

ON INFINITESIMALLY FLAT AFFINE GROUP SCHEMES WITH CONNECTED FIBERS OVER RINGS

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This paper explores Csima and Kottwitz’s comodule theory for representations of multiplicative affine group schemes, framed through comodules over their coordinate algebras. We extend this framework to broader classes of group schemes and provide concise proofs of Jantzen’s classical results concerning flat group schemes with infinitesimally flat structures and connected fibers. This establishes a comprehensive characterization of their representation categories. Using Néron blowup techniques, we reconstruct smooth group schemes with connected fibers and explicitly derive their coordinate algebras. The versatility of our approach is demonstrated through additive group scheme examples, mirroring Csima–Kottwitz’s multiplicative models.

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1. INTRODUCTION

The affine group schemes under consideration, while recently discussed in the context of representations by Csima and Kottwitz [2], have a foundational history. To the best of our knowledge, their first appearance over a discrete valuation ring is due to Waterhouse and Weisfeiler [11] in their study of one-dimensional affine group schemes. These objects subsequently played a central role in the work of Waterhouse [12], where he introduced and developed the theory of Kummer–Artin–Schreier sequence over rings. This line of research was further developed by Sekiguchi, Oort, and Suwa [9], who explored the deformation of $\mathbb{G}_a$ to $\mathbb{G}_m$, and more recently by Tossici [10] and Mézard–Romagny–Tossici [8] in the study of models of group schemes. A concrete prototype for the present study is the flat affine group scheme $\mathbb{G}_m^\lambda$, as introduced in [2]. For a commutative ring A and a non-zero divisor $\lambda \in A$, $\mathbb{G}_m^\lambda$ is the functor representing the group $\mathbb{G}_m^\lambda(R) = \{(a, b) \in R^\times \times R \mid a - 1 = \lambda b\}$ for any A -algebra R , with the group law $(a, b) \cdot (a', b') = (aa', ab' + b)$. It is easy to see that the identity is $(1, 0)$ and the inverse is $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$.

The coordinate algebra of $\mathbb{G}_m^\lambda$ is given by $A[\mathbb{G}_m^\lambda] = A[x, (1 + \lambda x)^{-1}]$, where x is the class of X in the quotient $A[T, T^{-1}, X]/(T - 1 - \lambda X)$. Its Hopf algebra structure is uniquely determined by $\Delta(x) = x \otimes 1 + 1 \otimes x + \lambda x \otimes x$, $\epsilon(x) = 0$, and the antipode $S(x) = -x(1 + \lambda x)^{-1}$.

When $\lambda \in A^\times$, $\mathbb{G}_m^\lambda$ coincides with

$$\mathbb{G}_m = \text{Spec } A[T, T^{-1}].$$

The Hopf algebra homomorphism $\varphi: A[T, T^{-1}] \rightarrow A[X]_{1+\lambda X}$, $T \mapsto 1 + \lambda X$, induces an isomorphism $\mathbb{G}_m^\lambda \otimes_A A[\lambda^{-1}] \cong \mathbb{G}_{m, A[\lambda^{-1}]}$.

Following the approach used in [2], the group scheme $\mathbb{G}_m^\lambda$ can be characterized using a family of linear forms. Let $\{L_n\}_{n \geq 0}$ be the linear forms $L_n: A[\lambda^{-1}][T, T^{-1}] \rightarrow A[\lambda^{-1}]$ defined by:

$$L_n \left(\sum_{i \in \mathbb{Z}} b_i T^i \right) = \sum_{i \in \mathbb{Z}} \lambda^n \binom{i}{n} b_i.$$

As shown in [2], the coordinate algebra is realized as

$$A[\mathbb{G}_m^\lambda] = \{g \in A[\lambda^{-1}][T, T^{-1}] \mid L_n(g) \in A \text{ for all } n \geq 0\}.$$

This explicit construction perfectly aligns with the general theory of infinitesimally flat group schemes over Dedekind rings ([7, Chapter 10]), where the coordinate algebra is recovered from its generic fiber via distributions:

$$A[G] = \{f \in K[G_K] \mid \nu(f) \in A \text{ for all } \nu \in \text{Dist}(G)\}.$$

In our case, the maps $\{\lambda^{-n} L_n\}_{n \geq 0}$ play the role of a basis for the algebra of distributions.

When A is a discrete valuation ring with uniformizer π and $\lambda = \pi^n$ for any non-negative integer n , gives a flat affine group schemes $G_n = \mathbb{G}^{\pi^n}$ over A has general fiber $\mathbb{G}_{m, K}$ (K is the fractional field of A). These form a projective system

$$\cdots \longrightarrow G_2 \xrightarrow{\times \pi} G_1 \xrightarrow{\times \pi} G_0 = \mathbb{G}_m.$$

Here, the group schemes G_n coincides with the Néron blowup at n times in succession at unit. This yields a flat group scheme $\mathbb{G}_{m, \infty} := \varprojlim_n G_n$ with general fiber $\mathbb{G}_{m, K}$ but it is not finite type [5, Example 1.1]. This is similar to the construction as in [2]. The coordinate algebras $A[G_n]$'s and $A[\mathbb{G}_{m, \infty}]$ corresponding to G_n 's and $\mathbb{G}_{m, \infty}$ is well known by Neron blowup theory, [5, Example 1.1 and Section 2.5]. At this point, we remark that $\mathbb{G}_{m, \infty}$ is an example of an infinitesimally flat group scheme.

The paper is organized as follows: in Section 2, we present an alternative proof of the results from [7, Chapter 10, Proposition 10.12] adapted to flat, infinitesimally flat affine group schemes with connected fibers, and we

describe their representation categories. Section 3 introduces the theory of Néron blowups to study smooth group schemes with connected fibers, providing a detailed description of their coordinate algebras. Finally, we conclude with explicit examples, including those of additive group schemes.

2. ON INFINITESIMALLY FLAT AFFINE GROUP SCHEMES

2.1. Affine group schemes, representations and distribution algebra

Definition 2.1 ([7]). An affine group scheme G over a commutative ring A is a functor

$$G: \text{Alg}_A \longrightarrow \text{Grp}$$

from the category of A -algebras to the category of groups that is representable. Specifically, there exists an A -algebra $A[G]$, called the coordinate algebra of G , equipped with the structure of a Hopf algebra (comultiplication Δ , counit ϵ , antipode S), such that for every A -algebra R ,

$$G(R) = \text{Hom}_{A\text{-Alg}}(A[G], R)$$

inherits a natural group structure from the Hopf algebra operations on $A[G]$.

The group axioms (associativity, unitality, invertically) correspond to the following Hopf algebra conditions: $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ (Coassociativity), $(\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta$ (Counitality), $m \circ (S \otimes \text{id}) \circ \Delta = \epsilon \cdot 1_{A[G]} = m \circ (\text{id} \otimes S) \circ \Delta$ (Antipode).

These are equivalently expressed by the commutativity of standard Hopf algebra diagrams:

$$\begin{array}{ccc} A[G] & \xrightarrow{\Delta} & A[G] \otimes A[G] \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ A[G] \otimes A[G] & \xrightarrow{\text{id} \otimes \Delta} & A[G] \otimes A[G] \otimes A[G] \end{array} \qquad \begin{array}{ccc} A[G] & \xrightarrow{\Delta} & A[G] \otimes A[G] \\ & \searrow \text{id} & \downarrow \epsilon \otimes \text{id} \\ & & A[G] \end{array}$$

$$\begin{array}{ccc} A[G] & \xrightarrow{\Delta} & A[G] \otimes A[G] \\ \downarrow \Delta & \searrow S & \downarrow m \circ (S \otimes \text{id}) \\ A[G] \otimes A[G] & \xrightarrow{m \circ (\text{id} \otimes S)} & A[G] \end{array}$$

We remark that the coordinate algebra of an affine group scheme is always a commutative Hopf algebra and G is called flat if $A[G]$ is as flat as an A -module.

