}$, $i = 1, 2, 3, \dots$, and $\{f_1, f_2, f_3, \dots\}$ forms a complete orthogonal basis of $\widehat{L}^2(\Omega)$. Then for each $k = 1, 2, 3, \dots$, the number of nodal domains of f_k is less than or equal to k . Similar conclusion can be obtained for the closed eigenvalue problem (7) as well, that is, for each $k = 0, 1, 2, 3, \dots$, the number of nodal domains of an eigenfunction of the k -th closed eigenvalue $\lambda_{k,\phi}^c$ is less than or equal to $k + 1$.*

Remark 1.3. (1) As the Laplacian case, one could also try to get Courant-type nodal domain results for the Neumann eigenvalue problem and the mixed eigenvalue problem of the Witten-Laplacian (by using a similar argument to the one given in Section 2). However, in this paper, we just focus on the Dirichlet eigenvalues and closed eigenvalues of the Witten-Laplacian.

(2) Maybe spectral geometers have already known the conclusion of Theorem 1.2, but for the completion of the structure of this paper, we still write it down formally. Another reason is that in another recent work [6], we have used the conclusion of Theorem 1.2 to get the Hong–Krahn–Szegő type isoperimetric inequalities for the second Dirichlet eigenvalue of the Witten-Laplacian on bounded domains in both the Euclidean spaces and the hyperbolic spaces. By making necessary changes to the proof of Courant-type theorem for the characterization of nodal domains to eigenfunctions of the Laplacian in the Riemannian case given by Bérard–Meyer [3], one might get our proof of Theorem 1.2 shown in Section 2.

Second, we can get several interesting properties concerning the nodal set and the multiplicity of eigenvalues.

THEOREM 1.4. *Assume that M^2 is a smooth Riemannian 2-manifold without boundary (not necessarily compact). If a smooth function $f \in C^\infty(M^2)$ satisfies $(\Delta_\phi + h(x))f = 0$, $h \in C^\infty(M^2)$, then except for a few isolated points, the set of nodes of f forms a smooth curve.*

COROLLARY 1.5. *Assume that M^2 is a Riemannian 2-manifold. Then, for any solution of the equation $(\Delta_\phi + h(x))f = 0$, $h \in C^\infty(M)$, one has:*

- *The critical points on the nodal lines are isolated.*
- *When the nodal lines meet, they form an equiangular system.*
- *The nodal lines consist of a number of C^2 -immersed closed smooth curves. Therefore, when M^2 is compact, they are a number of C^2 -immersed circles. A C^2 -immersed circle means $\Phi(\mathbb{S}^1)$, where $\Phi : \mathbb{S}^1 \rightarrow M^2$ is a C^2 immersion.*

THEOREM 1.6. *Suppose that M^2 is a compact Riemann surface of genus g . Then the multiplicity of the i -th closed eigenvalue $\lambda_{i,\phi}^c(M^2)$ is less than or equal to $(2g + i + 1)(2g + i + 2)/2$.*

This paper is organized as follows. The proof of Theorem 1.2 is given in Section 2. The proofs of Theorems 1.4 and 1.6, and Corollary 1.5 are shown in Section 3.

2. PROOF OF THEOREM 1.2

We divide the proof of Theorem 1.2 into two cases:

Case I. When f_k 's nodal domains are normal domains;

Case II. When there is no assumption made on the nodal domains, we only discuss the closed and Dirichlet eigenvalue problems of the Witten-Laplacian, i.e., the eigenvalue problems (1) and (7).

