ON THE ADJACENCY AND NET LAPLACIAN SPECTRA OF SOME COMPLETE SIGNED GRAPHS

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Given a signed graph $\Sigma = (G, \sigma)$, let $A(\Sigma)$ and $N(\Sigma)$ denote its adjacency and net Laplacian matrices, respectively. This paper provides a complete characterization of the adjacency and net Laplacian spectra of complete signed graphs (K_n, σ), where the negative edges (resp. positive edges) induce either a simple negative (resp. positive) corona signed subgraph $K_p^- \circ^- K_1$ (resp. $K_p \circ^+ K_1$) or a negative (resp. positive) Mycielskian signed subgraph $\mu^-(K_p^-)$ (resp. $\mu^+(K_p)$), for $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. Additionally, we identify specific values of p for which the adjacency spectrum of (K_n, σ) is symmetric and contains exactly 3, 4, or 5 distinct eigenvalues for the adjacency and net Laplacian matrices. These results contribute to the spectral understanding of signed graphs and highlight intriguing structural properties tied to their spectra.

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1. INTRODUCTION

A signed graph Σ is defined as an ordered pair (G, σ), where $G = (V, E)$ represents the graph known as the underlying graph of Σ, and $\sigma\colon E(G) \longrightarrow \{+1, -1\}$ is a sign function, referred to as the signature of Σ. Throughout this paper, all graphs under consideration are simple (having no loops or multiple edges) and undirected. For a vertex v, the net degree, denoted by $d_\Sigma^\pm(v)$, is given as $d_\Sigma^\pm(v) = d_\Sigma^+(v) - d_\Sigma^-(v)$, where $d_\Sigma^+(v)$ and $d_\Sigma^-(v)$ indicate the counts of positive and negative edges incident to v, respectively.

The adjacency matrix $A(\Sigma) = (a_{ij})$ of Σ is defined as the $n \times n$ matrix where $a_{ij} = \sigma(ij)$ if i and j are adjacent, and 0 otherwise. The eigenvalues of Σ, which are the eigenvalues of $A(\Sigma)$, constitute the adjacency spectrum of Σ. The net Laplacian matrix is expressed as $N(\Sigma) = D^\pm(\Sigma) - A(\Sigma)$, where $D^\pm(\Sigma)$ denotes the diagonal matrix of the net degrees of Σ. Consequently, the eigenvalues of $N(\Sigma)$ are referred to as the net Laplacian eigenvalues of Σ,

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forming the net Laplacian spectrum of Σ. The net Laplacian matrix serves as a parallel concept to the standard Laplacian matrix of a conventional graph. For an $n \times n$ matrix M (either $A(\Sigma)$ or $N(\Sigma)$), its spectrum is represented as $\mathrm{Spec}(M) = (\{a_1\}^{m(a_1)}, \{a_2\}^{m(a_2)}, \ldots, \{a_n\}^{m(a_n)})$, where $m(a_i)$ denotes the algebraic multiplicity of the eigenvalue a_i, $i \in \{1, 2, \ldots, n\}$. For eigenvalues with multiplicity one, the multiplicity is not indicated. The characteristic polynomials of Σ for the adjacency and net Laplacian matrices are denoted by $\Phi_\Sigma(x)$ and $\Psi_\Sigma(x)$, respectively.

A key property of signed graphs is the concept of switching the signature. For a signed graph $\Sigma = (G, \sigma)$ with vertex set $V(G)$, let $U \subset V(G)$ be a nonempty subset. The signed graph Σ^U is obtained from Σ by reversing the signs of all edges connecting vertices in U to those in $V(G) - U$. In this case, we say that Σ^U is switching equivalent to Σ.

In recent years, researchers have shown increasing interest in studying the spectra of different signed graphs partly due to their emerging applications; see, e.g., [12]. In [1], Akbari et al. analyzed the spectrum of complete signed graphs where the negative edges form an r-regular subgraph. Similarly, the spectrum of signed complete and complete bipartite graphs, where the negative edges constitute a matching, was investigated in [2]. Shamsher et al. [11] characterized the unicyclic signed graphs with minimal energies. More recently, Pirzada et al. [8] explored the spectrum of complete bipartite signed graphs where the negative edges induce either a complete bipartite subgraph, or a regular subgraph, or a path. For additional findings on the eigenvalues of signed graphs, readers are directed to [9, 10, 14, 15].

In [13, 16], Stanić presented several spectral properties of the net Laplacian matrix $N(\Sigma)$. In [7], Ou et al. examined the net Laplacian spectrum of complete signed graphs and characterized the signed graphs that possess exactly two distinct net Laplacian eigenvalues. For an ordinary graph $G = (V, E)$, the simple corona graph $G \circ K_1$ is obtained by attaching a positive pendant edge at every vertex of G [3]. If $V = \{v_1, v_2, \ldots, v_n\}$, then the Mycielskian $\mu(G)$ of G is a graph whose vertex set is the disjoint union of V, $V' = \{v'_1, v'_2, \ldots, v'_n\}$ and an isolated vertex w, and its edge set is

$$E \cup \{v'_i v_j : v_i v_j \in E\} \cup \{v'_i w : v'_i \in V'\} \quad \text{(see [4])}.$$

We begin by extending these definitions in the setting of signed graphs. Note that, in signed graphs, there are many different ways in which the newly introduced edges can be signed.