Example 2.2. (i) The additive group scheme $\mathbb{G}_a$ over A is defined by the functor

$$\mathbb{G}_a: \mathbf{Alg}_A \rightarrow \mathbf{Grp}, \quad R \mapsto (R^+, +),$$

where R^+ denotes the additive group of R . It is represented by the polynomial ring $A[T]$, equipped with the following Hopf algebra structure: coproduct $\Delta: A[T] \rightarrow A[T] \otimes A[T]$, $\Delta(T) = T \otimes 1 + 1 \otimes T$, the counit map $\epsilon: A[T] \rightarrow A$ is the map $\epsilon(T) = 0$ and the antipode is given by $S(T) = -T$.

(ii) The multiplicative group scheme $\mathbb{G}_m$ over A is defined by the functor

$$\mathbb{G}_m: \mathbf{Alg}_A \rightarrow \mathbf{Grp}, \quad R \mapsto (R^\times, \cdot),$$

where $R^\times$ denotes the group of units of R . It is represented by the Laurent polynomial algebra $A[T, T^{-1}]$, equipped with the Hopf algebra structure: $\Delta(T) = T \otimes T$ (Coproduct); $\epsilon(T) = 1$ (Counit); $S(T) = T^{-1}$ (Antipode).

Definition 2.3. Let G be a flat affine group scheme over A , and let $A[G]$ be its coordinate ring, which is flat as an A -module. Equip $A[G]$ with the coproduct $\Delta: A[G] \rightarrow A[G] \otimes A[G]$ and counit $\varepsilon: A[G] \rightarrow A$. A right $A[G]$ -comodule is a pair (M, ρ) , where M is an A -module and

$$\rho: M \rightarrow M \otimes A[G]$$

is an A -linear map satisfying:

$$\begin{cases} (\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta) \circ \rho; \\ (\text{id} \otimes \varepsilon) \circ \rho = \text{id}, \end{cases}$$

where id always denotes the identity map.

The category of representations $\mathbf{Rep}(G)$ (resp. $\mathbf{Rep}_f(G)$, $\mathbf{Rep}_f^{\text{flat}}(G)$) denotes the category of (finite, flat and finite) right $A[G]$ -comodules, where finite means M is finitely generated as an A -module. This follows from the duality between affine group schemes and their coordinate Hopf algebras, as detailed in [7, Section 2.8].

Definition 2.4 ([7]). Let $I = \ker \epsilon$ be the augmentation ideal of $A[G]$ (kernel of the counit map ϵ). We define the distribution algebra of G (the distribution on G with support at unit) to be

$$\mathbf{Dist}(G) = \{\sigma \in A[G]^* \mid \sigma(I^n) = 0 \text{ for some integer } n \geq 1\}.$$

In the literature, $\mathbf{Dist}(G)$ is also called *hyperalgebra* or *algebra of invariant differential operator* on G . Now, for each $n \geq 1$, the set

$$\mathbf{Dist}_n(G) = \{\sigma \in A[G]^* \mid \sigma(I^{n+1}) = 0\} = (A[G]/I^{n+1})^*.$$

The inclusion $\mathbf{Dist}_n(G) \cdot \mathbf{Dist}_m(G) \subset \mathbf{Dist}_{n+m}(G)$ follows from $(A[G]/I^n)^* \cdot (A[G]/I^m)^* \subset (A[G]/I^{n+m})^*$. Thus,

$$\mathbf{Dist}(G) = \varinjlim (A[G]/I^{n+1})^* = \varinjlim \mathbf{Dist}_n(G).$$

Remark 2.5. By [7, Section I.7.11], every comodule (M, ρ) has a natural $\mathbf{Dist}(G)$ -module structure. The action is defined by

$$\nu \rightharpoonup m = (\mathrm{id}_M \otimes \nu) \circ \rho(m) \quad \text{for } \nu \in \mathbf{Dist}(G), m \in M,$$

which corresponds to the composition:

$$M \xrightarrow{\rho} M \otimes A[G] \xrightarrow{\mathrm{id}_M \otimes \nu} M \otimes A \cong M.$$

Example 2.6. We compute distribution algebras for key group schemes (see [7, Example 7.8]):

1. Additive group $\mathbb{G}_{a,A} = \mathrm{Spec} A[T]$ has augmentation ideal $I = (T)$ and $A[T]/I^{n+1}$ has the residue classes $\{1, T, T^2, \dots, T^n\}$ as a set of basis. Let $\gamma_n \in \mathbf{Dist}(\mathbb{G}_{a,A})$ satisfy $\gamma_n(T^m) = \delta_{m,n}$. Then $\mathbf{Dist}(\mathbb{G}_{a,A})$ is a free A -module with basis $\{\gamma_n\}_{n \geq 0}$, and $\mathbf{Dist}_n(\mathbb{G}_{a,A})$ has basis $\{\gamma_m\}_{0 \leq m \leq n}$.

2. Multiplicative group $\mathbb{G}_{m,A} = \mathrm{Spec} A[T, T^{-1}]$ with augmentation ideal $I = (T - 1)$. The basis of $A[T, T^{-1}]/I^{n+1}$ is $\{1, T - 1, (T - 1)^2, \dots, (T - 1)^n\}$. Let $\delta_n \in \mathbf{Dist}(\mathbb{G}_{m,A})$ satisfy $\delta_n((T - 1)^m) = \delta_{n,m}$. Using $T^k = ((T - 1) + 1)^k$, we compute $\delta_i(T^k) = \binom{k}{i}$. The distribution algebra $\mathbf{Dist}(\mathbb{G}_{m,A})$ is free with basis $\{\delta_n\}_{n \geq 0}$, and $\mathbf{Dist}_n(\mathbb{G}_{m,A})$ has basis $\{\delta_i\}_{0 \leq i \leq n}$.

Definition 2.7 ([6] or [7]). We say that G is infinitesimally flat if $A[G]/I^n$ is A -finitely generated and A -projective for any $n \geq 1$.

This is equivalent to I^n/I^{n+1} if it is an A -finitely generated projective module for $n \geq 1$ (equivalently finitely and presented flat). The projectivity of $A[G]/I^n$ for all $n \geq 1$ and the exact sequence

$$0 \rightarrow I^n/I^{n+1} \rightarrow A[G]/I^{n+1} \rightarrow A[G]/I^n \rightarrow 0$$

yield I^n/I^{n+1} is an A -pure submodule of an A -finitely generated projective, and it is also an A -finitely generated projective module. Therefore, if G is an infinitesimally flat group scheme, then $\mathbf{Dist}(G)$ is projective as an A -module. We also remark that if A is a field, then any algebraic group schemes G is infinitesimally flat and if G is an A -smooth affine group scheme then G is an infinitesimally flat group scheme ([1, VIII.1.3]). Recall that the set of distributions $\mathbf{Dist}(G)$ supported at the identity of G possesses a Hopf algebra structure, where the multiplication and comultiplication are induced by the group scheme structure of G (see, e.g., [7, I.2.10]). For our specific case $\mathbb{G}_m^\lambda$, using the basis $\{L_n\}$, the structure is given by:

1. Multiplication: $L_n * L_m = \binom{n+m}{n} L_{n+m}$;
2. Counit: $\epsilon(L_n) = \delta_{n,0}$;
3. Antipode: $S(L_n) = (-1)^n \sum_{j=0}^n \binom{n+j-1}{j} \lambda^j L_{n+j}$.