For Case I, let $G_1, G_2, \dots, G_k, G_{k+1}, \dots$ be the nodal domains of f_k , and define

$$\varphi_j := f_k|_{G_j}, \quad \text{on } G_j$$

and

$$\varphi_j := 0, \quad \text{on } \bar{\Omega} \setminus G_j$$

for each $j = 1, 2, \dots, k$. It is not hard to know that there exists a nontrivial function

$$(2.1) \quad \mathcal{F} := \sum_{j=1}^k \alpha_j \varphi_j$$

satisfying

$$(\widehat{\mathcal{F}, f_1}) = (\widehat{\mathcal{F}, f_2}) = \dots = (\widehat{\mathcal{F}, f_{k-1}}) = 0,$$

which is equivalent with

$$(2.2) \quad \sum_{j=1}^k \alpha_j (\widehat{\varphi_j, f_l}) = 0, \quad l = 1, 2, \dots, k-1.$$

In fact, if one treats $\alpha_1, \dots, \alpha_k$ as unknowns and $(\widehat{\varphi_j, f_l})$ as coefficients, then system (2.2) has more unknowns than equations, which implies a nontrivial solution $(\alpha_1, \alpha_2, \dots, \alpha_k)$ must exist. Then using a very similar argument to the proof of Rayleigh's theorem of the Laplacian (see, e.g., [5, Chapter I]), one can obtain

$$(2.3) \quad \lambda_{k,\phi}(\Omega) \leq \frac{\int_{\Omega} |\nabla \mathcal{F}|^2 d\eta}{\int_{\Omega} \mathcal{F}^2 d\eta}.$$

On the other hand, by the definition of $\mathcal{F}$ (i.e., (2.1)), it is not hard to see that

$$(2.4) \quad \begin{aligned} \frac{\int_{\Omega} |\nabla \mathcal{F}|^2 d\eta}{\int_{\Omega} \mathcal{F}^2 d\eta} &= \frac{\sum_{j,l=1}^k \int_{\Omega} \alpha_j \nabla \varphi_j \alpha_l \nabla \varphi_l d\eta}{\sum_{j,l=1}^k \int_{\Omega} \alpha_j \varphi_j \alpha_l \varphi_l d\eta} = \frac{\sum_{j,l=1}^k (\sum_i \int_{G_i} \alpha_j \nabla \varphi_j \alpha_l \nabla \varphi_l d\eta)}{\sum_{j,l=1}^k (\sum_i \int_{G_i} \alpha_j \varphi_j \alpha_l \varphi_l d\eta)} \\ &= \frac{\sum_{i=1}^k \int_{G_i} \alpha_i^2 |\nabla \varphi_i|^2 d\eta}{\sum_{i=1}^k \int_{G_i} \alpha_i^2 \varphi_i^2 d\eta}. \end{aligned}$$

Since for each $i = 1, 2, \dots, k$, by the divergence theorem and the definition of φ_i , it follows that

$$-\int_{G_i} \varphi_i \Delta_\phi \varphi_i d\eta = \int_{G_i} |\nabla \varphi_i|^2 d\eta = \lambda_{k,\phi} \int_{G_i} \varphi_i^2 d\eta,$$

combining this fact with (2.4) yields

$$(2.5) \quad \frac{\int_\Omega |\nabla \mathcal{F}|^2 d\eta}{\int_\Omega \mathcal{F}^2 d\eta} = \lambda_{k,\phi}.$$

Therefore, from (2.3) and (2.5), one easily knows that $\mathcal{F}$ is an eigenfunction corresponding to $\lambda_{k,\phi}$, and $\mathcal{F}|_{G_{k+1}} = 0$. Then using the maximum principle of the second-order elliptic PDEs to system (1), it follows that $\mathcal{F} \equiv 0$ in Ω . This is a contradiction. So, this completes the proof of the assertion of Theorem 1.2 for Dirichlet eigenvalues of the Witten-Laplacian in Case I.

Making suitable minor changes to the above argument, one can get the nodal domain result for closed eigenvalues of the Witten-Laplacian directly.

In Case II, the divergence theorem cannot be directly used for calculations. Therefore, we first need to establish an integral formula for the nodal domains of eigenfunctions.

