

ON THE ROOTS OF CERTAIN HILBERT QUASIPOLYNOMIALS AND THEIR Q -ANALOGUES

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In this paper, we investigate the roots of Hilbert quasipolynomials arising from certain rational generating functions and their reciprocity property. We also demonstrate their potential connection to zeta functions enumerating algebraic subobjects from the perspective of q -analogue of rational generating functions.

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1. INTRODUCTION

Whenever a counting problem pertaining to some mathematical object Γ produces a sequence of non-negative integers $(H_\Gamma(n))$ for $n \geq 0$, one may study this problem by encoding this sequence into a generating function

$$P_\Gamma(t) := \sum_{n=0}^{\infty} H_\Gamma(n)t^n.$$

One simply writes $P(t)$ and $H(n)$ if Γ is understood well in the context.

It is a well-known fact (e.g., [1, Proposition 4.5.1]) that such generating function is, in fact, a rational generating function of the form

$$(1) \quad P(t) := \frac{A(t)}{\prod_{i=1}^d (1 - t^{k_i})} = \frac{U(t)}{(1 - t^k)^d}, \quad U(1) \neq 0,$$

where $d, k_1, \dots, k_d \in \mathbb{N}$, $k = \text{l.c.m.}(k_1, \dots, k_d)$, and

$$U(x) = c_0 + c_1x + \dots + c_ex^e \in \mathbb{C}[x]$$

of degree $e < dk$, if and only if $H(n)$ is a *quasipolynomial* with period k of degree $\leq d - 1$, i.e., there exist polynomials $H_0(n), \dots, H_{k-1}(n)$ of degree $\leq d - 1$ such that for $0 \leq i \leq k - 1$,

$$H(n) = H_i(n) \text{ if } n \equiv i \pmod{k},$$

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where polynomials $H_0(n), \dots, H_{k-1}(n)$ are the *constituents* of $H(n)$.

A typical example of such rational generating function is the *Hilbert series* $P_A(t)$ of a finitely generated graded algebra A over a field k . It is defined as

$$P_A(t) = \sum_{n=0}^{\infty} \dim_k(A_n) t^n = \sum_{n=0}^{\infty} H_A(n) t^n,$$

where each A_n consists of the elements of A of degree n and $H_A(n) := \dim_k(A_n)$ is called the *Hilbert polynomial*. In particular, if A is *standard* of Krull dimension d , then

$$(2) \quad P_A(t) = \frac{\prod_{j=1}^r (1 + t + \dots + t^{n_j})}{(1 - t)^d}$$

for some positive integers $n_1, \dots, n_r$; See [3, 4], for details.

Note that (2) is a special case of (1), where $k = k_1 = \dots = k_d = 1$. In general, suppose $P(t)$ is a rational generating function in t of the form

$$(3) \quad P(t) = \sum_{n=0}^{\infty} H(n) t^n = \frac{U(t)}{(1 - t)^d}, \quad U(1) \neq 0,$$

where $d \in \mathbb{N}$ and $U(x) \in \mathbb{C}[x]$ of degree $e \leq d - 1$. In such case, $H(n)$ is simply a polynomial of degree $d - 1$.

In [2], Rodriguez-Villegas studied the polynomials $H(n)$ coming from rational generating functions of the form (3), and showed that for certain cases the roots of $H(n)$ lie on the vertical line. To be precise, for $a \in \mathbb{Z}$, let

$$h_a(x) = \begin{cases} (x+1)(x+2) \cdots (x+a), & a \geq 1, \\ 1, & a < 1, \end{cases}$$

and

$S_a = \{p \in \mathbb{C}[x] \setminus \{0\} \mid p = h_a v, v \in \mathbb{C}[x] \text{ has all its roots on } \Re(x) = -(a+1)/2\}$.

In [2], Rodriguez-Villegas proved that if $U(t)$ has all its roots on the unit circle, then $H \in S_{d-1-e}$.

1.1. Main results

In this article, we study the roots of Hilbert quasipolynomials of rational generating series of the form (1), generalizing the result by Rodriguez-Villegas [2], and extend our results to q -analogue cases.