In this paper, we define the simple negative (resp. positive) corona signed graph $\Sigma \circ^- K_1$ (resp. $\Sigma \circ^+ K_1$) of $\Sigma = (G, \sigma)$ as the signed graph obtained by attaching a negative (resp. positive) pendant edge at every vertex of Σ and the

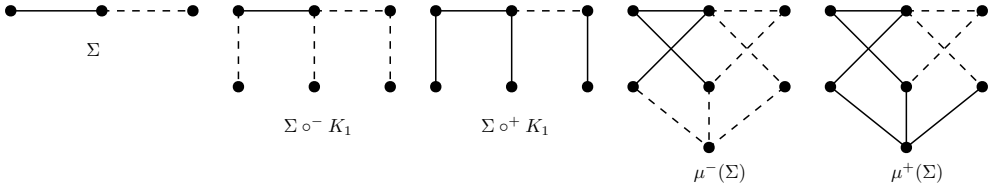

Figure 1 – Signed path, the simple negative corona, the simple positive corona, the negative Mycielskian, and the positive Mycielskian of a signed path. Negative edges are dashed.

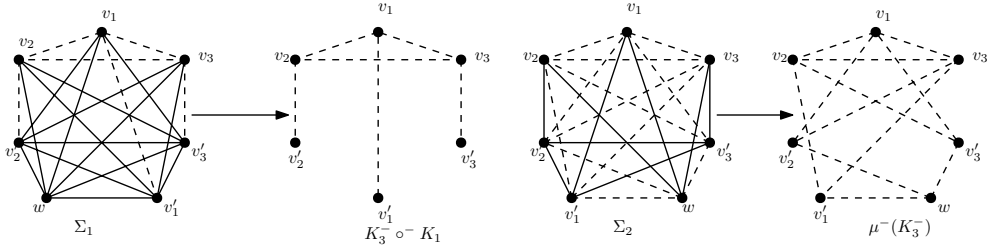

Figure 2 – The complete signed graph $K_\circ^3 = (\Sigma_1, K_3^- \circ^- K_1)$ and $K_\mu^3 = (\Sigma_2, \mu^-(K_3^-))$.

negative (resp. positive) Mycielskian signed graph of Σ as the signed graph $\mu^-(\Sigma) = (\mu(G), \sigma_\mu^-)$ (resp. $\mu^+(\Sigma) = (\mu(G), \sigma_\mu^+)$), where the signature σ_μ^* is defined as $\sigma_\mu^*(v_iv_j) = \sigma_\mu^*(v_i'v_j) = \sigma(v_iv_j)$ and $\sigma_\mu^*(v_i'w) = *1$, where $*$ is a fixed element of $\{-, +\}$. An illustration is given in Figure 1. For any signed graph Σ, it is straightforward to verify that the signed graph $\Sigma \circ^- K_1$ (resp. $\mu^-(\Sigma)$) is switching equivalent to the signed graph $\Sigma \circ^+ K_1$ (resp. $\mu^+(\Sigma)$).

Inspired by the aforementioned works on the spectrum of complete signed graphs, we examine the adjacency and net Laplacian spectra of complete signed graphs, where the negative edges form either a simple negative corona signed subgraph $K_p^- \circ^- K_1$ or a negative Mycielskian signed subgraph $\mu^-(K_p^-)$, for $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$, where K_p^- is a complete signed graph with all negative signature. We denote the former complete signed graph as $K_\circ^p = ((K_n, \sigma), K_p^- \circ^- K_1)$ and the latter as $K_\mu^p = ((K_n, \sigma), \mu^-(K_p^-))$. For an illustration, see Figure 2. Also, in recent years, there has been a significant study of signed graphs with few distinct eigenvalues. In this work, we also identify some values of p for which the complete signed graphs $K_\circ^p$ and K_μ^p have 3, 4, or 5 distinct eigenvalues of their adjacency or net Laplacian matrices.

We observe that all of our subsequent results can be analogously proven for complete signed graphs (K_n, σ), where the positive edges induce either a simple positive corona subgraph $K_p \circ^+ K_1$ or positive Mycielskian subgraph

$\mu^+(K_p)$, by considering the signed graph $(K_n, -\sigma)$. Section 2 is devoted to preliminaries. Our results and some consequences are given in Section 3.

2. PRELIMINARIES

We use O, J, I, and $\mathbf{j}$ to represent the all-zero matrix, the all-one matrix, the identity matrix, and the all-one column vector, respectively. The size of these matrices or vectors is specified in the subscript when necessary.

Let M be a real symmetric matrix of order n. Considering M as a matrix over the field of rational functions $\mathbb{C}(x)$, its characteristic matrix $xI - M$ has a determinant $\det(xI - M) = \Phi_M(x) \neq 0$. The M-coronal $\chi_M(x) \in \mathbb{C}(x)$ is defined as the sum of all entries in the matrix $(xI_n - M)^{-1}$. Specifically, $\chi_M(x) = \mathbf{j}_n^{\intercal}(xI_n - M)^{-1}\mathbf{j}_n$ [6]. When M has a constant row sum k, it follows that $\chi_M(x) = \frac{n}{x-k}$.

