

NEW STRONGLY REGULAR GRAPHS ON 53 VERTICES

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The only known strongly regular graph with parameters $(53, 26, 12, 13)$ is the one coming from the famous Paley construction. In this paper, we provide new examples of strongly regular graphs with 53 vertices. We established that if an automorphism group of prime order p acts on such a strongly regular graph, then $p \in \{2, 3, 13, 53\}$. We have constructed all strongly regular graphs on 53 vertices with Z_{13} or Z_6 as an automorphism group. We also give examples of new regular two-graphs with 54 vertices that have at least one descendant whose automorphism group is either an abelian group of order six or a cyclic group of order 13. Finally, we have proved the existence of 643 mutually nonisomorphic strongly regular graphs with parameters $(53, 26, 12, 13)$ as their descendants.

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Key words: strongly regular graphs, Paley graphs, Godsil–McKay switching, regular two-graphs, orbit matrices.

1. INTRODUCTION

A *strongly regular graph* with parameters (v, k, λ, μ) ($\text{SRG}(v, k, \lambda, \mu)$) is a simple regular graph with v vertices of valency k such that any two adjacent vertices are together adjacent to λ vertices, while any two nonadjacent vertices are together adjacent to μ vertices. Solving the problem of existence and enumeration of strongly regular graphs with certain parameters is an active and important area of research in graph theory (see [4, 5, 6]).

Recently, Ihringer constructed a large number of new SRGs with less than 100 vertices using a refined form of the Godsil–McKay and Wang–Qiu–Hu switching [11]. The same paper lists the parameter sets of SRGs for which these switching techniques have been applied but have not yielded new graphs. One of these cases is the $\text{SRG}(53, 26, 12, 13)$, for which, to our knowledge, the only known example is the Paley graph with 53 vertices $P(53)$ (see [4]). Its vertices can be considered as elements of the finite field of order 53, and two of them are adjacent if their difference is a square in the field. The full automorphism group of $P(53)$ is the group $Z_{53} : Z_{26}$ of order 1378.

In this paper, we first consider strongly regular graphs with parameters $(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$ and Godsil–McKay (GM) switching of partition type 4, $n-4$, and show, that, when starting from $\text{SRG}(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$, such a switching does not generate new graphs if 8 does not divide $n-1$ (this includes the case $P(53)$).

Since in the case of $\text{SRGs}(53, 26, 12, 13)$, GM switching of partition type 4, $n-4$ does not produce any new graphs we turn to another construction method. For the construction of new $\text{SRGs}(53, 26, 12, 13)$, we use techniques that have previously been successfully applied for the construction and partial classification of strongly regular graphs with up to 50 vertices (see [8, 12, 13, 14]). More precisely, we investigate possible automorphisms of a strongly regular graph with parameters $(53, 26, 12, 13)$ and use presumed action of an automorphism group for a construction of new $\text{SRGs}(53, 26, 12, 13)$ and their partial classification. We proved that if an automorphism of prime order p acts on such a graph, then $p \in \{2, 3, 13, 53\}$. Using orbit matrices constructed for the presumed automorphism group action on strongly regular graphs, as described in [1, 2, 9], we constructed new $\text{SRGs}(53, 26, 12, 13)$ and classified all such graphs that have either Z_{13} or Z_6 as an automorphism group.

In addition, we use a connection between regular two-graphs and strongly regular graphs established by Taylor in [19] to construct more strongly regular graphs $\text{SRGs}(53, 26, 12, 13)$ from regular two-graphs on 54 vertices. In this way, we have proved the existence of 643 mutually nonisomorphic strongly regular graphs with parameters $(53, 26, 12, 13)$.

In our work, we have used a system for computational discrete algebra GAP [20] and its package GRAPE [17].

2. ON THE GODSIL–MCKAY SWITCHING WITH A PARTITION OF TYPE 4, $n-4$

Let $\{C_1, \dots, C_\ell, D\}$ be a partition of the vertex set X of a graph Γ such that

- any two vertices in C_i have the same number of neighbours in C_j for all $i, j \in \{1, \dots, \ell\}$,
- each $x \in D$ has either 0, $\frac{1}{2}|C_i|$, or $|C_i|$ neighbours in C_i for every $i \in \{1, \dots, \ell\}$.