Let $P(t)$ be a rational generating function in t of the form

$$P(t) := \frac{A(t)}{\prod_{i=1}^d (1 - t^{k_i})} = \frac{U(t)}{(1 - t^k)^d},$$

as above, where $U(1) \neq 0$. For a given $U(t) = c_0 + c_1t + \cdots + c_et^e$ and $0 \leq i \leq k-1$, let

$$\begin{aligned} U_i(t) &= c_it^i + c_{k+i}t^{k+i} + \cdots + c_{q_ik+i}t^{q_ik+i} \\ &= t^i \cdot (c_i + c_{k+i}t^k + \cdots + c_{q_ik+i}t^{q_ik}) \end{aligned}$$

denote the polynomial that consists of all the terms in $U(t)$ of degree congruent to i modulo k . If $U_i(t) \neq 0$, we have a unique integer $q_i \geq 0$, where

$$e_i = q_ik + i \leq e$$

is the degree of $U_i(t)$. Also, for $a \in \mathbb{Z}, k \in \mathbb{N}$, and $0 \leq i \leq k-1$, let

$$h_{a,k,i}(x) = \begin{cases} (x+k-i)(x+2k-i) \cdots (x+ak-i), & a \geq 1, \\ 1, & a < 1, \end{cases}$$

and

$$S_{a,k,i} = \{p \in \mathbb{C}[x] \setminus \{0\} \mid p = h_{a,k,i}v, v \in \mathbb{C}[x]\}$$

has all its roots on the line $\Re(x) = -((a+1)k-2i)/2$.

Under this setting, we study the roots of $H_i(n)$.

THEOREM 1.1. *Let*

$$P(t) = \frac{U(t)}{(1-t^k)^d} = \sum_{n=0}^{\infty} H(n)t^n,$$

and let the notation be as above. For any i , suppose that $U_i(1) \neq 0$ and all the roots of $U_i(t)/t^i$ lie on the unit circle. Then

$$H_i \in S_{d-q_i-1,k,i}.$$

THEOREM 1.2. *For $n \in \mathbb{Z}$, let $H(n) = H_i(n)$ for $n \equiv i \pmod k$. Then $H(n) = 0$ for $n \in \{-1, -2, \dots, -(dk-e-1)\}$.*

Theorem 1.1 generalizes the result by F. Rodriguez-Villegas in [2]. In particular, $k=1$ recovers the main result in [2]. Theorem 1.2 describes the integral roots of $H(n)$ in general, and does not really depend on each $U_i(t)$. This can be interpreted as a property of the whole rational generating function $P(t)$.

In Section 4, drawing on Stanley's reciprocity theorem ([3, Theorem 4.6]), we offer new proofs of Proposition 4.1 and Theorem 1.2. Finally, in Section 5, we explore how our results can be extended to q -analogues of rational generating functions.

2. PRELIMINARIES

In this section, we prove some useful results to prove the main theorems.

LEMMA 2.1. *Let*

$$P(t) = \frac{c}{(1 - t^k)^d}.$$

Then we have $H_0(n) = \frac{c}{(d-1)!k^{d-1}}(n+k)(n+2k)\cdots(n+(d-1)k)$ and $H_1(n) = \cdots = H_{k-1}(n) = 0$.

Proof. Testing that $H_1(n) = \cdots = H_{k-1}(n) = 0$ is trivial. We need to show that

$$H_0(n) = c(d-1)!k^{d-1}(n+k)(n+2k)\cdots(n+(d-1)k).$$

Note that one can write

$$P(t) = \frac{c}{(1 - t^k)^d} = c \sum_{n=0}^{\infty} \binom{n+d-1}{n} t^{kn}.$$

By noting that

$$\binom{n+d-1}{n} = \binom{n+d-1}{d-1} = \frac{(n+1)\cdots(n+d-1)}{(d-1)!},$$

we have

$$H_0(n) = c \binom{(n/k) + d - 1}{d - 1} = \frac{c}{(d-1)!k^{d-1}}(n+k)(n+2k)\cdots(n+(d-1)k). \quad \square$$

Example 2.2. Let

$$P(t) = \frac{1}{(1 - t^2)^2} = \sum_{n=0}^{\infty} H(n)t^n.$$

Then

$$H(n) = \begin{cases} \frac{1}{2}(2+n) & n \equiv 0 \pmod{2}, \\ 0 & n \equiv 1 \pmod{2}. \end{cases}$$

COROLLARY 2.3. *For* $0 \leq j \leq dk - 1$, *where* $j \equiv i \pmod{k}$ *for any* $0 \leq i \leq k - 1$, *let*

$$P(t) = \frac{c_j t^j}{(1 - t^k)^d}.$$

Then, we have

$$H_i(n) = \frac{c_j}{(d-1)!k^{d-1}}(n+k-j)(n+2k-j)\cdots(n+(d-1)k-j)$$

and all other constituents are zero.