For signed graphs Σ_1 and Σ_2, the positive join (resp. negative join) $\Sigma_1\nabla^+\Sigma_2$ (resp. $\Sigma_1\nabla^-\Sigma_2$) is the signed graph obtained by adding all possible positive (resp. negative) edges between vertices of Σ_1 and vertices of Σ_2. We proceed with the following lemma.

LEMMA 2.1 ([13, Theorem 3.4]). *Let Σ_1 and Σ_2 be two signed graphs with net Laplacian eigenvalues $\nu_1(\Sigma_1)$, $\nu_2(\Sigma_1), \ldots, \nu_{n_1}(\Sigma_1) = 0$ and $\nu_1(\Sigma_2)$, $\nu_2(\Sigma_2), \ldots, \nu_{n_2}(\Sigma_2) = 0$, respectively, and let $*$ be a fixed element of $\{+, -\}$. The net Laplacian eigenvalues of $\Sigma_1\nabla^*\Sigma_2$ are: $*(n_1+n_2)$, $\nu_1(\Sigma_1)*n_2$, $\nu_2(\Sigma_1)*n_2, \ldots, \nu_{n_1-1}(\Sigma_1)*n_2$, $\nu_1(\Sigma_2)*n_1$, $\nu_2(\Sigma_2)*n_1, \ldots, \nu_{n_2-1}(\Sigma_2)*n_1$ and 0.*

Next, we state the lemma from the matrix theory.

LEMMA 2.2 (Schur complement formula, [5, Lemma 2.2]). *Let M_1, M_2, M_3, M_4 be the $p\times p, p\times q, q\times p$, and $q\times q$ matrices, with M_1 and M_4 being invertible. Then*

$$\begin{aligned}\det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} &= \det(M_1)\det(M_4 - M_3{M_1}^{-1}M_2) \\ &= \det(M_4)\det(M_1 - M_2{M_4}^{-1}M_3).\end{aligned}$$

3. MAIN RESULTS

3.1. Adjacency and net Laplacian spectra of $K_\circ^p = ((K_n, \sigma), K_p^- \circ^- K_1)$

In this section, we discuss the results related to the computation of adjacency and net Laplacian spectrum of the complete signed graph $K_\circ^p$. We start with obtaining the adjacency spectrum of $K_\circ^p$, in the following theorem.

THEOREM 3.1. *For a complete signed graph $K_\circ^p$, the adjacency spectrum is given by*

(i) -1 *with multiplicity* $n-2p-1$,

(ii) $\pm\sqrt{5}$ *with multiplicity* $p-1$,

(iii) *the three roots of cubic equation*

$$\begin{aligned} x^3 &+ (2p+1-n)x^2 + \big((p-1)(1+2p-n) - p(n-2p) \\ &+ (1-p)(3p-n) - np + p^2 + 4p - 4\big)x \\ &+ (1-p)\big((p-1)(1+2p-n) - p(n-2p)\big) \\ &- (p(p-4)+4)(1+2p-n) - p(n-2p)(3p-5) = 0. \end{aligned}$$

Proof. With a suitable labelling of the vertices, the adjacency matrix of $K_\circ^p$ has the form

$$A(K_\circ^p) = \begin{pmatrix} I_p - J_p & J_p - 2I_p & J_{p\times(n-2p)} \\ J_p - 2I_p & J_p - I_p & J_{p\times(n-2p)} \\ J_{(n-2p)\times p} & J_{(n-2p)\times p} & J_{n-2p} - I_{n-2p} \end{pmatrix}.$$

Therefore, for the corresponding characteristic polynomial, we find

$$\begin{aligned} \Phi_{K_\circ^p}(x) &= \det\big(xI_n - A(K_\circ^p)\big) \\ &= \det\begin{pmatrix} (x-1)I_p + J_p & 2I_p - J_p & -J_{p\times(n-2p)} \\ 2I_p - J_p & (x+1)I_p - J_p & -J_{p\times(n-2p)} \\ -J_{(n-2p)\times p} & -J_{(n-2p)\times p} & (x+1)I_{n-2p} - J_{n-2p} \end{pmatrix}. \end{aligned}$$

By performing row operations $R_i + J_{p\times(n-2p)}((x+1)I_{n-2p} - J_{n-2p})^{-1}R_3 \to R_i$, $i \in \{1,2\}$ and using Lemma 2.2, we obtain

$$\Phi_{K_\circ^p}(x) = (x+1)^{n-2p-1}(x+1+2p-n)\det(M), \tag{1}$$

where

$$\begin{aligned} \det(M) &= \det\big(((x-1)I_p + J_p - \chi_{J_{n-2p}}(x+1)J_p)((x+1)I_p - J_p \\ &\quad - \chi_{J_{n-2p}}(x+1)J_p) - (2I_p - J_p - \chi_{J_{n-2p}}(x+1)J_p)^2\big) \\ &= \det\big(((x-1)I_p + (1-\chi_{J_{n-2p}}(x+1))J_p)((x+1)I_p - J_p) \\ &\quad - (p-4)J_p - 4I_p - ((x-1)I_p + J_p)\chi_{J_{n-2p}}(x+1)J_p \\ &\quad - 2(p-2)\chi_{J_{n-2p}}(x+1)J_p\big) \\ &= \big((x-1)(x+1) - 4\big)^{p-1}\big((x-1+(1-\chi_{J_{n-2p}}(x+1))p) \\ &\quad \times (x+1-p) - (p-4)p - 4 - (x-1+p)\chi_{J_{n-2p}}(x+1)p \\ &\quad - 2(p-2)\chi_{J_{n-2p}}(x+1)p\big) \end{aligned}$$