The construction known as the *Godsil–McKay switching* was introduced in [10]. We construct a new graph Γ' from Γ by applying the Godsil–McKay switching as follows: for each vertex $x \in D$ and each $i \in \{1, \dots, \ell\}$ such that x has

exactly half of the vertices in C_i as neighbours, we switch the adjacency and nonadjacency between $x \in D$ and the vertices in C_i .

In many applications of the Godsil–McKay switching, the case $\ell = 1$ is considered, so that $C = C_1$. In this case, the subset C is called (*Godsil–McKay switching set*), and it is said to be the Godsil–McKay switching with a partition of type $t, n - t$ ($t = |C|, n = |X|$). In particular, many interesting results have been obtained when the switching set is of size 4, that is from the Godsil–McKay switching with a partition of type 4, $n - 4$ (see, for example, [3, 11]).

Since cospectral strongly regular graphs (SRGs) have the same parameters, the Godsil–McKay switching can generate new SRGs with the same parameters as the initial graph (see [10, Theorem 2.2]). In the following theorem, we show that for strongly regular graphs with parameters

$$\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right),$$

where $n \not\equiv 1 \pmod{8}$, the Godsil–McKay switching with the partition of type 4, $n - 4$ does not yield any new graphs.

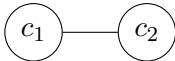
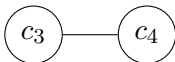
THEOREM 2.1. *A Godsil–McKay switching set of size 4 does not exist in SRGs($n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}$) when $n \not\equiv 1 \pmod{8}$.*

Proof. Let Γ be an $\text{SRG}(v, k, \lambda, \mu)$, where $v = n, k = \frac{n-1}{2}, \lambda = \frac{n-5}{4}, \mu = \frac{n-1}{4}$, whose vertex set is partitioned as $C \cup D$, where $C = \{c_1, c_2, c_3, c_4\}$ is a Godsil–McKay switching set of size 4. Let Γ_C denotes the subgraph of Γ induced by the set of vertices C . Since C is a switching set, the graph Γ_C must be regular. Therefore, without loss of generality, the following cases must be considered.

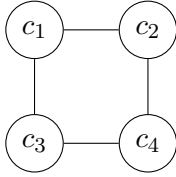
1. Γ_C is 0-regular and it is given by



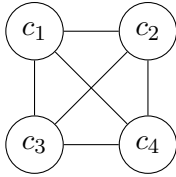
2. Γ_C is 1-regular and it is given by



3. Γ_C is 2-regular and it is given by



4. Γ_C is 3-regular, that is, vertices c_1 , c_2 , c_3 and c_4 are all mutually adjacent.



For a vertex x of Γ , denote by $\Gamma(x)$ the set of all neighbours of x . Let

$$\begin{aligned} C_1 &= \Gamma(c_1) \setminus C, \\ C_2 &= C_1 \cap \Gamma(c_2) \cap \Gamma(c_3)^C \cap \Gamma(c_4)^C \\ C_3 &= C_1 \cap \Gamma(c_2)^C \cap \Gamma(c_3) \cap \Gamma(c_4)^C \\ C_4 &= C_1 \cap \Gamma(c_2)^C \cap \Gamma(c_3)^C \cap \Gamma(c_4) \\ C_{234} &= C_1 \cap \Gamma(c_2) \cap \Gamma(c_3) \cap \Gamma(c_4). \end{aligned}$$

Note that $C_1 = C_2 \cup C_3 \cup C_4 \cup C_{234}$. Now, we have the following:

1. If Γ_C is 0-regular, then we have $|C_1| = k = \frac{n-1}{2}$ and

$$\begin{aligned} |C_2| + |C_{234}| &= |C_3| + |C_{234}| = |C_4| + |C_{234}| = \mu = \frac{n-1}{4} \\ |C_2| + |C_3| + |C_4| + |C_{234}| &= k = \frac{n-1}{2}. \end{aligned}$$

By solving these equations, we get $|C_2| = |C_3| = |C_4| = |C_{234}| = \frac{n-1}{8}$.