Note that for a rational function

$$P(t) = \frac{U(t)}{(1-t^k)^d},$$

one can write

$$P_i(t) := \frac{U_i(t)}{(1-t^k)^d}, \quad P(t) = \sum_{i=0}^{k-1} P_i(t) = \sum_{i=0}^{k-1} \frac{U_i(t)}{(1-t^k)^d}.$$

Each constituent $H_i(n)$ only comes from each $P_i(t)$.

3. ROOTS OF HILBERT QUASIPOLYNOMIALS

In this section, we prove our main results. For $a \in \mathbb{Z}, k \in \mathbb{N}$, and any $0 \leq i \leq k-1$, let

$$h_{a,k,i}(x) = \begin{cases} (x+k-i)(x+2k-i) \cdots (x+ak-i), & a \geq 1, \\ 1, & a < 1, \end{cases}$$

and

$$S_{a,k,i} = \{p \in \mathbb{C}[x] \setminus \{0\} \mid p = h_{a,k,i}v, v \in \mathbb{C}[x]\}$$

has all its roots on the line $\Re(x) = -((a+1)k-2i)/2$.

LEMMA 3.1. *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, and $f \in S_{a,k,i}$ for some a, k, i . Then*

$$f(x-k) - \alpha f(x) \in S_{a-1,k,i}.$$

Proof. This proof closely follows the proof of the Lemma in [2]. Note that $h_{a-1,k,i}$ divides both $f(x)$ and $f(x-k)$. Let

$$\begin{aligned} f(x) &= h_{a-1,k,i}(x)r(x) \\ f(x-k) &= h_{a-1,k,i}(x)s(x) \\ g(x) &= s(x) - \alpha r(x). \end{aligned}$$

We need to prove that g has all its roots on the line $\Re(x) = -(ak-2i)/2$. By our definitions, for every root μ of $r(x)$ that lies on the left half plane $\Re(x) < -(ak-2i)/2$ there is a corresponding root ν of $s(x)$ located symmetrically on the right half plane $\Re(x) > -(ak-2i)/2$, and vice versa.

Let $\beta \in \mathbb{C}$ a root of g . Then $g(\beta) = s(\beta) - \alpha r(\beta) = 0$, giving

$$|s(\beta)| = |r(\beta)|.$$

If $\Re(\beta) < -(ak-2i)/2$ say, then

$$|\beta - \mu| < |\beta - \nu|,$$

for all pairs of corresponding roots μ, ν , giving a contradiction as

$$|r(\beta)| = \prod_{\mu} |\beta - \mu| < \prod_{\nu} |\beta - \nu| = |s(\beta)|.$$

An analogous argument shows that there are also no roots with

$$\Re(\beta) > -(ak - 2i)/2.$$

Hence, all the roots of g lie on the line $\Re(x) = -(ak - 2i)/2$ as required. $\square$

With this lemma, we prove Theorem 1.1.

Proof of Theorem 1.1. Let

$$P(t) = \frac{U(t)}{(1 - t^k)^d} = \sum_{n=0}^{\infty} H(n)t^n,$$

and let the notation be as above. For any i , suppose that $U_i(1) \neq 0$ and all the roots of $U_i(t)/t^i$ are on the unit circle.

We proceed by induction on q_i to show that $H_i \in S_{d-q_i-1,k,i}$

For $q_i = 0$, $U_i(t) = c_i t^i$ and Corollary 2.3 gives

$$H_i(n) = \frac{c_i}{(d-1)!k^{d-1}}(n+k-i)(n+2k-i)\cdots(n+(d-1)k-i),$$

which belongs to $S_{d-1,k,i}$.