2.2. On infinitesimally flat affine group schemes with connected fibers

In this subsection, we give a short proof for [7, Proposition 10.12] and apply it to describe the category $\mathbf{Rep}_f^{\text{flat}}(G)$ of representations of G using its distribution algebra.

Following [7, Section 10.11], if G is smooth then $A[G]$ is flat over A and finitely presented as an A -algebra. The natural map from $A[G]$ to $K[G_K]$ is injective, and we can identify $A[G]$ with its image. Since I^n/I^{n+1} is an A -finitely generated projective module for all $n \geq 1$, G is infinitesimally flat and $\mathbf{Dist}_n(G)$ can be embedded into $\mathbf{Dist}_n(G_K) \cong \mathbf{Dist}_n(G) \otimes_A K$ (similarly for $\mathbf{Dist}(G) \hookrightarrow \mathbf{Dist}(G_K)$). Here, G_K denotes the scalar extension of G to K , referred to as the generic fiber of G .

The dual space $\mathbf{Dist}(G)^*$ naturally inherits an associative algebra structure from the Hopf structure of $\mathbf{Dist}(G)$. As Jantzen demonstrates, the operations defined in Sections 7.3–7.6 make $\mathbf{Dist}(G)$ a Hopf algebra over R [7, I.7.7], whereby the comultiplication of $\mathbf{Dist}(G)$ dually induces the multiplication on $\mathbf{Dist}(G)^*$. In this context, we observe that the natural pairing $\langle \cdot, \cdot \rangle : A[G] \times \mathbf{Dist}(G) \rightarrow A$ is non-degenerate.

The following proposition is an extension of [4, Proposition 3.1.7], dropping the finite type assumption on G .

PROPOSITION 2.8. *Let A be a Dedekind ring and G be a flat, infinitesimally flat group scheme over A . If the generic fiber G_K is connected and of finite type, then the coordinate algebra $A[G]$ is a specially locally finite coalgebra. That is, $A[G]$ is the union of subcoalgebras, each of which is a saturated A -submodule of $A[G]$.*

Proof. Since G_K is a connected group scheme of finite type over a field K , it is automatically irreducible (cf. [3, SGA 3, Example VI_A, 2.1.2]). By [6, Lemma II.2.1.15], the irreducibility of G_K implies that the canonical inclusion $K[G_K] \hookrightarrow \mathbf{Dist}(G_K)^*$ is dense in the sense of Hashimoto [6, II.2.2.7]. This means that its image is dense with respect to the dual topology, or equivalently, that the natural pairing between $\mathbf{Dist}(G_K)$ and $K[G_K]$ is non-degenerate. Let $I = \ker \epsilon$ be the augmentation ideal of $A[G]$. Then $\bigcap_n (I_K)^n = 0$, where $I_K = I \otimes_A K$.

For any $c \in A[G] \subset K[G]$, there exists a finite-dimensional subcoalgebra $C_K \subset K[G]$ containing c by the fundamental theorem of coalgebras. Let

$$C = C_K \cap A[G].$$

Then $A[G]/C$ is torsion-free, hence A -flat.

Since $\bigcap_n (I_K)^n = 0$, there exists $n \geq 1$ such that $C_K \cap (I_K)^n = 0$. As $(I_K)^n = I^n \otimes_A K$, we have

$$(C \cap I^n) \otimes_A K = C_K \cap (I^n \otimes_A K) = 0.$$

Thus, $C \cap I^n$ is torsion. But $C \cap I^n \subset C$ is A -flat (hence torsion-free), forcing $C \cap I^n = 0$. The injection $C \hookrightarrow A[G]/I^n$ then holds. Since G is infinitesimally flat, $A[G]/I^n$ is finitely generated over the Noetherian ring A . Thus, C is finitely generated. By [4, Proposition 3.1.5], $A[G]$ is a *specially locally finite coalgebra* (consequently, $A[G]$ is a Mittag-Leffler A -module). $\square$

COROLLARY 2.9. *Let G be a flat, infinitesimally flat group scheme of finite type with connected general fiber over a Dedekind ring A . The coordinate algebra $A[G]$ is projective as a A -module.*

Proof. We note that $A[G]$ is countably finitely generated over A since G is of finite type. So the Mittag-Leffler property of $A[G]$ allows us to conclude that $A[G]$ is a projective module. $\square$

LEMMA 2.10. *Let G be a flat, infinitesimally flat group scheme with general fiber G_K connected group scheme of finite type (for instance, G_K is irreducible) over a Noetherian domain A .*

1. *Let*

$$L := \{f \in K[G_K] \mid \nu(f) \in A \text{ for all } \nu \in \mathbf{Dist}(G)\}.$$

Then L is A -subalgebra of $K[G]$ which contains $A[G]$ and $L \otimes K \cong K[G]$.

2. *The natural map $\phi : L \longrightarrow \mathbf{Dist}(G)^*$ defined by $\phi(f)(\nu) = \nu(f)$ for all $f \in L$, is an injection.*

Proof. 1. Obviously; 2. Suppose $\phi(f)(\nu) = \nu(f) = 0$ for all $\nu \in \mathbf{Dist}(G)$. We show $f = 0$. For each $n \geq 0$, choose $\nu \in \mathbf{Dist}_n(G)$, which annihilates I^{n+1} . By hypothesis, $\nu(f) = 0$. Since $\mathbf{Dist}_n(G_K) \cong \mathbf{Dist}_n(G) \otimes_A K$ (via flat base change), this implies $f \in I_K^n$ for all n . By the Krull intersection theorem over the field K , $\bigcap_{n \geq 0} I_K^n = 0$. Thus $f = 0$, as required. $\square$

In what follows, we frequently utilize the notion of A -purity. Recall that a morphism of A -modules $f : M \rightarrow N$ is said to be A -pure in the sense of Hashimoto [6] if for every A -module P , the map $f \otimes 1_P : M \otimes_A P \rightarrow N \otimes_A P$ is injective. Specifically, we have the following crucial lemma.

LEMMA 2.11 ([6, Lemma 2.1.4]). *Let $\psi : L' \rightarrow L$ be an A -pure homomorphism of algebras over any Noetherian domain A with fraction field K , and let L be an A -flat module. If $\psi_K = \psi \otimes_A K$ is an isomorphism then ψ is an isomorphism.*

Proof. Let $\psi: L' \rightarrow L$ be an A -pure homomorphism. Identifying L' as a subalgebra of L , the cokernel $\text{Coker } \psi = L/L'$ is A -flat by [6, Lemma 2.1.4]. Tensoring the exact sequence

$$0 \rightarrow L' \xrightarrow{\psi} L \rightarrow \text{Coker } \psi \rightarrow 0$$

with K yields

$$0 \rightarrow L' \otimes_A K \xrightarrow{\psi \otimes K} L \otimes_A K \rightarrow \text{Coker } \psi \otimes_A K \rightarrow 0.$$

If $\psi \otimes K$ is surjective, then $\text{Coker } \psi \otimes_A K = 0$. As $\text{Coker } \psi$ is flat and torsion-free, this forces $\text{Coker } \psi = 0$, so ψ is an isomorphism. $\square$

PROPOSITION 2.12 ([7, Propositions 10.12 and 10.13]). *Suppose that G is flat, infinitesimally a flat affine group scheme of finite type with connected fibers (for instance, irreducible fibers) over a Dedekind ring. Then*

1. $A[G]$ is projective as an A -module and

$$A[G] = \{f \in K[G_K] \mid \nu(f) \in A \text{ for all } \nu \in \mathbf{Dist}(G)\}.$$

2. Let (V, ρ) be a finite dimensional representation of G_K . Suppose that M is an A -module such that M is flat finitely generated as an A -module, $M_K := M \otimes_A K \cong V$ and $\mathbf{Dist}(G)M \subset M$. Then M is a representation of G .