$$
\begin{aligned}
&= (x^2-5)^{p-1}\Big((x+1-p)\Big(x-1+p-\frac{p(n-2p)}{x+1+2p-n}\Big)\\
&\quad - p(p-4)-4-p(x-1+p)\frac{(n-2p)}{x+1+2p-n}\\
&\quad - 2p(p-2)\frac{(n-2p)}{x+1+2p-n}\Big)\\
&= \frac{(x^2-5)^{p-1}}{x+1+2p-n}\big((x+1-p)((x-1+p)(x+1+2p-n)\\
&\quad - p(n-2p))-p(x-1+p)(n-2p)-(p(p-4)+4)\\
&\quad \times (x+1+2p-n)-2p(p-2)(n-2p)\big)\\
&= \frac{(x^2-5)^{p-1}}{x+1+2p-n}\big((x+1-p)(x^2+(3p-n)x+(p-1)(1+2p-n)\\
&\quad - p(n-2p))-(np-2p^2+p^2-4p+4)x\\
&\quad - (p(p-4)+4)(1+2p-n)-(n-2p)(p^2-p+2p^2-4p)\big)\\
&= \frac{(x^2-5)^{p-1}}{x+1+2p-n}\big(x^3+(2p+1-n)x^2+((p-1)(1+2p-n)\\
&\quad - p(n-2p)+(1-p)(3p-n)-np+p^2+4p-4)x+(1-p)\\
&\quad \times((p-1)(1+2p-n)-p(n-2p))-(p(p-4)+4)\\
&\quad \times (1+2p-n)-p(n-2p)(3p-5)\big). \qquad (2)
\end{aligned}
$$

Hence, the result follows by equations (1) and (2). □

As a consequence of Theorem 3.1, we obtain the following result.

COROLLARY 3.2. *For a complete signed graph $K_\circ^p$, if*

(i) *n is even and $p=\frac{n}{2}$, then the adjacency spectrum of the graph $K_\circ^p = ((K_n,\sigma), K_{\frac{n}{2}}^- \circ^- K_1)$ is given by*

$$\mathrm{Spec}(K_\circ^p) = \Big(\{\pm\sqrt{5}\}^{\frac{n}{2}-1}, \pm\sqrt{\frac{n^2-6n+10}{2}}\Big),$$

(ii) *n is odd and $p=\frac{n-1}{2}$, then the adjacency spectrum of the graph $K_\circ^p = ((K_n,\sigma), K_{\frac{n-1}{2}}^- \circ^- K_1)$ is given by*

$$\mathrm{Spec}(K_\circ^p) = \big(\{\pm\sqrt{5}\}^{\frac{n-1}{2}-1}, \alpha_1, \alpha_2, \alpha_3\big),$$

where α_i, $i\in\{1,2,3\}$ are the roots of $x^3+(2n-7-2(\frac{n-1}{2})^2)x-2(\frac{n-1}{2})^2+2n-2$.

Proof. The proof follows directly by plugging $p=\frac{n}{2}$ (resp. $p=\frac{n-1}{2}$) in Theorem 3.1 and engaging in long but straight forward calculations. □

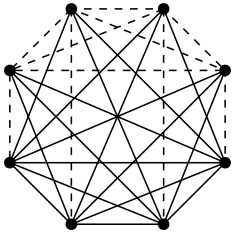

Figure 3 – The complete signed graph $((K_8,\sigma), K_4^- \circ^- K_1)$ with 4 distinct adjacency eigenvalues $(\{\sqrt{5}\}^3, \{-\sqrt{5}\}^3, \sqrt{13}, -\sqrt{13})$.

From Corollary 3.2(i), it is worth noting that the complete signed graph $((K_n,\sigma), K_{\frac{n}{2}}^- \circ^- K_1)$ has symmetric spectrum and exactly 4 distinct adjacency eigenvalues for $n \neq 6$. Figure 3 depicts complete signed graph $((K_8,\sigma), K_4^- \circ^- K_1)$ with 4 distinct adjacency eigenvalues.

In the next theorem, we determine the net Laplacian spectrum of $K_\circ^p$, but with less details in the proof.