Now, if $H_i \in S_{d-q_i-1,k,i}$ is the Hilbert polynomial corresponding to $U_i(t)$, then one can check that the Hilbert polynomial corresponding to $(t^k - \alpha)U_i(t)$ is $H_i^*(x) = H_i(x - k) - \alpha H_i(x)$. By the hypothesis, $|\alpha| = 1$ and Lemma 3.1 gives $H_i^* \in S_{d-q_i-2,k,i}$ as required. $\square$

As discussed, this can be seen as a generalization of the result in [2]. It shows how the roots of each *local* constituent H_i of H are arranged nicely on one critical line $\Re(x) = -((d - q_i)k - 2i)/2$, when all the roots of $U_i(t)/t^i$ are on the unit circle.

4. INTEGRAL ROOTS OF $H_i(n)$ IN GENERAL CASES

In this section, we investigate the integer roots of $H_i(n)$ in general cases, when the roots of $U_i(t)/t^i$ are not necessarily on the unit circle.

PROPOSITION 4.1. *Let the notation be as above. Suppose $U_i(t) \neq 0$ with degree $e_i = q_i k + i$. Then*

$$-(k - i), -(2k - i), \dots, -((d - 1 - q)k - i)$$

are members of the roots satisfying $H_i(n) = 0$.

Proof. For each i , let $e_i = q_i k + i$. Using Corollary 2.3, we have

$$\begin{aligned} H_i(n) &= \frac{1}{(d-1)!k^{d-1}} \sum_{a=0}^{q_i} c_{ak+i}(n+k-(ak+i)) \cdots (n+(d-1)k-(ak+i)) \\ &= \frac{C_i(n)}{(d-1)!k^{d-1}} \prod_{j=1}^{d-1-q_i} (n+jk-i), \end{aligned}$$

where $C_i(x) \in \mathbb{C}[x]$ since $\prod_{j=1}^{d-1-q_i} (n+jk-i)$ is a common factor for

$$(n+k-(ak+i))(n+2k-(ak+i)) \cdots (n+(d-1)k-(ak+i)).$$

Therefore,

$$n = -(k-i), -(2k-i), \dots, -((d-1-q_i)k-i)$$

satisfies $H_i(n) = 0$. $\square$

Note that Proposition 4.1 has no restriction on the roots of U_i . For $n \in \mathbb{Z}$, let $H(n) = H_i(n)$ for $n \equiv i \pmod k$. With Proposition 4.1, we show that $H(n) = 0$ for $n \in \{-1, -2, \dots, -(dk-e-1)\}$, proving Theorem 1.2.

Proof of Theorem 1.2. Proposition 4.1 implies that

$$n = -(k-i), -(2k-i), \dots, -((d-1-q_i)k-i)$$

satisfies $H_i(n) = 0$. As $e = \deg(U(t))$, there has to be an integer r such that $e = e_r = q_r k + r$, $0 \leq q_0, \dots, q_{r-1} \leq q_r$, and $0 \leq q_{r+1}, \dots, q_{k-1} < q_r$. Therefore, looking at the roots of $H_i(n)$, we have at least

$$-1, -2, \dots, -((d-1-(q_r-1))k-(r+1)) = -(dk-q_r k-r-1) = -(dk-e-1)$$

satisfying $H(n) = H_i(n) = 0$ as required. $\square$

Theorem 1.2 implies that no matter how your rational function

$$P(t) = \frac{U(t)}{(1-t^k)^d} = \sum_{n=0}^{\infty} H(n)t^n$$

looks like, globally $H(n)$ takes trivial integer roots

$$n \in \{-1, -2, \dots, -(dk-e-1)\},$$

which are only determined by the degree of the numerator and the denominator of $P(t)$. In particular, this implies that the condition (10) in the Proposition of [2] is always satisfied.

4.1. More on the reciprocity

Here, we provide other proofs for Proposition 4.1 and Theorem 1.2 using the reciprocity of the series.

Given a series of the form

$$A(x) = \sum_{i=0}^{\infty} a_i x^i,$$

let $[x^k](A(x))$ denote the coefficient a_k of x^k in $A(x)$. Recall the following theorem by Popoviciu.

THEOREM 4.2 ([3], Theorem 4.6 (Popoviciu)). *Let $\alpha_1, \dots, \alpha_m$ be distinct nonzero complex numbers, and let $p_1, \dots, p_m$ be polynomials with complex coefficients. Define for all $n \in \mathbb{Z}$, $H(n) = \sum_{i=1}^m p_i(n) \alpha_i^n$. Let*

$$F(\lambda) = \sum_{n=0}^{\infty} H(n) \lambda^n \quad \text{and} \quad \bar{F}(\lambda) = \sum_{n=1}^{\infty} H(-n) \lambda^n.$$

Then, as a rational function of λ , we have $\bar{F}(\lambda) = -F(1/\lambda)$.