Proof. 1. The projectivity of $A[G]$ follows from Corollary 2.9. As G is an infinitesimally flat affine group scheme of finite type with connected fibers, the inclusion $\mathbf{Dist}(G) \hookrightarrow A[G]^*$ is universally dense by [6, II. Theorem 2.2.8]. By [6, Lemma I.3.8.5], the map

$$\phi: A[G] \rightarrow \mathbf{Dist}(G)^*, \quad \phi(f)(\nu) = \nu(f),$$

is pure. Consider the embeddings $A[G] \hookrightarrow L \hookrightarrow \mathbf{Dist}(G)^*$, where L is as in Lemma 2.10, such that $\mathbf{Dist}(G)^*/A[G]$ is A -flat. Since A is Dedekind, $L/A[G]$ is also A -flat, implying $A[G]$ is pure in L ([6, I. Lemma 2.1.4]). By Lemma 2.11, the canonical inclusion $A[G] \hookrightarrow L$ induces an isomorphism $A[G] \otimes K \cong L \otimes K$ upon base change to K , thereby concluding the proof.

2. The proof is a variant of [7, Proposition 10.13] and, for the sake of convenience, we give more details here. Since A is a Dedekind ring and M is projective of finite rank, we can assume M is free on A . Thus, we can choose a basis $\{e_j\}_{j=1}^n$ of M such that it is also a basis of V . By V , it is a comodule on $K[G_K]$, and we have $a'_{ij}s \in K[G_K]$, such that

$$\rho_V(e_j) = \sum_{i=1}^r e_i \otimes a_{ij}, \text{ for all } i = \overline{1, r};$$

hence

$$\nu \rightharpoonup e_j = (\mathrm{id}_V \otimes \nu) \rho_V(e_j) = (\mathrm{id}_V \otimes \nu) \left(\sum_{i=1}^r e_i \otimes a_{ij} \right) = \sum_{i=1}^r \nu(a_{ij}) e_i.$$

According to the first part, M is an $A[G]$ -comodule if $\nu(a_{ij}) \in A$. $\square$

Remark 2.13. Proposition 2.12 remains in force if G is smooth with connected fibers over a Dedekind ring.

The notation $\mathcal{C}$ is the category of objects and morphisms defined as follows:

- $\mathcal{C}_1$) An object in $\mathcal{C}$ is a pair (M, V) where V is a finite dimensional representation of G_K , M is a flat finite A -submodule of V such that $M \otimes K \cong V$ and $\mathbf{Dist}(G)M \subset M$.
- $\mathcal{C}_2$) A morphism in $\mathcal{C}$ is a homomorphism between (M, V) and (M', V') defined as $\sigma : V \rightarrow V'$ in $\mathbf{Rep}_f G_K$ such that $\sigma(M) \subset M'$.

At this point, it is worth noticing from Proposition 2.12 that the objects of $\mathcal{C}$ make sense. Assuming that $(M, \rho) \in \mathbf{Rep}_f^{\mathrm{flat}} G$, the conditions of the comodule in the above definition are satisfied. The homomorphism $M \xrightarrow{\rho} M \otimes A[G]$ can be scalarly extended onto K to $M_K \xrightarrow{\rho_K} M_K \otimes_K K[G]$ and the conditions of the comodule are always satisfied. Hence, (M_K, ρ_K) is a representation of G_K . Thus, we can define the functor

$$\mathcal{F} : \mathbf{Rep}_f^{\mathrm{flat}} G \rightarrow \mathcal{C}, M \mapsto (M, M \otimes_A K).$$

We easily obtain the following proposition.

PROPOSITION 2.14. *Assuming that A is a Dedekind ring, the functor*

$$\mathcal{F} : \mathbf{Rep}_f^{\mathrm{flat}} G \rightarrow \mathcal{C}$$

is a categorical equivalence.

Remark 2.15. Let A be a Dedekind ring, for any representation M of G (not necessarily flat as an A -module), there exists a short exact sequence in the category of all representations of G :

$$0 \rightarrow M_2 \rightarrow M_1 \rightarrow M \rightarrow 0,$$

in which $M_1, M_2 \in \mathbf{Rep} G$ whose underlying A -modules are flat. If the representation $M \in \mathbf{Rep}_f G$ then so are M_1, M_2 . This can be argued similarly to [2, Lemma 1.4.1].

3. NÉRON BLOWUP OF FLAT AFFINE GROUP SCHEME OVER DISCRETE VALUATION RING AND ADDITIVE GROUP SCHEMES

3.1. Affine group scheme on a DVR generated by Neron blow-ups

We work with a discrete valuation ring A (DVR) having uniformizer π , quotient field K , and residue field k , adhering to the framework in [5] for this section.

Let G be a smooth affine group scheme of finite type on A . By flatness, we have $A[G] \subset K[G]$. Let H_0 be a subgroup of the closed fiber G_k of G . Then H_0 is defined by a proper ideal J containing π , J can be obtained by taking the inverse image in $A[G]$ of the defining ideal of H_0 in $k[G_k]$. The Neron blowup of G corresponding to H_0 is a flat affine group scheme of finite type G^{H_0} with the coordinate algebra generated by $A[G]$ and $\pi^{-1}J$:

$$A[G^{H_0}] := A[G][\pi^{-1}J] \subset K[G].$$

From the definition, we immediately have that G^{H_0} is flat on A and the monomorphism $A[G] \rightarrow A[G^{H_0}]$ induces a homomorphism between the group schemes $G^{H_0} \rightarrow G$. Furthermore, $G_K^{H_0} \cong G_K$ (see [11, Proposition 1.1]).

Example 3.1. 1. Given $G = \mathbb{G}_a$ and $H_0 = \{e\} \subset G_k$, then

$$J = (\pi, T) \subset A[T] \text{ and } G^{\{e\}} = \text{Spec}(A[T][\pi^{-1}T]) \cong \text{Spec}(A[X]),$$

and the mapping $G^{\{e\}} \rightarrow G$ is defined by $A[T] \rightarrow A[X]$, $T \mapsto \pi X$.

2. Given $\mathbb{G}_{m,A} = \text{Spec}(A[T, T^{-1}])$, $H_0 = \{e\} \subset G_k$, then

$$J = (\pi, T^{-1}, T - 1).$$

Therefore, $\mathbb{G}_m^{\{e\}}$ is the spectrum of the ring $A[T^{-1}, \pi^{-1}(T - 1)]$. By changing the variables $X = T^{-1}$, $Y = \pi^{-1}(T - 1)$, $T = \pi Y + 1$, we have

$$\mathbb{G}_m^{\{e\}} = \text{Spec}(A[X, Y]/X(\pi Y + 1) = 1))$$

with the coalgebra structure defined by

$$\Delta(X) = X \otimes X, \Delta(Y) = Y \otimes 1 + 1 \otimes Y + \pi Y \otimes Y.$$

If we blowup at n times in succession at the unit, we get a formalism affine group scheme $G_n = \text{Spec}(A[X, Y]/X(\pi^n Y + 1) = 1))$, with the general fiber $G_n \otimes K \cong \mathbb{G}_{m,K}$.