THEOREM 3.3. *For a complete signed graph $K_\circ^p$, the net Laplacian spectrum is given by*

$$\mathrm{Spec}(K_\circ^p) = \big(0, n-4, \{n\}^{n-2p}, \{n-p-2-\sqrt{p^2+4}\}^{p-1}, \{n-p-2+\sqrt{p^2+4}\}^{p-1}\big).$$

Proof. With consistent vertex labelings, the diagonal matrix of net degrees of $K_\circ^p$ has the form

$$D^{\pm}(K_\circ^p) = \mathrm{diag}\big((n-2p-1)I_p, (n-3)I_p, (n-1)I_{n-2p}\big).$$

The net Laplacian matrix of $K_\circ^p$ is given by

$$N(K_\circ^p) = \begin{pmatrix} (n-2p-2)I_p + J_p & 2I_p - J_p & -J_{p\times(n-2p)} \\ 2I_p - J_p & (n-2)I_p - J_p & -J_{p\times(n-2p)} \\ -J_{(n-2p)\times p} & -J_{(n-2p)\times p} & nI_{n-2p} - J_{n-2p} \end{pmatrix}.$$

So, for the corresponding net Laplacian characteristic polynomial, $\Psi_{K_\circ^p}(x)$ is equal to

$$\det \begin{pmatrix} (x-n+2p+2)I_p - J_p & J_p - 2I_p & J_{p\times(n-2p)} \\ J_p - 2I_p & (x-n+2)I_p + J_p & J_{p\times(n-2p)} \\ J_{(n-2p)\times p} & J_{(n-2p)\times p} & (x-n)I_{n-2p} + J_{n-2p} \end{pmatrix}.$$

By performing row operations $R_i - J_{p\times(n-2p)}((x-n)I_{n-2p} + J_{n-2p})^{-1}R_3 \to R_i$, $i \in \{1,2\}$ and using Lemma 2.2, we obtain

$$\begin{aligned}\Psi_{K_\circ^p}(x) &= \det\big((x-n)I_{n-2p} + J_{n-2p}\big) \\ &\quad \times \det \begin{pmatrix} (x-n+2p+2)I_p - J_p - J_p\chi & J_p - 2I_p - J_p\chi \\ J_p - 2I_p - J_p\chi & (x-n+2)I_p + J_p - J_p\chi \end{pmatrix}\end{aligned}$$

$$= (x-n)^{n-2p-1}(x-2p)\det(M), \tag{3}$$

where for brevity, we write $\chi = \chi_{(-J_{n-2p})}(x-n)$ and

$$\begin{aligned}\det(M) &= \det\big(((x-n+2p+2)I_p - J_p - J_p\chi)((x-n+2)I_p + J_p - J_p\chi)\\ &\quad - (J_p - 2I_p - J_p\chi)^2\big)\\ &= \big((x-n+2p+2)(x-n+2) - 4\big)^{p-1}((x-n+2p+2-p)\\ &\quad \times (x-n+2+p-p\chi) - p(x-n+2+p)\chi - (p-2)^2 + 2p(p-2)\chi\big).\end{aligned}$$

Now, using the value of $\chi = \chi_{(-J_{n-2p})}(x-n) = \frac{n-2p}{x-2p}$ in the above equation and solving further, we obtain

$$\begin{aligned}\det(M) &= \frac{1}{x-2p}\big(x^2 - 2(n-p-2)x + n(n-2p-4) + 4p\big)^{p-1}\big((x-n+p+2)\\ &\quad \times ((x-n+p+2)(x-2p) - p(n-2p)) - (p-2)^2(x-2p)\\ &\quad - p(n-2p)(x-n+p+2) + 2p(p-2)(n-2p)\big)\\ &= \frac{1}{x-2p}(x-n+p+2)(x-n+p+2+\sqrt{p^2+4})^{p-1}(x-n+p+2\\ &\quad - \sqrt{p^2+4})^{p-1} \times \big(x^2 - (n-2-p+2p)x + 2p(n-2-p) - p(n-2p)\\ &\quad - p^2 - 4 + 4p - pn + 2p^2\big)\\ &= \frac{x}{x-2p}(x-n+p+2+\sqrt{p^2+4})^{p-1}(x-n+p+2-\sqrt{p^2+4})^{p-1}\\ &\quad \times \big(x^2 - 2(n-2)x + n^2 - 4n\big)\\ &= \frac{x(x-n)(x-n+4)}{x-2p}(x-n+p+2+\sqrt{p^2+4})^{p-1}(x-n+p+2\\ &\quad - \sqrt{p^2+4})^{p-1}.\end{aligned}$$

Using the value of $\det(M)$ in equation (3), the result follows. □

From Theorem 3.3, we observe the following.

Remark 3.4. For a complete signed graph $K_\circ^p$, if

(i) n is odd and $p \neq 1$, then it has exactly 5 distinct net Laplacian eigenvalues,

(ii) $n \neq 4$ is even and $p \notin \{1, \frac{n}{2}\}$, then it has exactly 5 distinct net Laplacian eigenvalues,

(iii) $n \neq 4$ is even and $p = \frac{n}{2}$, $p \neq 1$, then it has exactly 4 distinct net Laplacian eigenvalues.

3.2. **Adjacency and net Laplacian spectra of $K_{\mu}^{p} = ((K_n, \sigma), \mu^{-}(K_p^{-}))$**

In this section, we study the adjacency and net Laplacian spectrum of the complete signed graph K_{μ}^{p}.