For $P(t) = U(t)/(1 - t^k)^d = \sum_{n=0}^{\infty} H(n) t^n$ with $\deg(U) = e < dk$, we can show that $H(n)$ can be represented in the above form, by considering the partial fraction decomposition

$$P(t) = \frac{U(t)}{\prod_{i=0}^{k-1} (1 - \zeta_k^i t)^d} = \sum_{i=0}^{k-1} \sum_{j=1}^d \frac{c_{i,j}}{(1 - \zeta_k^i t)^j}$$

($\zeta_k = e^{2\pi i/k}$) and using the binomial expansion. In particular, we can write

$$H(n) = \sum_{j=0}^{k-1} p_j(n) \zeta_k^{jn}$$

for polynomials $p_0, \dots, p_{k-1}$. Thus, we can apply Theorem 4.2, giving

$$\begin{aligned} H(-n) &= [t^n](-P(t^{-1})) = [t^n] \left(\frac{-U(t^{-1})}{(1 - t^{-k})^d} \right) \\ &= [t^n] \left(\frac{(-1)^{d+1} t^{kd-e} \cdot \tilde{U}(t)}{(1 - t^k)^d} \right) \end{aligned}$$

where $\tilde{U}(t) = t^e U(t^{-1})$ becomes a polynomial in t . This shows that if $0 < n < dk - e$, then $H(-n) = 0$, obtaining Theorem 1.2.

Meanwhile, we showed that $H(n)$ becomes quasipolynomial for $n \geq 0$ of period dividing k , and we could find polynomials $H_0, H_1, \dots, H_{k-1}$ such that

$H_i(km + i) = H(km + i)$ for $m \geq 0$. By considering the expression

$$H(n) = \sum_{j=0}^{k-1} p_j(n) \zeta_k^{jn},$$

substituting $n = km + i$ gives

$$H_i(km + i) = \sum_{j=0}^{k-1} p_j(km + i) \zeta_k^{ij}$$

for $m \geq 0$. As both sides are polynomials in m , the above equation should hold for any $m \in \mathbb{Z}$, and we have $H_i(km + i) = H(km + i)$ for all $m \in \mathbb{Z}$. Thus, we obtain the result of Proposition 4.1 that: if $-(dk - e) < n < 0$ and $n \equiv i \pmod{k}$, then $H_i(n) = H(n) = 0$.

5. GENERALIZATION TO A q -ANALOGUE AND FURTHER STUDIES

In this section, we record some observations and computations to extend the results in this article to q -analogues of rational generating functions.

Consider a q -analogue of the series of a form

$$(4) \quad P(t; q) = \frac{U(t; q)}{\prod_{i=1}^l (1 - q^{a_i} t^k)^{d_i}} = \sum_{n \geq 0} H(n; q) t^n.$$

Here, a_i are all distinct, $d = \sum_{i=1}^l d_i$, and $U(t; q)$ is defined to be a polynomial in q and t whose degree in t is $e < dk$.

If we consider the left side as a rational expression in t having coefficients in $\mathbb{C}(q^{1/k})$, we can factorize the denominator with linear factors of the form $(1 - q^{a_i/k} \zeta_k^j)$, allowing the following partial fraction decomposition

$$P(t; q) = \sum_{i=1}^l \sum_{j=0}^{k-1} \sum_{e=1}^{d_i} \frac{c_{i,j,e}}{(1 - q^{a_i/k} \zeta_k^j t)^e}, \quad c_{i,j,e} = c_{i,j,e}(q^{1/k}) \in \mathbb{C}(q^{1/k}).$$

Applying the binomial expansion gives

$$H(n; q) = \sum_{i=1}^l \sum_{j=0}^{k-1} (q^{1/k})^{a_i n} \zeta_k^{jn} p_{i,j}(n; q^{1/k})$$

where $p_{i,j}(n; q^{1/k})$ is polynomial in n with coefficients in $\mathbb{C}(q^{1/k})$. If we define $H(n; q)$ as in the above expression for all $n \in \mathbb{Z}$, then we can show that Theorem 4.2 is also applicable to $H(n; q)$, by considering q as a fixed real constant. This implies the next proposition.