Remark 3.2. Let G be a smooth affine group scheme with connected fibers such that $G_K \cong \mathbb{G}_{m,K}$. Then $A[G] \cong \text{Spec}A[T^{-1}, \pi^{-n}(T - 1)] = A[G_n]$ by [11, Theorem 2.5].

The construction of Csima and Kottwitz in [2, Section 1] is a particular case of the following definition.

Definition 3.3 ([5, Section 2]). Let G be a flat group scheme over A . The automatic blowup $G_\infty \rightarrow G$ of the identity is the limit of the diagram

$$\cdots \longrightarrow G_n \longrightarrow \cdots G_1 \longrightarrow G$$

where $G_{n+1} \rightarrow G_n$ is the Néron blowup of $\{e\} \subset G_n \otimes k$. If $A[G][\pi^{-n}I] = A[G_n]$, then $A[G_{n+1}] = A[G_n][\pi^{-1}I_n]$ is just $A[G_n][\pi^{-n-1}I]$, which equals $A[G][\pi^{-n-1}I]$.

Remark 3.4 ([5, Proposition 2.20]). If I is the augmentation ideal of $A[G]$, then the algebra $A[G_\infty]$ of the automatic blowup of the identity is the $A[G]$ -subalgebra of $K[G]$ generated by $\cup_n \pi^{-n}I$. The latter union also generates the augmentation ideal of $A[G_\infty]$.

The following proposition is similar to Proposition 2.12.

PROPOSITION 3.5. *Assume that G is a smooth affine group scheme with connected fibers and let (V, ρ) be a finite dimensional representation of G_K . Suppose M is an A -module such that M is a flat finite A -module, $M \otimes_A K \cong V$. Then M is a representation of $G_\infty = \varprojlim G_n$ if and only if M is unitary as a $\mathbf{Dist}(G)$ -module (i.e., $1_{A[G]^*}M = M$).*

Proof. Since A is a DVR, M is projective of finite rank, hence free on A . Thus, we can choose a basis $\{e_i\}_{i=1}^n$ of M such that it is also a basis of V . As V is a comodule, then $a'_{ij}s \in K[G] \cong A[G_\infty] \otimes K$ such that $\rho_V(e_j) = \sum_1^r e_i \otimes a_{ij}$. From the conditions of the comodule, we have

$$e_j = (\mathrm{id}_V \otimes \epsilon_K)\rho_V(e_j) = (\mathrm{id}_V \otimes \epsilon_K)\left(\sum_{i=1}^r e_i \otimes a_{ij}\right) = \sum_{i=1}^r \epsilon_K(a_{ij})e_i.$$

By the proof of [5, Corollary 2.21], the coordinate algebra of G_∞ is

$$A[G_\infty] = \cup_n A[G][\pi^{-n}I] = \{f \in K[G] : \epsilon_K(f) \in A\}.$$

Consequently, M is an $A[G_\infty]$ -comodule if $\epsilon_K(a_{ij}) \in A$. $\square$

3.2. Additive group schemes via Néron blowups and representations

Example 3.6 (Continued from Example 3.1). Let A be a discrete valuation ring with uniformizer π , and consider the additive group scheme $\mathbb{G}_a = \mathrm{Spec}, A[T]$. Let $H_0 = \{e\} \subset \mathbb{G}_{a,k}$ be the trivial subgroup of the special fiber. The defining ideal of H_0 in $k[\mathbb{G}_{a,k}] = k[T]$ is (T) , so the corresponding ideal in $A[\mathbb{G}_a] = A[T]$ is $J = (\pi, T)$.

The Néron blowup $\mathbb{G}_a^{H_0}$ has the coordinate ring $A[\mathbb{G}_a^{H_0}] = A[T][\pi^{-1}T] \cong A[X]$, where $X = \pi^{-1}T$. The comultiplication is inherited from $\mathbb{G}_a$: $\Delta(X) = X \otimes 1 + 1 \otimes X$, with counit $\epsilon(X) = 0$ and antipode $S(X) = -X$. This yields a flat group scheme over A isomorphic to $\mathbb{G}_{a,A}$ itself, demonstrating that blowing up the additive group at the trivial subgroup recovers the original scheme.

More interestingly, consider successive Néron blowups at the identity. Let $G_0 = \mathbb{G}_{a,A}$ and define G_{n+1} as the blowup of G_n at $\{e\} \subset G_{n,k}$. The coordinate ring becomes $A[G_n] = A[\pi^{-n}T] \subset K[T]$, forming a tower:

$$\cdots \rightarrow G_2 \xrightarrow{\times\pi} G_1 \xrightarrow{\times\pi} G_0 = \mathbb{G}_a.$$

The projective limit $G_\infty = \varprojlim G_n$ has coordinate ring

$$A[G_\infty] = \bigcup_{n \geq 0} A[\pi^{-n}T] = A[\pi^{-\infty}T],$$

which is a non-finitely generated flat Hopf algebra over A . The generic fiber $G_{\infty,K} \cong \mathbb{G}_{a,K}$ remains additive, while the special fiber becomes trivial at all finite levels.

Remark 3.7. The coordinate algebra $A[G_\infty] = A[\pi^{-\infty}T]$ consists of “infinite Laurent polynomials” in T with coefficients divisible by arbitrarily high powers of π . While not finitely generated, it remains flat over A as a colimit of free modules. This mirrors the multiplicative case in [2, Section 1], where $A[\mathbb{G}_{m,\infty}]$ is constructed as a union of rings $A[\pi^{-n}(T-1)]$.

Now, let $\mathbb{G}_a^\pi = \text{Spec} A[\pi^{-1}T]$ be the deformed additive group scheme from Section 3.1. Its distribution algebra $\mathbf{Dist}(\mathbb{G}_a^\pi)$ has basis $\{\gamma_n\}_{n \geq 0}$ dual to the monomials $(\pi^{-1}T)^n$, with $\gamma_n((\pi^{-1}T)^m) = \delta_{n,m}$. By Proposition 2.12, flat finite representations of $\mathbb{G}_a^\pi$ correspond to A -lattices $M \subset V$ in finite-dimensional K -representations V of $\mathbb{G}_{a,K}$, stable under $\mathbf{Dist}(\mathbb{G}_a^\pi)$. For the standard representation $V = Ke_1 \oplus \cdots \oplus Ke_r$ with $\rho_V(e_j) = \sum_{i=1}^r e_i \otimes (\pi^{-1}T)a_{ij}$, the lattice $M = \bigoplus_{i=1}^r Ae_i$ is stable if and only if $a_{ij} \in \pi A$ for all i, j . This recovers the classical result of the representations of deformed additive groups.

PROPOSITION 3.8. *The category $\mathbf{Rep}_f^{\text{free}}(\mathbb{G}_a^\pi)$ is equivalent to the category of finite free A -modules equipped with A -linear endomorphisms whose structure constants lie in πA .*

Proof. The proof follows from Proposition 2.12 and the comodule structure $\rho_M(e_j) = \sum e_i \otimes \pi^{-1}Ta_{ij}$ with $a_{ij} \in A$. Flatness ensures M is free, and $\pi^{-1}Ta_{ij} \in A[\pi^{-1}T]$ iff $a_{ij} \in \pi A$. $\square$

We proceed to consider the more general case.