LEMMA 2.1. *Assume that $u \in C^2$, $v \in C^1$, and D is a nodal domain of the eigenfunction u . Then, we have*

$$-\int_D v \Delta_\phi u d\eta = \int_D \nabla u \cdot \nabla v d\eta.$$

Proof. We may assume that u is positive in D . For each $\epsilon > 0$, define $D_\epsilon := \{x \in D | u(x) > \epsilon\}$ and a function u_ϵ as follows:

$$(2.6) \quad u_\epsilon = \begin{cases} u - \epsilon, & \text{in } D_\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $\{\epsilon_i\}$ is a non-increasing sequence of regular values of u , and ϵ_i approaches 0 as i tends to infinity (by the way, the existence of this sequence can be assured by Sard's theorem). Let $u_i = u_{\epsilon_i}$, $D_i = D_{\epsilon_i}$, and ∂D_i be the smooth boundary of D_i . Applying Green's formula to D_i , we have

$$(2.7) \quad -\int_{D_i} v \Delta_\phi u_i d\eta = \int_{D_i} \nabla u_i \cdot \nabla v d\eta.$$

Using (2.6) and (2.7), one can further obtain

$$\left| -\int_D v \Delta_\phi u d\eta - \int_D \nabla v \cdot \nabla u d\eta \right| = \left| \int_D v \Delta_\phi u d\eta - \int_{D_i} v \Delta_\phi u_i d\eta - \int_{D_i} \nabla u_i \cdot \nabla v d\eta + \int_D \nabla v \cdot \nabla u d\eta \right|$$

$$\begin{aligned}
& \leq \left| \int_{D \setminus D_i} v \Delta_\phi u d\eta \right| + \left| \int_{D \setminus D_i} \nabla v \cdot \nabla u d\eta \right| \\
(2.8) \quad & \leq c |D \setminus D_i|_\phi,
\end{aligned}$$

where c is a positive constant, and $|\cdot|_\phi$ stands for the weighted volume of a prescribed geometric object. Since $D = \bigcup_{i=1}^\infty D_i$, it is easy to know that $|D \setminus D_i|_\phi$ tends to 0 as i tends to infinity. This fact, together with (2.8), implies the conclusion of Lemma 2.1 directly. $\square$

LEMMA 2.2. *Assume that u is an eigenfunction of the eigenvalue problem (1). Let λ be the eigenvalue corresponding to u , and D be a nodal domain of u . Then, we have $\lambda = \lambda_{1,\phi}(D)$, where, by the convention of the usage of notations in Section 1, $\lambda_{1,\phi}(D)$ naturally denotes the first Dirichlet eigenvalue of the Witten-Laplacian on D .*

Proof. It is not hard to see that the eigenfunction u is at least C^2 . By applying Lemma 2.1, one has

$$\lambda \int_D u^2 d\eta = - \int_D u \Delta_\phi u d\eta = \int_D |\nabla u|^2 d\eta \geq \int_{D_\epsilon} |\nabla u|^2 d\eta \geq \lambda_{1,\phi}(D_\epsilon) \int_{D_\epsilon} u_\epsilon^2 d\eta,$$

with the domain D_ϵ and the function u_ϵ constructed as we see in the proof of Lemma 2.1. Due to the fact that $\overline{D_\epsilon} \subset D$, and using the Domain monotonicity of Dirichlet eigenvalues of the Witten-Laplacian (see, e.g., [10, Lemma 1.5]), one knows $\widehat{W}_0^{1,2}(D_\epsilon) \subset \widehat{W}_0^{1,2}(D)$ and $\lambda_{1,\phi}(D_\epsilon) \geq \lambda_{1,\phi}(D)$. Therefore, it follows that

$$\lambda \int_D u^2 d\eta \geq \lambda_{1,\phi}(D) \int_{D_\epsilon} u_\epsilon^2 d\eta.$$