PROPOSITION 5.1. *For $-(dk - e) < n < 0$, we have $H(n; q) = 0$.*

One can interpret Proposition 5.1 as a q -analogue of Theorem 1.2.

Meanwhile, if we fix the residue of $n \bmod k$, then $H(n; q)$ behaves nicely. Let $n = mk + r$ for $m \geq 0$, $0 \leq r \leq k - 1$. Then

$$H(mk + r; q) = \sum_{i=1}^l \sum_{j=0}^{k-1} q^{a_i m} q^{a_i r/k} \zeta_k^{rj} p_{i,j}(mk + r) = H_r(q^m, m; q)$$

becomes a polynomial in q^m and m with coefficients in $\mathbb{C}(q^{1/k})$. Thus, if we define H_r as a polynomial satisfying the above equation, then we have $H(mk + r; q) = H_r(q^m, m; q)$ for any $m \in \mathbb{Z}$.

We show that $H_r(q^m, m; q)$ is a rational function in q .

PROPOSITION 5.2. *The coefficients of $H_r(q^m, m; q)$ are rational functions in q .*

Proof. We use the approach from Section 2 to evaluate the coefficient. As H_r depends linearly in U , it suffices to prove the statement for $U(t; q) = t^\alpha q^\beta$ for $0 \leq \alpha < dk$ and $\beta \geq 0$. For such case, we have

$$[t^n] \left(\frac{t^\alpha q^\beta}{\prod_{i=1}^l (1 - q^{a_i} t^k)^{d_i}} \right) = [t^{n-\alpha}] \left(\frac{q^\beta}{\prod_{i=1}^l (1 - q^{a_i} t^k)^{d_i}} \right),$$

so $H(n; q)$ is zero unless $n - \alpha$ is divisible by k . Assume $\alpha = \beta = 0$ for simplicity. Then $H_r = 0$ for $r \neq 0$, and

$$H_0(q^m, m; q) = H(mk; q) = [T^m] \left(\frac{1}{\prod_{i=1}^l (1 - q^{a_i} T)^{d_i}} \right)$$

for $T = t^k$. We consider the right-hand side as a rational function in T with $\mathbb{C}(q)$ -coefficients and conduct the partial fraction decomposition in T . Then there exists $c_{i,j}(q) \in \mathbb{C}(q)$ such that

$$\frac{1}{\prod_{i=1}^l (1 - q^{a_i} T)^{d_i}} = \sum_{i=1}^l \sum_{j=1}^{d_i} \frac{c_{i,j}(q)}{(1 - q^{a_i} T)^j},$$

which implies

$$\begin{aligned} [T^m] \left(\frac{1}{\prod_{i=1}^l (1 - q^{a_i} T)^{d_i}} \right) &= \sum_{i=1}^l \sum_{j=1}^{d_i} [T^m] \frac{c_{i,j}(q)}{(1 - q^{a_i} T)^j} \\ &= \sum_{i=1}^l \sum_{j=1}^{d_i} q^{a_i m} c_{i,j}(q) \binom{j + m - 1}{j - 1}. \end{aligned}$$

This is a polynomial in q^m and m with coefficients in $\mathbb{C}(q)$ as required. $\square$

It turns out that we can even give a universal bound for denominators for all $\mathbb{C}(q)$ -coefficients appearing in the expansions of $H_r(q^m, m; q)$ (in polynomials of q^m and m).

PROPOSITION 5.3. *Denote $c_{i,j}(q)$ as above in the proof of Proposition 5.2, and assume $a_1 < \dots < a_l$. Then $c_{i,j}(q) \cdot \prod_{1 \leq i < j \leq l} (1 - q^{a_j - a_i})^d$ is polynomial in $\mathbb{Z}[q]$.*

Proof. Note that if $l = 1$, then $1/(1 - q^{a_1}T)^{d_1}$ itself is a partial fraction decomposition. Hence, $c_{1,j} = 0$ or 1 and we are done.