2. If Γ_C is 1-regular, then $|C_1| = k - 1 = \frac{n-3}{2}$ and

$$\begin{aligned} |C_2| + |C_{234}| &= \lambda = \frac{n-5}{4} \\ |C_3| + |C_{234}| &= |C_4| + |C_{234}| = \mu = \frac{n-1}{4} \\ |C_2| + |C_3| + |C_4| + |C_{234}| &= k - 1 = \frac{n-3}{2}. \end{aligned}$$

In this case, we have $|C_3| = |C_4| = |C_{234}| = \frac{n-1}{8}$, $|C_2| = \frac{n-9}{8}$.

3. For Γ_C is 2-regular, we have $|C_1| = k - 2 = \frac{n-5}{2}$ and

$$|C_2| + |C_{234}| = |C_3| + |C_{234}| = \lambda = \frac{n-5}{4}$$

$$|C_4| + |C_{234}| = \mu - 2 = \frac{n-9}{4}$$

$$|C_2| + |C_3| + |C_4| + |C_{234}| = k - 2 = \frac{n-5}{2}.$$

It follows: $|C_2| = |C_3| = \frac{n-1}{8}, |C_4| = |C_{234}| = \frac{n-9}{8}$.

4. If Γ_C is 3-regular, then $|C_1| = k - 3 = \frac{n-7}{2}$ and

$$|C_2| + |C_{234}| = |C_3| + |C_{234}| = |C_4| + |C_{234}| = \lambda - 2 = \frac{n-13}{4}$$

$$|C_2| + |C_3| + |C_4| + |C_{234}| = k - 3 = \frac{n-7}{2}.$$

In this case, we have $|C_2| = |C_3| = |C_4| = \frac{n-1}{8}, |C_{234}| = \frac{n-25}{8}$.

For a switching set of size four, $n - 1$ must therefore, be divisible by eight. $\square$

Remark 2.2. Note that according to Theorem 2.1, the Godsil–McKay switching with a partition of type 4, $n - 4$ on a strongly regular graph

$$\text{SRG}\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right),$$

where $n \not\equiv 1 \pmod{8}$ results in a switching class consisting of a single graph. In particular, the Godsil–McKay switching with a partition of type 4, $n - 4$ on a Paley graph $P(n)$, where $n \not\equiv 1 \pmod{8}$, does not lead to new graphs. However, if $n \equiv 1 \pmod{8}$ a switching class of a $\text{SRG}(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$ can consist of more than one graph. For example, we take the third graph in the list of $\text{SRGs}(41, 20, 9, 10)$ available at:

http://www.math.uniri.hr/~mmaksimovic/nontrivial_41.txt

(see [14, Theorem 3]) as the seed graph, and we applied three iterations of GM switching of type 4, 37 to it. After applying an isomorphism test, we obtained 52 nonisomorphic $\text{SRGs}(41, 20, 9, 10)$.

3. AUTOMORPHISMS OF STRONGLY REGULAR GRAPHS WITH PARAMETERS (53, 26, 12, 13)

In this section, we analyse the possible automorphisms of prime order p acting on a strongly regular graph with parameters (53, 26, 12, 13). Since the full automorphism group of the Paley graph $P(53)$ has order $1378 = 53 \cdot 13 \cdot 2$, we know that such an automorphism exists for $p \in \{2, 13, 53\}$. Moreover,

$P(53)$ is the only strongly regular graph on 53 vertices with an automorphism of order 53, up to isomorphism.

Our approach uses the method of constructing orbit matrices for the presumed action of an automorphism group of a strongly regular graph (see [1, 2, 9]). In particular, we use column orbit matrices in our construction (see, for example, [9, 14]).

Definition 3.1. A $(b \times b)$ -matrix $C = [c_{ij}]$ with entries satisfying conditions:

$$(1) \quad \sum_{i=1}^b c_{ij} = \sum_{j=1}^b \frac{n_j}{n_i} c_{ij} = k$$

$$(2) \quad \sum_{s=1}^b \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k - \mu) + \mu n_i + (\lambda - \mu) c_{ij}$$

where $0 \leq c_{ij} \leq n_i$, $0 \leq c_{ii} \leq n_i - 1$ and $\sum_{i=1}^b n_i = v$, is called a column orbit matrix for a strongly regular graph with parameters (v, k, λ, μ) and the orbit lengths distribution $(n_1, \dots, n_b)$.