3.3. Representations of deformed additive groups over general bases

In this subsection, we extend our results to the case where the base A is an arbitrary commutative ring with identity. Unlike the previous sections, we no longer assume A to be an integral domain. This allows for the study of group schemes over rings that may contain zero divisors, where the classical techniques for Dedekind domains (such as Proposition 2.12) are not directly applicable.

We first recall the additive group scheme as a functor $\mathbb{G}_{a,A} : R \mapsto (R^+, +)$ for $R \in \mathbf{Alg}_A$. The coordinate ring of $\mathbb{G}_{a,A}$ is the polynomial ring $A[T]$ with the standard Hopf algebraic structure:

1. Coproduct $\Delta(T) = T \otimes 1 + 1 \otimes T$;
2. Counit $\epsilon(T) = 0$;
3. Antipode $S(T) = -T$.

Suppose λ is an arbitrary element in A . We define the group scheme $\mathbb{G}_{a,A}^\lambda$ via its coordinate ring $A[\mathbb{G}_{a,A}^\lambda] := A[X, T]/(T - \lambda X)$. In this algebra, the generator X is assumed to be a primitive element, which uniquely determines the Hopf structure on T via the relation $T = \lambda X$. Specifically, we have:

$$\begin{cases} \Delta(X) = X \otimes 1 + 1 \otimes X, \\ \epsilon(X) = 0, \\ S(X) = -X. \end{cases}$$

It follows that T also satisfies the primitive relation $\Delta(T) = T \otimes 1 + 1 \otimes T$. When $\lambda = 0$, the relation $T = \lambda X$ implies $T = 0$ in the coordinate ring. In this case, the algebra $A[X, T]/(T)$ is naturally isomorphic to the polynomial ring $A[X]$. Since X is a primitive element, this structure represents the standard additive group scheme $\mathbb{G}_{a,A}$ in the variable X .

From now on, we assume that λ is a non-zero divisor in A . Let $B = A[\lambda^{-1}]$ be the localization of A at λ . Under this assumption, $A[\mathbb{G}_{a,A}^\lambda]$ is identified with the A -subalgebra $A[\lambda^{-1}T]$ of $B[T]$. The Hopf algebra

$$A[X, T]/(T - \lambda X) = A[\lambda^{-1}T]$$

is clearly flat as a module on A and finitely generated as an A -algebra. The group scheme $\mathbb{G}_{a,A}^\lambda := \mathrm{Spec}(A[\lambda^{-1}T])$ represents the functor

$$\mathbb{G}_{a,A}^\lambda(R) = \{(a, b) \in R \times R : a = \lambda b\}, \quad R \in \mathbf{Alg}_A,$$

with the group structure defined by the addition of pairs:

1. Composition law: $(a, b) + (a', b') = (a + a', b + b')$;
2. Unit: $(0, 0)$;
3. Inverse: $-(a, b) = (-a, -b)$.

Remark 3.9. The flatness of $A[\mathbb{G}_{a,A}^\lambda]$ over A follows from the fact that it is isomorphic to the polynomial ring $A[X]$, which is a free (and thus, flat) A -module. Furthermore, the map $\phi: A[T] \rightarrow A[\mathbb{G}_{a,A}^\lambda]$ sending $T \mapsto \lambda X$ induces a group scheme homomorphism $\phi^*: \mathbb{G}_{a,A}^\lambda \rightarrow \mathbb{G}_{a,A}$. After scalar extension to $B = A[\lambda^{-1}]$, this map becomes an isomorphism:

$$\mathbb{G}_{a,A}^\lambda \times_{\text{Spec } A} \text{Spec } B \cong \mathbb{G}_{a,B}.$$

This shows that $\mathbb{G}_{a,A}^\lambda$ is a flat family of group schemes whose fiber over B is the standard additive group.

Example 3.10. Consider the case where A is a discrete valuation ring with the fractional field K . Let π be a uniformizer of A , i.e., a generator of the maximal ideal of A , clearly π elements have no divisors of zero. Now replacing λ by π we get the group scheme which coincides with the scheme

$$\mathbb{G}_{a,A}^{\{e\}} = \text{Spec } A[T][\pi^{-1}T] \cong \text{Spec}(A[T_1]) := G_1$$

obtained by the Neron blowup of the group scheme $\mathbb{G}_{a,A}$ at the closed subgroup of the unit $\{e\} \subset \mathbb{G}_{a,k}$ of the special fiber (see [11, Section 1]). Furthermore, the homomorphism $G_1 = G_{a,A}^{\{e\}} \rightarrow G_0 = G_{a,A}$ defined by the multiplicative mapping by π and is isomorphism after scalar extension to K . Similarly, with λ being a power of π , $\lambda = \pi^n$ ($n \in \mathbb{N}$), the group scheme G_n coincides with the Neron blowup at n times in succession at the unit. Then $\{G_n\}_{n \geq 0}$ is a sequence of finite type flat group schemes on A which defines a flat group scheme $\mathbb{G}_{a,\infty} := \varprojlim_n G_n$ but is not finite type with general fiber $\mathbb{G}_{a,K}$, [5, Example 1.1]. This is similar to the construction as in Example 3.6.

This contrasts with the multiplicative case [2, Section 1.3], where representations involve binomial coefficients scaled by powers of λ . Both cases illustrate how distribution algebras and Néron blowups provide unified frameworks for studying group schemes over general bases. This construction parallels Csima–Kottwitz’s multiplicative model [2], demonstrating the versatility of Néron blowups in building both additive and multiplicative group schemes with controlled fibers. The key distinction lies in the coproduct structure: while $\mathbb{G}_{m,A}^\lambda$ acquires nonlinear terms (e.g., $\Delta(X) = X \otimes 1 + 1 \otimes X + \lambda X \otimes X$), the additive version maintains linearity throughout the blowup process.

Next, we build the representations for the group scheme $\mathbb{G}_{a,A}^\lambda$ using the language of comodules over its coordinate algebra.

3.3.1 The coordinate algebra of the group $\mathbb{G}_{a,A}^\lambda$

For each non-negative integer n , we define the B -module homomorphism as follows:

$$Q_n: B[T] \longrightarrow B$$

$$\sum_{i \geq 0} b_i T^i \longmapsto \begin{cases} \lambda^n b_n, & \text{if } i = n; \\ 0, & \text{if } i \neq n, \end{cases}$$

where $i \in \mathbb{N}$ and $b_i = 0$ for all but finitely many i . Since λ is assumed to be a non-zero divisor in A , the set $\{Q_n\}_{n \geq 0}$ is linearly independent over A . Thus, we may naturally identify $\mathbf{Dist}(\mathbb{G}_{a,A}^\lambda)$ as the free A -module with basis $\{Q_n\}_{n \geq 0}$. Although $A[\mathbb{G}_{a,A}^\lambda] \cong A[X]$ as A -algebras, we distinguish $\mathbb{G}_{a,A}^\lambda$ as a scaled deformation to capture the behavior of the distributions $\{Q_n\}$ relative to λ . While each Q_n vanishes for $n > 0$ when $\lambda = 0$ (reflecting the structure's degeneration), the assumption that λ is a non-zero divisor ensures that the set $\{Q_n\}$ remains a valid basis, preserving the link between the parameter λ and the group's infinitesimal structure.