For general cases, we use induction on d . If $d = 1$ then it implies $l = 1$, so our statement holds. For the inductive step, $l = 1$ was already shown, so it suffices to only consider cases $l \geq 2$. Note the following identity for integers $0 \leq a < b$:

$$\frac{1}{(1 - q^a T)(1 - q^b T)} = \frac{1}{1 - q^{b-a}} \frac{1}{1 - q^a T} - \frac{q^{b-a}}{1 - q^{b-a}} \frac{1}{1 - q^b T}.$$

This implies

$$\begin{aligned} & \frac{1}{(1 - q^{a_1} T)^{d_1} (1 - q^{a_2} T)^{d_2} \prod_{i \geq 3} (1 - q^{a_i} T)^{d_i}} \\ &= \frac{1}{1 - q^{a_2 - a_1}} \frac{1}{(1 - q^{a_1} T)^{d_1} (1 - q^{a_2} T)^{d_2 - 1} \prod_{i \geq 3} (1 - q^{a_i} T)^{d_i}} \\ & \quad - \frac{q^{a_2 - a_1}}{1 - q^{a_2 - a_1}} \frac{1}{(1 - q^{a_1} T)^{d_1 - 1} (1 - q^{a_2} T)^{d_2} \prod_{i \geq 3} (1 - q^{a_i} T)^{d_i}}. \end{aligned}$$

Denominators of both terms

$$\frac{1}{(1 - q^{a_1} T)^{d_1} (1 - q^{a_2} T)^{d_2 - 1} \prod_{i \geq 3} (1 - q^{a_i} T)^{d_i}}$$

and

$$\frac{1}{(1 - q^{a_1} T)^{d_1 - 1} (1 - q^{a_2} T)^{d_2} \prod_{i \geq 3} (1 - q^{a_i} T)^{d_i}}$$

are of degree $d - 1$ in T , so we can apply the inductive hypothesis for them. As there are additional factors having denominator $(1 - q^{a_2 - a_1})$ in front of both of them, both coefficients of $1/(1 - q^{a_i} T)^j$ in each term are in $\mathbb{Z}(q)$ with denominators dividing

$$(1 - q^{a_2 - a_1}) \cdot \prod_{1 \leq i < j \leq l} (1 - q^{a_j - a_i})^{d-1}.$$

As $(1 - q^{a_2 - a_1}) \cdot \prod_{1 \leq i < j \leq l} (1 - q^{a_j - a_i})^{d-1}$ divides

$$(1 - q^{a_2 - a_1}) \cdot \prod_{1 \leq i < j \leq l} (1 - q^{a_j - a_i})^d,$$

it shows $c_{i,j}(q)$ is in $\mathbb{Z}(q)$ with denominator dividing $\prod_{1 \leq i < j \leq l} (1 - q^{a_j - a_i})^d$, proving our inductive step. $\square$

PROPOSITION 5.4. *Suppose $d_1 = d_2 = \cdots = d_l = 1$. Then H_r becomes polynomial in q^m with q -rational coefficients. For $-(dk - e) < -hk + r < 0$, H_r is divisible by $q^{m+h} - 1$.*

Proof. The first statement follows from the fact that the degree of $p_{i,j}(n; q)$ is at most $d_i - 1$. The second statement follows from that:

$$H(-tk + r; q) = H_r(q^{-t}; q) = 0. \quad \square$$

Remark 5.5. For $d_i > 1$, H_r is not always divisible by $q^{m+t} - 1$. For example, let

$$P(t; q) = \frac{1}{(1-t)(1-qt^3)^2} = \frac{1+t+t^2}{(1-t^3)(1-qt^3)^2}.$$

We have $d_1 = 1, d_2 = 2, d = 3, k = 3$, and $e = 2$. So, we have

$$H(-1; q) = H(-2; q) = \cdots = H(-6; q) = 0.$$

Meanwhile, one can compute

$$\begin{aligned} H(3m; q) &= H_0(q^m, m; q) = 1 + 2q + 3q^2 + \cdots + (m+1)q^m \\ &= \frac{1 - (m+2)q^{m+1} + (m+1)q^{m+2}}{(1-q)^2} \end{aligned}$$

and it is not divisible by $q^{m+1} - 1$ or $q^{m+2} - 1$. This does not contradict $H(-3; q) = H_0(q^{-1}, -1; q) = 0$ and $H(-6; q) = H_0(q^{-2}, -2; q) = 0$.

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