This definition results from the properties of orbit matrices obtained from strongly regular graphs. A fixed row of the orbit matrix is a row that corresponds to an orbit of length 1, while the other rows are referred to as nonfixed rows. Every orbit matrix obtained from a strongly regular graph fulfils all the conditions given in Definition 3.1. However, not every matrix that fulfils these conditions necessarily corresponds to an orbit matrix of a strongly regular graph.

An important role in the construction is played by the concept of a row prototype for a column orbit matrix $C = [c_{ij}]$ of a strongly regular graph admitting an automorphism of prime order p introduced in [1]. Let p be a prime number. A prototype for a fixed row associated with an orbit lengths distribution $(d_1 \times 1, d_p \times p)$, where d_i denotes the number of orbits of length $i \in \{1, p\}$, is a nonnegative integer solution $(x_0, x_1; y_0, y_1)$ to the system:

$$(3) \quad \begin{aligned} x_0 + x_1 &= d_1 \\ y_0 + y_1 &= d_p \\ x_1 + py_1 &= k, \end{aligned}$$

where x_0 and x_1 denote the number of zeros and ones, respectively, in the fixed part of the row, and y_0 and y_1 the corresponding counts for the nonfixed part.

Similarly, a prototype for a nonfixed row for the same orbit lengths dis-

tribution is a nonnegative integer solution $(x_0, x_p; y_0, y_1, \dots, y_p)$ to the system:

$$\begin{aligned}
 (4) \quad & x_0 + x_p = d_1 \\
 & y_0 + y_1 + \dots + y_p = d_p \\
 & x_p + y_1 + 2y_2 + \dots + py_p = k \\
 & px_p + y_1 + 4y_2 + \dots + p^2y_p = \frac{(k - \mu)p + \mu p^2 + (\lambda - \mu)c_{rr}p}{p},
 \end{aligned}$$

where x_0 and x_p are the numbers of zeros and p 's, respectively, in the fixed part of the row r , and $y_0, \dots, y_p$ count the occurrences of the values $0, 1, \dots, p$ in the nonfixed part.

Our first result eliminates the cases in which the assumed automorphism is of prime order p , where $p \notin \{2, 3, 13, 53\}$.

THEOREM 3.2. *Let an automorphism group of prime order $1 < p < 53$ acts on a strongly regular graph with parameters $(53, 26, 12, 13)$. Then the prime p satisfies $p \in \{2, 3, 13\}$.*

Proof. We have to prove that $p \notin \{5, 7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$. The starting point for constructing a strongly regular graph with parameters $(53, 26, 12, 13)$ that admits an automorphism of prime order p is to identify all feasible orbit lengths distributions of the form $(d_1 \times 1, d_p \times p)$, where $2 \leq p \leq 47$ and $d_1 + pd_p = 53$. According to [1, Theorem 3.7], a nontrivial automorphism acting on an SRG with parameters (v, k, λ, μ) and eigenvalues $s < r < k$ can fix at most

$$\frac{\max(\lambda, \mu)}{\frac{k-r}{v}}v$$

vertices. Applying this bound to our case, we conclude that $d_1 \leq 30$.

Once the distribution of orbit lengths is determined, the next step is to apply the conditions (3) and (4) to check whether prototypes exist for both the fixed and nonfixed rows of the orbit matrix. By applying these conditions, we get the following:

- For primes $p \geq 29$, the only possible orbit lengths distributions are: $(d_1 \times 24, d_{29} \times 1)$, $(d_1 \times 22, d_{31} \times 1)$, $(d_1 \times 16, d_{37} \times 1)$, $(d_1 \times 12, d_{41} \times 1)$, $(d_1 \times 10, d_{43} \times 1)$. However, since each vertex fixed by an automorphism of order p must be adjacent to either all or none of the vertices of an orbit of length p , and each vertex has exactly k neighbouring vertices, there can be no such graph for $p \in \{29, 31, 37, 41, 43, 47\}$.