THEOREM 3.5. *For a complete signed graph K_{μ}^{p}, the adjacency spectrum is comprised of*

(i) -1 *with multiplicity* $n-2p-2$,

(ii) $\pm\sqrt{5}$ *with multiplicity* $p-1$,

(iii) *the four roots of*

$$\begin{aligned}
&\big(x^2-(1-p)^2\big)\big(x(x+2+2p-n)-n+2p+1\big)\\
&\quad -p(x+1-p)(x+2+2p-n)-p\big(x(x-1+p)-p\big)(n-2p-1)\\
&\quad -p(x+2)(x+1-p)(n-2p-1)\\
&\quad -p(x-1+p)\big(x+2+2p-n-2(n-2p-1)\big)\\
&\quad +p^2(n-2p-1)-\big(p-2\big)^2(x(x+2+2p-n)-n+2p+1\big)\\
&\quad +2p(p-2)(n-2p-1)x\\
&\quad -2p\big((p-2)(x+2+2p-n)-p(n-2p-1)\big)=0.
\end{aligned}$$

Proof. With suitable labeling of the vertices, the adjacency matrix of K_{μ}^{p} has the form

$$A(K_{\mu}^{p}) = \begin{pmatrix} I_p - J_p & 2I_p - J_p & \mathbf{j}_p & J_{p\times(n-2p-1)} \\ 2I_p - J_p & J_p - I_p & -\mathbf{j}_p & J_{p\times(n-2p-1)} \\ \mathbf{j}_p^{\intercal} & -\mathbf{j}_p^{\intercal} & 0 & \mathbf{j}_{n-2p-1}^{\intercal} \\ J_{(n-2p-1)\times p} & J_{(n-2p-1)\times p} & \mathbf{j}_{n-2p-1} & J_{n-2p-1} - I_{n-2p-1} \end{pmatrix}.$$

Therefore, for the corresponding characteristic polynomial, we find

$$\begin{aligned}
\Phi_{K_{\mu}^{p}}(x) &= \det\big(xI_n - A(K_{\mu}^{p})\big)\\
&= \det\begin{pmatrix} (x-1)I_p + J_p & J_p - 2I_p & -\mathbf{j}_p & -J_{p\times(n-2p-1)} \\ J_p - 2I_p & (x+1)I_p - J_p & \mathbf{j}_p & -J_{p\times(n-2p-1)} \\ -\mathbf{j}_p^{\intercal} & \mathbf{j}_p^{\intercal} & x & -\mathbf{j}_{n-2p-1}^{\intercal} \\ -J_{(n-2p-1)\times p} & -J_{(n-2p-1)\times p} & -\mathbf{j}_{n-2p-1} & \begin{matrix}(x+1)I_{n-2p-1}\\ -J_{n-2p-1}\end{matrix} \end{pmatrix}.
\end{aligned}$$

By performing row operations

$$R_i + J_{p\times(n-2p-1)}\big((x+1)I_{n-2p-1} - J_{n-2p-1}\big)^{-1}R_4 \to R_i, \quad i \in \{1,2\},$$

and

$$R_3 + \mathbf{j}_{n-2p-1}^{\intercal}\big((x+1)I_{n-2p-1} - J_{n-2p-1}\big)^{-1}R_4 \to R_3$$

and using Lemma 2.2, we obtain

$$\begin{aligned}\Phi_{K_\mu^p}(x) &= \det\big((x+1)I_{n-2p-1} - J_{n-2p-1}\big)\\ &\quad\times \det\begin{pmatrix}(x-1)I_p + J_p - \chi J_p & J_p - 2I_p - \chi J_p & -\mathbf{j}_p - \chi\mathbf{j}_p\\ J_p - 2I_p - \chi J_p & (x+1)I_p - J_p - \chi J_p & \mathbf{j}_p - \chi\mathbf{j}_p\\ -\mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & \mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & x-\chi\end{pmatrix}\\ &= \big(x+1\big)^{n-2p-2}\big(x-n+2+2p\big)\det\big(M\big), \qquad (4)\end{aligned}$$

where, for brevity, we write $\chi = \chi_{J_{n-2p-1}}(x+1)$ and

$$\det\big(M\big) = \det\begin{pmatrix}(x-1)I_p + J_p - \chi J_p & J_p - 2I_p - \chi J_p & -\mathbf{j}_p - \chi\mathbf{j}_p\\ J_p - 2I_p - \chi J_p & (x+1)I_p - J_p - \chi J_p & \mathbf{j}_p - \chi\mathbf{j}_p\\ -\mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & \mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & x-\chi\end{pmatrix}.$$

On application of $R_1 + \frac{(1+\chi)}{x-\chi}\mathbf{j}_p R_3 \to R_1$ and $R_2 - \frac{(1-\chi)}{x-\chi}\mathbf{j}_p R_3 \to R_2$, the RHS of the above equation becomes

$$\det\begin{pmatrix}(x-1)I_p+J_p-\chi J_p-\frac{(1+\chi)^2}{x-\chi}J_p & J_p - 2I_p - \chi J_p + \frac{(1-\chi^2)}{x-\chi}J_p & \mathbf{0}_p\\ J_p - 2I_p - \chi J_p + \frac{(1-\chi^2)}{x-\chi}J_p & (x+1)I_p - J_p - \chi J_p - \frac{(1-\chi)^2}{x-\chi}J_p & \mathbf{0}_p\\ -\mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & \mathbf{j}_p^\intercal - \chi\mathbf{j}_p^\intercal & x-\chi\end{pmatrix}.$$