Then, letting u_ϵ approach u in the sense of $\widehat{L}^2$, one has

$$(2.9) \quad \lambda \geq \lambda_{1,\phi}(D).$$

On the other hand, define a positive function $v_\epsilon \in \widehat{W}_0^{1,2}(D_\epsilon)$ which satisfies the equation $\Delta_\phi v_\epsilon = -\lambda_{1,\phi}(D_\epsilon)v_\epsilon$. Using the fact that u is an eigenfunction of λ directly yields

$$(2.10) \quad - \int_{D_\epsilon} v_\epsilon \Delta_\phi u d\eta = \lambda \int_{D_\epsilon} v_\epsilon u d\eta.$$

Since the boundary ∂D_ϵ is smooth, applying the Green's formula, it is not hard to get

$$\begin{aligned}
(2.11) \quad & - \int_{D_\epsilon} v_\epsilon \Delta_\phi u d\eta = - \int_{D_\epsilon} u \Delta_\phi v_\epsilon d\eta + \int_{\partial D_\epsilon} u \frac{\partial v_\epsilon}{\partial \vec{\nu}} d\eta \\
& = \lambda_{1,\phi}(D_\epsilon) \int_{D_\epsilon} u v_\epsilon d\eta + \int_{\partial D_\epsilon} u \frac{\partial v_\epsilon}{\partial \vec{\nu}} d\eta,
\end{aligned}$$

where $\vec{\nu}$ denotes the outward unit normal vector field along ∂D_ϵ . Since $\frac{\partial v_\epsilon}{\partial \vec{\nu}} \leq 0$, we have from equations (2.10) and (2.11) that $\lambda_{1,\phi}(D_\epsilon) \geq \lambda$. For any positive number $a > 0$, by using the characterization (6) with $\Omega = D$, there should exist $\rho \in C^\infty(D)$, which is compactly supported on D , such that

$$\lambda_{1,\phi}(D) \geq \inf \left\{ \frac{\int_D |\nabla \rho|^2 d\eta}{\int_D \rho^2 d\eta} \right\} - a.$$

Based on the choice of the function ρ , there must exist $\epsilon > 0$ for which $\text{supp}(\rho) \subseteq D_\epsilon$, and then by using the characterization (6) with $\Omega = D_\epsilon$, one gets $\lambda_{1,\phi}(D_\epsilon) \leq \inf \left\{ \frac{\int_D |\nabla \rho|^2 d\eta}{\int_D \rho^2 d\eta} \right\}$ directly. So, for a given $a > 0$, there exists $\epsilon > 0$ such that

$$(2.12) \quad \lambda_{1,\phi}(D) \geq \lambda_{1,\phi}(D_\epsilon) - a \geq \lambda - a.$$

Combining equations (2.9) and (2.12), using the fact that $\lambda_{1,\phi}(D_\epsilon)$ is increasing with respect to ϵ , and the arbitrariness of $a > 0$, one can easily obtain that $\lambda_{1,\phi}(D) = \lambda$. $\square$

As one knows that if D is a bounded domain, $u \in \widehat{W}^{1,2}(D) \cap C^0(\overline{D})$ and $u|_{\partial D} = 0$, then $u \in \widehat{W}_0^{1,2}(D)$. So, using Lemmas 2.1 and 2.2, and a similar argument to the proof of Case I, we can get the nodal domain theorem for the Witten-Laplacian in Case II.

3. NODAL LINES AND MULTIPLICITIES OF EIGENVALUES OF THE WITTEN-LAPLACIAN

Assume that f satisfies the equation $(\Delta_\phi + h(x))f = 0$, $h \in C^\infty(M^n)$, and assume that x_0 is a zero point of $f(x)$, i.e., $f(x_0) = 0$. So, one can assume that M^n locates within a small neighborhood of x_0 . We use normal coordinates around x_0 and hence, we can assume we are working in a small open set of the origin of $\mathbb{R}^n$. Using the results of N. Aronszajn [1], f can only vanish up to finite order around the origin. Hence, by L. Bers' main result in [4] (see also [8, Theorem 2.1]), it follows that

$$(3.1) \quad f(x) = \mathcal{P}_N(x) + O(|x|^{N+\varepsilon}),$$

where $\mathcal{P}_N$ is a homogenous polynomial of degree N and $\varepsilon \in (0, 1)$. Besides, $\mathcal{P}_N$ also satisfies the osculating equation at the origin. Since the normal coordinates around x_0 have been used, $\mathcal{P}_N$ should be a spherical harmonic polynomial of degree N . In other words, $\mathcal{P}_N$ satisfies

$$\left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right) \mathcal{P}_N(x) = 0.$$