Remark 3.11. By simple calculation, we can easily check that $Q_n(gg') = \sum_{i=0}^n Q_i(g)Q_{n-i}(g')$ for all $g, g' \in B[T]$.

PROPOSITION 3.12. *The ring $A[\mathbb{G}_{a,A}^\lambda]$ is equal to*

$$\{g \in B[T]: Q_n(g) \in A, \forall n \geq 0\}$$

as a subalgebra of $B[T]$.

Proof. For all $n \geq 0$, we have $Q_n(\lambda^{-1}T) = 0$ or $1 \in A$. So $\lambda^{-1}T$ is in the right-hand equality of Proposition 3.12. On the other hand, for every $g = \sum_{i \geq 0} b_i T^i$, ($b_i \in B$), we can write

$$g = \sum_{i \geq 0} Q_i(g) \lambda^{-i} T^i = \sum_{i \geq 0} Q_i(g) (\lambda^{-1}T)^i \in A[\mathbb{G}_{a,A}^\lambda]$$

since $Q_i(g) \in A$ for all $i \geq 0$ by the assumption. Now, it is easy to check that the right-hand side is a submodule of $B[T]$ and a subring of $B[T]$ by the above remark. So the equality in the proposition is equal to the algebras. $\square$

Example 3.13. The projective system in Example 3.10 corresponds to the inductive system

$$A[T] = A[T_0] \longrightarrow A[T_1] \longrightarrow A[T_2] \longrightarrow \cdots$$

defined by $T_n \mapsto \pi T_{n+1}$. Then the coordinate algebra

$$A[G_{a,\infty}] = \varinjlim_n A[G_n] = \{P \in K[T] \mid P(0) \in A\}.$$

The above equality also holds under the initial assumption of A . In addition, the A -module $A[G_{a,\infty}]$ is isomorphic to $A \oplus K \oplus K \oplus \cdots$ and is flat but not projective as an A -module, [5, Example 1.1].

Let M be an A -module such that λ is a non-zero divisor of M and therefore, λ is a non-zero divisor of $M \otimes B$ ([2, Lemma 1.1.1]). By the assumption of λ , we can use the injection $M \hookrightarrow M_B := M \otimes B, m \mapsto m \otimes 1$. This can be obtained by taking the tensor product of the embedding $A \hookrightarrow B$ with the module M on A and hence, we can identify M with the A -submodule of M_B . From the flatness of $A[\mathbb{G}_{a,A}^\lambda]$ and of B on A , we have the following lemma.

LEMMA 3.14. *Let $M \otimes A[\mathbb{G}_{a,A}^\lambda]$ be an A -submodule of $M_B \otimes_B B[T]$ and coincides after scalar extension from A to B , $(M \otimes A[\mathbb{G}_{a,A}^\lambda]) \otimes B = M_B \otimes_B B[T]$.*

Proof. The embedding $M \hookrightarrow M_B$ can be viewed as a monomorphism between A -modules. Taking the tensor product with $A[\mathbb{G}_{a,A}^\lambda]$ over A , we get the embedding $M \otimes A[\mathbb{G}_{a,A}^\lambda] \hookrightarrow M_B \otimes A[\mathbb{G}_{a,A}^\lambda]$ between A -modules, that is, $M \otimes A[\mathbb{G}_{a,A}^\lambda]$ is an A -submodule of $M_B \otimes A[\mathbb{G}_{a,A}^\lambda]$. The first assertion is deduced from the equality

$$M_B \otimes A[\mathbb{G}_{a,A}^\lambda] = (M \otimes_A B) \otimes_B (B \otimes_A A[\mathbb{G}_{a,A}^\lambda]) = M_B \otimes_B B[T].$$

Similarly, thanks to the property of tensor products, we can easily obtain

$$(M \otimes A[\mathbb{G}_{a,A}^\lambda]) \otimes B = M \otimes (A[\mathbb{G}_{a,A}^\lambda] \otimes B) = M \otimes B[T]$$

and thus, the second assertion follows from the equality

$$M \otimes B[T] = M \otimes_A B[T] = M \otimes_A B \otimes_B B[T] = (M \otimes_A B) \otimes_B B[T] = M_B \otimes_B B[T],$$

completing the proof. $\square$

Each element in $M_B \otimes_B B[T]$ is of the form $\sum_{i \geq 0} x_i \otimes T^i$ with $x_i \in M_B$ equal to 0 for all but finitely number, see [7, Section 7.12]. For each non-negative integer n , we define the homomorphisms of the B -module as follows:

$$\begin{aligned} \mathbf{Q}_n: M_B \otimes_B B[T] &\longrightarrow M_B \\ \sum_{i \geq 0} x_i \otimes T^i &\longmapsto \begin{cases} \lambda^n x_n, & \text{if } i = n; \\ 0, & \text{if } i \neq n. \end{cases} \end{aligned}$$

PROPOSITION 3.15. *The module $M \otimes A[\mathbb{G}_{a,A}^\lambda]$ is equal to*

$$\{x \in M_B \otimes_B B[T] \mid \mathbf{Q}_n(x) \in M, \forall n \geq 0\}$$

as an A -module.

Proof. The proof imitated [2, Lemma 1.2.2]. For each $n \geq 0$, by considering the composition map of Q_n with the canonical projection $B[T] \xrightarrow{Q_n} K \xrightarrow{p} B/A$, we get a homomorphism

$$B[T] \xrightarrow{Q} \prod_{n \geq 0} B/A$$

with $Q := \prod_n (p \circ Q_n)$. Furthermore, it is clear that Q has an image in $\oplus_{n \geq 0} B/A$ by the definition of the maps Q_n . At the same time, it is also easy to see that $B[T] \xrightarrow{Q} \oplus_{n \geq 0} B/A$ is a surjection since Q_n is surjection for all n . From Proposition 3.12, we have the kernel of Q is $A[\mathbb{G}_{a,A}^\lambda]$ and the result is a short exact sequence of A -modules

$$0 \rightarrow A[\mathbb{G}_{a,A}^\lambda] \hookrightarrow B[T] \xrightarrow{Q} \bigoplus_{n \geq 0} B/A \rightarrow 0.$$

Taking the tensor product of the above exact sequence with M , we get an exact sequence of A -modules

$$M \otimes A[\mathbb{G}_{a,A}^\lambda] \rightarrow M \otimes B[T] \xrightarrow{\text{id}_M \otimes Q} M \otimes \left(\bigoplus_{n \geq 0} B/A \right).$$

On the other hand, since λ is a non-zero divisor of M , $M \otimes B/A = (M \otimes B)/(M \otimes A) = M_B/M$ so $M \otimes (\oplus_{n \geq 0} B/A) = \oplus_{n \geq 0} M_B/M$. Hence, from the second exact sequence above and the last part of the proof of Lemma 3.14, $M \otimes B[T] = M_B \otimes_B B[T]$, we have $\ker(\text{id}_M \otimes Q)$ which coincides with

$$\{x \in M_B \otimes_B B[T] \mid \mathbf{Q}_n(x) \in M, \forall n \geq 0\}.$$

Also, from Lemma 3.14, we can identify $M \otimes A[\mathbb{G}_{a,A}^\lambda]$ with A -submodule of $M_B \otimes_B B[T]$ and therefore, $M \otimes A[\mathbb{G}_{a,A}^\lambda] = \ker(\text{id}_M \otimes Q)$. This completes the proof. $\square$