Solving further, we obtain

$$\begin{aligned}\det(M) &= (x-\chi)\det\Big(\Big((x-1)I_p + J_p - \chi J_p - \frac{(1+\chi)^2}{x-\chi}J_p\Big)\Big((x+1)I_p - J_p\\ &\quad - \chi J_p - \frac{(1-\chi)^2}{x-\chi}J_p\Big) - \Big(J_p - 2I_p - \chi J_p + \frac{(1-\chi^2)}{x-\chi}J_p\Big)^2\Big)\\ &= (x-\chi)\det\Big(\Big((x-1)I_p + J_p - \chi J_p - \frac{(1+\chi)^2}{x-\chi}J_p\Big)((x+1)I_p - J_p - \chi J_p)\\ &\quad - ((x-1)I_p + J_p - \chi J_p)\frac{(1-\chi)^2}{x-\chi}J_p - (J_p - 2I_p - \chi J_p)^2 - 2\frac{(1-\chi^2)}{x-\chi}\\ &\quad \times (pJ_p - 2J_p - p\chi J_p)\Big)\\ &= \frac{1}{(x-\chi)^{p-1}}\big((x-\chi)(x-1)(x+1) - 4(x-\chi)\big)^{p-1}\big(((x-1+p)(x-\chi)\\ &\quad - p\big)(x+1-p) - p\big((x-1+p)x - p\big)\chi - p(x+2)(x+1-p)\chi\\ &\quad - (x-1+p)(1-2\chi)p + p^2\chi - (p-2)^2(x-\chi) + 2p(p-2)\chi x\\ &\quad - 2p(p-2-p\chi)\big).\end{aligned}$$

Now, using value of $\chi = \chi_{J_{n-2p-1}}(x+1) = \frac{n-2p-1}{x-n+2+2p}$, we obtain

$$\det(M) = \frac{(x^2-5)^{p-1}}{x-n+2+2p}\big((x^2-(1-p)^2)(x(x+2+2p-n) - n + 2p + 1)$$

$$\begin{aligned}
&\times -p(x+1-p)(x+2+2p-n)-p(x(x-1+p)-p)(n-2p-1)\\
&-p(x+2)(x+1-p)(n-2p-1)-p(x-1+p)(x+2+2p-n\\
&-2(n-2p-1))+p^2(n-2p-1)+2p(n-2p-1)(p-2)x-(p-2)^2\\
&\times (x(x+2+2p-n)-n+2p+1)-2p((p-2)(x+2+2p-n)\\
&-p(n-2p-1))\big). \qquad (5)
\end{aligned}$$

Using equations (5) in (4), the proof is complete. □

COROLLARY 3.6. *For a complete signed graph K_μ^p, if*

(i) *n is even and $p=\frac{n-2}{2}$, then the adjacency spectrum of $K_\mu^p=((K_n,\sigma),\mu^-(K^-_{\frac{n-2}{2}}))$ is given by* $\mathrm{Spec}(K_\mu^p)=(\{\pm\sqrt{5}\}^{\frac{n}{2}-1},\pm\sqrt{\frac{n^2-6n+10}{2}})$,

(ii) *n is odd and $p=\frac{n-1}{2}$, then the adjacency spectrum of $K_\mu^p=((K_n,\sigma),\mu^-(K^-_{\frac{n-1}{2}}))$ is given by* $\mathrm{Spec}(K_\mu^p)=(\{\pm\sqrt{5}\}^{\frac{n-1}{2}-1},\alpha_1,\alpha_2,\alpha_3)$, *where α_i, $i\in\{1,2,3\}$ are the roots of $x^3+(2n-7-2(\frac{n-1}{2})^2)x-2(\frac{n-1}{2})^2+2n-2$.*

By Corollaries 3.2 and 3.6, we conclude that the adjacency spectrum of $((K_n,\sigma),K_{\frac{n}{2}}\circ^- K_1)$ is the same as that of $((K_n,\sigma),\mu^-(K^-_{\frac{n-2}{2}}))$ and the adjacency spectrum of $((K_n,\sigma),K^-_{\frac{n-1}{2}}\circ^- K_1)$ is same as that of $((K_n,\sigma),\mu^-(K^-_{\frac{n-1}{2}}))$.

Finally, we compute the net Laplacian spectrum of K_μ^p.

THEOREM 3.7. *For a complete signed graph K_μ^p, the net Laplacian spectrum consists of*

(i) 0 *with multiplicity* 1,

(ii) *n with multiplicity $n-2p-2$,*

(iii) *$n-3p+1\pm\sqrt{p^2-2p+5}$ with multiplicity $p-1$,*

(iv) *α_i, $i\in\{1,2,3\}$ where α_i are the three zeros of the cubic polynomial*

$$\begin{aligned}
p(x)=&\big((x-n+2p+1)(x-2p-1)-n+2p+1\big)x\\
&+(6p-2n-2)(x-n+2p+1)(x-2p-1)\\
&-2p(n-2p-1)(x-n+2p+1)+(n-3p+2)(n-3p)(x-2p-1)\\
&-p\big(x-2p-1-2(n-2p-1)\big)-(p^2-4p+4)(x-2p-1)\\
&-p(x-n+p-3)+p(2n-6p+2)(n-2p-1)\\
&-(n-2p-1)(n-3p+2)(n-3p)+p(n-3p)\\
&-(2p^2-4p)(n-2p-1)+(n-2p-1)(p^2-4p+4)\\
&-2p(2-p)-2p(n-2p-1)-(n-2p-1)(6p-2n-2).
\end{aligned}$$