If $N = 1$, $\mathcal{P}_N(x)$ is a linear polynomial and $df(0) \neq 0$, the set of nodes around 0 is a smooth curve. Below, we discuss the case that N is strictly greater than 1.

First, we need the following result.

LEMMA 3.1 ([8, Lemma 2.3]). *Assume that $\mathcal{P}_N$ is a homogeneous harmonic polynomial of degree N , $N > 1$. Then the set of nodes of $\mathcal{P}_N(x)$ around the origin has a singularity at 0.*

We also need the next lemma.

LEMMA 3.2 ([8, Lemma 2.4]). *Suppose that f and p are smooth function in $\mathbb{R}^n$,*

$$\begin{aligned} f(x) &= p(x) + O(|x|^{N+\varepsilon}) \\ \frac{\partial f(x)}{\partial x_i} &= \frac{\partial p(x)}{\partial x_i} + O(|x|^{N-1+\varepsilon}), \quad N \geq 1, \quad \varepsilon \in (0, 1) \\ \frac{\partial^v}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} p(0) &= 0, \quad 0 \leq v \leq N-1 \end{aligned}$$

and

$$|\nabla p| \geq \text{const}|x|^{N-1}.$$

Then there exists a local C^1 diffeomorphism Φ fixing the origin such that

$$f(x) = p(\Phi(x)).$$

Proof of Theorem 1.4. The nodal set at the origin for $\mathcal{P}_N$ is equal to $\{tx : t > 0, \mathcal{P}_N|_{\mathbb{S}^1}(x) = 0\}$. Assume that π is a set consisting of single points, and then $\mathcal{P}_N^{-1}(0) \setminus \pi = \gamma_0$ is a smooth curve. Therefore, by Lemma 3.2, we can obtain that

$$f^{-1}(0) \setminus \Phi^{-1}(\pi) = \Phi^{-1}(\gamma_0)$$

and $\Phi^{-1}(\gamma_0)$ is C^1 curve. Assume $y \in \Phi^{-1}(\gamma_0)$, and then we have $f(y) = 0$ and $\Phi(y) \in \gamma_0$. So, by L. Bers' main result in [4] (see also [8, Theorem 2.1]), we have $f(x) \sim \mathcal{P}_{\hat{N}}(x)$ around the point y for some $\hat{N} > 0$. Assume that $\hat{N} > 1$. Because $\Phi^{-1}(\gamma_0)$ is a C^1 diffeomorphism in a small neighborhood near y and the nodal set of $\mathcal{P}_{\hat{N}}$ near the origin is C^1 diffeomorphic, but Lemma 3.1 shows that if $\hat{N} > 1$, then $\mathcal{P}_{\hat{N}}$ should have singularity at 0, which contradicts with the fact that $\Phi^{-1}(\gamma_0)$ is a C^1 smooth curve. This means our assumption cannot be true, and so $\hat{N} = 1$. Summing up, $\Phi^{-1}(\gamma_0)$ can only be a smooth curve, which completes the proof of Theorem 1.4. $\square$

The assertions of Corollary 1.5 can be obtained by directly applying Theorem 1.4.

At the end, we would like to give a characterization to multiplicities to the closed eigenvalues of the Witten-Laplacian on compact Riemann surface with genus g . However, first we need the following proposition, which is a direct consequence of Theorem 1.2.