3.3.2 Comodules and categorical equivalence of representations

As in [2], we denote the category $\mathbf{Rep}^{\text{flat}} G$ as the category of representations (M, ρ) of G such that λ is a non-zero divisor of M when considering M as a module on A . In the case of the group scheme $G = \mathbb{G}_{a,B}$, each representation (V, ρ) of $\mathbb{G}_{a,B}$ can be viewed as a homomorphism between B -modules $\rho_V : V \rightarrow V \otimes_B B[T]$ satisfying conditions of comodule. On the other hand, since $B[T]$ is a free module on B with basis $\{T^i\}_{i \geq 0}$, each of these representations can be written in the form

$$\rho(v) = \sum_{i \geq 0} v_i \otimes_B T^i$$

with $v \in V$ and $v_i = 0$ for all but a finite number i . Now for each $n \geq 0$, consider the composition

$$V \xrightarrow{\rho} V \otimes_B B[T] \xrightarrow{\text{id} \otimes_B Q_n} V \otimes_B B \cong V$$

which yields the mapping $D_n = (\text{id} \otimes Q_n) \circ \rho: V \rightarrow V$. Thus, we obtain a family of B -homomorphisms $\{D_n\}_{n \geq 0}$ (here it is defined similar to the family $\{C_n\}_{n \geq 0}$ as in [2, Section 1.3]). The group scheme $\mathbb{G}_{a,A}^\lambda$ constructed in the previous section has fiber $\mathbb{G}_{a,A}^\lambda \otimes B \cong \mathbb{G}_{a,B}$ on B . From the family of homomorphisms defined above, we have the following proposition.

PROPOSITION 3.16. *Let (V, ρ) be a representation of $\mathbb{G}_{a,B}$. Suppose that M is an A -module with a non-zero divisor λ such that $M \otimes_A B \cong V$ and $D_n(M) \subset M$, $\forall n \geq 0$. Then M is a representation of $\mathbb{G}_{a,A}^\lambda$.*

Proof. Since $M \hookrightarrow M \otimes_A B \cong V$, then for each $v \in M \subset V$, we have

$$\rho(v) = \sum_{i \geq 0} v_i \otimes_B T^i \in M_B \otimes_B B[T]$$

with $v_i = 0$ for all but a finite number i . On the other hand, for each $n \geq 0$, we have

$$\begin{aligned} Q_n(\rho(v)) &= Q_n\left(\sum_{i \geq 0} v_i \otimes_B T^i\right) = \lambda^n v_n \\ &= v^n \otimes_B \lambda^n = v_n \otimes_B Q_n(T^n) \\ &= (\text{id} \otimes_B Q_n)\left(\sum_{i \geq 0} v_i \otimes_B T^i\right) \\ &= (\text{id} \otimes_B Q_n) \circ \rho(v) = D_n(v) \in M \end{aligned}$$

by the assumption. By Proposition 3.15, we have $\rho(v) \in M \otimes_A A[\mathbb{G}_{a,A}^\lambda]$ for all $v \in M$, therefore, $\rho(M) \subset M \otimes_A A[\mathbb{G}_{a,A}^\lambda]$. This leads to the existence of a homomorphism $M \rightarrow M \otimes_A A[\mathbb{G}_{a,A}^\lambda]$ denoted by ρ_M , when the scalar extension to B coincides with ρ . The remaining thing we have to check is that (M, ρ_M) is a comodule. Note that $M_B = M \otimes B \cong V$ together with ρ being a comodule of $\mathbb{G}_{a,B}$. Therefore, these can be described as two commutative diagrams:

$$\begin{array}{ccc} M_B & \xrightarrow{\rho} & M_B \otimes_B B[T] \\ \rho \downarrow & & \downarrow \text{id} \otimes_B \Delta \\ M_B \otimes_B B[T] & \xrightarrow{\rho \otimes \text{id}} & M_B \otimes_B B[T] \otimes_B B[T] \end{array} \quad \begin{array}{ccc} M_B & \xrightarrow{\rho} & M_B \otimes_B B[T] \\ \searrow \cong & & \swarrow \text{id} \otimes \epsilon_B \\ & M_B \otimes B & \end{array}$$

The identity in Lemma 3.14 says that $M \otimes_A A[\mathbb{G}_{a,A}^\lambda]$ is an A -submodule of $M_B \otimes_B B[T]$. Since λ is a non-zero divisor of M and $A[\mathbb{G}_{a,A}^\lambda] \otimes A[\mathbb{G}_{a,A}^\lambda]$ is flat on A , λ

is a non-zero divisor of $M \otimes A[\mathbb{G}_{a,A}^\lambda] \otimes A[\mathbb{G}_{a,A}^\lambda]$. Hence, $M \otimes A[\mathbb{G}_{a,A}^\lambda] \otimes A[\mathbb{G}_{a,A}^\lambda]$ is identified with the submodule of $M_B \otimes_B B[T] \otimes_B B[T]$. Therefore, we obtain the two commutative diagrams below:

$$\begin{array}{ccc}
 M & \xrightarrow{\rho_M} & M \otimes A[\mathbb{G}_{a,A}^\lambda] \\
 \rho_M \downarrow & & \downarrow \text{id} \otimes \Delta \\
 M \otimes A[\mathbb{G}_{a,A}^\lambda] & \xrightarrow{\rho \otimes \text{id}} & M \otimes A[\mathbb{G}_{a,A}^\lambda] \otimes A[\mathbb{G}_{a,A}^\lambda]
 \end{array}
 \qquad
 \begin{array}{ccc}
 M & \xrightarrow{\rho_M} & M \otimes A[\mathbb{G}_{a,A}^\lambda] \\
 \searrow \cong & & \swarrow \text{id} \otimes \epsilon \\
 & M \otimes A &
 \end{array}$$

So M together with the restricted map $\rho|_M = \rho_M$ is a comodule for $A[\mathbb{G}_{a,A}^\lambda]$ and after the scalar extension (M, ρ_M) onto B coincides with (V, ρ) . $\square$

The notation $\mathcal{D}$ is the category of objects and morphisms defined as follows:

- $\mathcal{D}_1$) An object in $\mathcal{D}$ is a pair (M, V) where V is a representation of $\mathbb{G}_{a,B}$, M is an A -module such that λ is a non-zero divisor of M , $M \otimes_A B \cong V$ and $D_n(M) \subset M$, $\forall n \geq 0$;
- $\mathcal{D}_2$) A morphism in $\mathcal{D}$ is a homomorphism between (M, V) and (M, V') defined as $\sigma : V \rightarrow V'$ in $\mathbf{Rep} \mathbb{G}_{a,B}$ such that $\sigma(M) \subset M'$.

Suppose that (M, ρ) is in $\mathbf{Rep}^{\text{flat}} \mathbb{G}_{a,A}^\lambda$, the conditions of the comodule in the above definition are satisfied. The homomorphism $M \xrightarrow{\rho} M \otimes A[\mathbb{G}_{a,A}^\lambda]$ can be scalarly extended on B to $M_B \xrightarrow{\rho_B} M_B \otimes_B B[T]$, $M_B \cong M \otimes B$ and the conditions of the comodule are always satisfied. Hence, (M_B, ρ_B) is a representation of $\mathbb{G}_{a,B}$ such that $D_n(M) \subset M$, $\forall n \geq 0$. Finally, arguing as in Proposition 2.14, we get the following result.

PROPOSITION 3.17. *The functor*

$$\mathcal{F} : \mathbf{Rep}^{\text{flat}} \mathbb{G}_{a,A}^\lambda \rightarrow \mathcal{D}, M \mapsto (M, M \otimes_A B)$$

is a categorical equivalence.

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