Proof. The diagonal matrix of the net degrees of K_μ^p has the form

$$D^{\pm}(K_\mu^p) = \operatorname{diag}\big((n-4p+3)I_p, (n-2p-1)I_p, (n-2p-1)I_1, (n-1)I_{n-2p-1}\big).$$

Therefore, the net Laplacian matrix of K_μ^p is given by

$$N(K_\mu^p) = \begin{pmatrix} (n-4p+2)I_p + J_p & J_p - 2I_p & -\mathbf{j}_p & -J_{p\times(n-2p-1)} \\ J_p - 2I_p & (n-2p)I_p - J_p & \mathbf{j}_p & -J_{p\times(n-2p-1)} \\ -\mathbf{j}_p^{\intercal} & \mathbf{j}_p^{\intercal} & n-2p-1 & -\mathbf{j}_{n-2p-1}^{\intercal} \\ -J_{(n-2p-1)\times p} & -J_{(n-2p-1)\times p} & -\mathbf{j}_{n-2p-1} & nI_{n-2p-1} - J_{n-2p-1} \end{pmatrix}.$$

We omit the rest of the details in the proof since it follows the line of the proof of Theorem 3.5. □

We conclude this section with the following result.

COROLLARY 3.8. *For a complete signed graph K_μ^p, if*

(i) *n is odd and $p = \frac{n-1}{2}$, then the net Laplacian spectrum of the graph $K_\mu^p = ((K_n, \sigma), \mu^-(K^-_{\frac{n-1}{2}}))$ is given by*

$$\operatorname{Spec}(K_\mu^p) = \left(0, \left\{\frac{5-n \pm \sqrt{n^2-6n+25}}{2}\right\}^{\frac{n-1}{2}}\right),$$

(ii) *n is even and $p = \frac{n-2}{2}$, then the net Laplacian spectrum of $K_\mu^p = ((K_n, \sigma), \mu^-(K^-_{\frac{n-2}{2}}))$ is given by*

$$\operatorname{Spec}(K_\mu^p) = \left(0, \left\{\frac{8-n \pm \sqrt{n^2-8n+32}}{2}\right\}^{\frac{n}{2}-2}, n, \alpha_1, \alpha_2\right),$$

where n, α_1 and α_2 are the zeros of the cubic polynomial $p(x)$ given in Theorem 3.7(iv).

Proof. In view of Theorem 3.7, for $p = \frac{n-1}{2}$, $(x-n)^{n-2p-2} = (x-n)^{-1}$. Consequently, $x = n$ is the only pole of $\Psi_{K_\mu^p}(x)$. Also, the cubic part $p(x)$ of the net Laplacian characteristic polynomial transforms to

$$\begin{aligned}(x-n)\Big(x^2 + (n-5)x + \Big(n - 3\Big(\frac{n-1}{2}\Big) + 2\Big)\Big(n - 3\Big(\frac{n-1}{2}\Big)\Big) - \Big(\frac{n-1}{2}\Big) \\ - \Big(\Big(\frac{n-1}{2}\Big)^2 - 4\Big(\frac{n-1}{2}\Big) + 4\Big) - \Big(\frac{n-1}{2}\Big)\Big) \\ = (x-n)\big(x^2 + (n-5)x - n\big),\end{aligned}$$

completing the proof of (i).

To prove (ii), observe that if $p = \frac{n-2}{2}$, then there is exactly one vertex, say v, in K_μ^p whose all incident edges are positive. Consequently, the graph

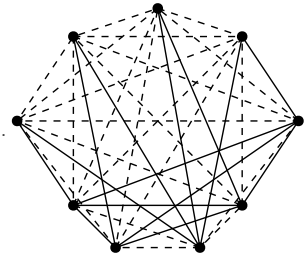

Figure 4 – The complete signed graph $((K_9, \sigma), \mu^-(K_4^-))$ with 3 distinct net Laplacian eigenvalues $(0, \{-2+\sqrt{13}\}^4, \{-2-\sqrt{13}\}^4)$.

$K_\mu^p = K_\mu^p \backslash \{v\} \nabla^+ \{v\}$ and in light of Lemma 2.1, n is one of the zero of the cubic polynomial $p(x)$ given in Theorem 3.7(iv). The other two zeros α_1 and α_2 of $p(x)$ can be obtained by employing the Vieta theorem. □

It is worth noting that in view of Corollary 3.8(i),

$$\big((K_n, \sigma), \mu^-(K^-_{\frac{n-1}{2}})\big)$$

has exactly 3 distinct net Laplacian eigenvalues. Figure 4, shows complete signed graph $((K_9, \sigma), \mu^-(K_4^-))$ with 3 distinct net Laplacian eigenvalues.

In the spirit of the obtained results, we formulate a research problem.

Problem 3.9. Determine the spectrum or the net Laplacian spectrum of a signed complete graph (K_p, σ) whose negative edges induce a tree.

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