PROPOSITION 3.3. *Assume that M^2 is a compact 2-dimensional smooth manifold with or without boundary. We have:*

(1) *If the boundary $\partial M^2 \neq \emptyset$ is non-empty, then the number of nodal domains of the i -th Dirichlet eigenvalue $\lambda_{i,\phi}$ is less than or equal to i ;*

(2) *If $\partial M^2 = \emptyset$, then the number of nodal domains of the i -th closed eigenvalue $\lambda_{i,\phi}^c$ is less than or equal to $i + 1$.*

We also need the following lemma.

LEMMA 3.4 ([8, Lemma 3.1]). *Assume that M^2 is a compact Riemann surface with genus g , and $\psi_i : \mathbb{S}^1 \rightarrow M^2$ is a closed C^1 curve on M^2 . If $\psi_i(\mathbb{S}^1) \cap \psi_k(\mathbb{S}^1)$, $i \neq k$, consists of only a finite number of points, then*

$$M^2 \setminus \psi_1(\mathbb{S}^1) \cup \cdots \cup \psi_{2g+t}(\mathbb{S}^1)$$

is at least $(t + 1)$ -connected.

Assume that u is an eigenfunction of $\lambda_{i,\phi}^c$. If x_0 is a zero point of u (i.e., $u(x_0) = 0$), and the zero order of x_0 is k , then by Corollary 1.5, we know that there are k nodal lines starting from x_0 . So, if $k > 2g + i$, then by Lemma 3.4, one has that M^2 is at least $(i + 1)$ -connected. However, by Proposition 3.3, M^2 should be at most i -connected. This is a contradiction, and then we have $k \leq 2g + i$.

Proof of Theorem 1.6. We first prove the case $g = 0$ and $i = 1$. Based on the previous argument about the order of the zeros of eigenfunctions, the order of vanishing on the nodal lines is less than or equal to 1. If the multiplicity of the first eigenvalue $\lambda_{1,\phi}^c(M^2)$ is 4, then we can find 4 linearly independent eigenfunctions ψ_i satisfying

$$\Delta_\phi \psi_i + \lambda_{1,\phi}^c(M^2) \psi_i = 0, \quad i = 1, 2, 3, 4.$$

One can find $a_i, b_i \in \mathbb{R}$, $1 \leq i \leq 3$, such that

$$a_i^2 + b_i^2 \neq 0$$

and

$$(a_i \psi_{i+1} - b_i \psi_1)(x_0) = 0.$$

Notice that $a_i\psi_{i+1} - b_i\psi_1$, $i = 1, 2, 3$, are again linearly independent. Because the tangent space is 2-dimensional, we can find 3 real numbers c_1, c_2, c_3 , which are not all zero, such that

$$\sum_{i=1}^3 c_i d(a_i\psi_{i+1} - b_i\psi_1)(x_0) = 0$$

and

$$\sum_{i=1}^3 c_i (a_i\psi_{i+1} - b_i\psi_1)(x_0) = 0.$$

This contradicts the fact that the order of the zero point of the first non-trivial eigenfunction at x_0 must be less than or equal to 1. Therefore, in the case $g = 0$ and $i = 1$, the multiplicity of $\lambda_{1,\phi}^c(M^2)$ is less than or equal to 3. The other cases can be proven almost the same. This is because on $\mathbb{R}^2$ the dimension of the space of constant coefficient partial differential operator of order less than or equal to k is equal to $\sum_{i=1}^{k+1} i$. The proof of Theorem 1.6 is finished. $\square$

Acknowledgments. This research was supported in part by the NSF of China (Grant Nos. 11801496 and 11926352), the Fok Ying-Tung Education Foundation (China), the Key Project of Jiangxi Provincial Natural Science Foundation (Grant No. 20252BAC250003), Hubei Key Laboratory of Applied Mathematics (Hubei University), and Key Laboratory of Intelligent Sensing System and Security (Hubei University), Ministry of Education.

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Received 10 June 2026

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