- For primes $11 \leq p \leq 23$, the possible orbit lengths distributions are: $(d_1 \times 9, d_{11} \times 4)$, $(d_1 \times 20, d_{11} \times 3)$, $(d_1 \times 19, d_{17} \times 2)$, $(d_1 \times 15, d_{19} \times 2)$, $(d_1 \times 7, d_{23} \times 2)$, $(d_1 \times 30, d_{23} \times 1)$. However, in each of these cases, there is no valid solution for a prototype of a nonfixed row (the conditions (4)).

• For $p = 7$, the possible orbit lengths distributions are $(d_1 \times 4, d_7 \times 7)$, $(d_1 \times 11, d_7 \times 6)$, $(d_1 \times 18, d_7 \times 5)$, $(d_1 \times 25, d_7 \times 4)$. For $d_1 > 4$, the system of equations (4) for a prototype of a nonfixed row has no solution, and for the case $d_1 = 4$, there is no prototype for a fixed row since there is no solution for $x_1 + 7y_1 = 26, x_1 \leq 3$.

• For $p = 5$, the possible orbit lengths distributions are: $(d_1 \times 3, d_5 \times 10)$, $(d_1 \times 8, d_5 \times 9)$, $(d_1 \times 13, d_5 \times 8)$, $(d_1 \times 18, d_5 \times 7)$, $(d_1 \times 23, d_5 \times 6)$, $(d_1 \times 28, d_5 \times 5)$. For $d_1 > 8$, the prototype system has no solution for a nonfixed row. For the distribution $(d_1 \times 3, d_5 \times 10)$, a prototype for a fixed row implies that, if such a graph existed, each fixed vertex would be adjacent to exactly one other fixed vertex. This contradicts the parameters $\mu = 13$ and $\lambda = 12$, so that such a graph cannot exist. For the distribution $(d_1 \times 8, d_5 \times 9)$, a prototype for a fixed row also shows that each fixed vertex would be adjacent to exactly one or exactly six other fixed vertices, which again contradicts the parameters $\mu = 13$ and $\lambda = 12$. Therefore, there is no such strongly regular graph in this case either.

This completes the proof. $\square$

We are now left with the primes $p \in \{2, 3, 13\}$. Since the structure of the full automorphism group of the Paley graph $P(53)$ is known, we know that there is at least one strongly regular graph with parameters $(53, 26, 12, 13)$ whose full automorphism group contains Z_{13} , and at least one whose full automorphism group contains Z_2 , but we do not know if there are more such graphs.

Let us first consider the case of a strongly regular graph with parameters $(53, 26, 12, 13)$ that admits Z_{13} as an automorphism group.

THEOREM 3.3. *There are exactly three strongly regular graphs with parameters $(53, 26, 12, 13)$ with Z_{13} as an automorphism group. Their full automorphism groups are listed in Table 1.*

$\text{Aut}(\Gamma)$	$ \text{Aut}(\Gamma) $	#SRGs
D_{26}	26	2
$Z_{53} : Z_{26}$	1378	1

Table 1 – SRGs(53, 26, 12, 13) with Z_{13} as an automorphism group.

Proof. For $p = 13$, we have $d_1 \in \{1, 14, 27\}$. For $d_1 = 14$ and $d_1 = 27$, there is no valid solution for a prototype of a nonfixed row.

For $d_1 = 1$, there is only one prototype $(1, 0; 2, 2)$ for a fixed row, and four prototypes for a nonfixed row:

$$(1, 1; 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0), (1, 1; 0, 0, 0, 0, 0, 1, 2, 0, 1, 0, 0, 0, 0, 0), \\ (1, 1; 0, 0, 0, 0, 1, 0, 0, 2, 1, 0, 0, 0, 0, 0), (1, 1; 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0).$$

Using these prototypes, only one orbit matrix can be constructed, up to isomorphism, that fulfils conditions (1) and (2). We give this matrix here.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 13 & 4 & 8 & 6 & 7 \\ 13 & 8 & 4 & 7 & 6 \\ 0 & 6 & 7 & 8 & 5 \\ 0 & 7 & 6 & 5 & 8 \end{bmatrix}$$

From this orbit matrix, we constructed the corresponding strongly regular graphs using the method that is given in [1] and obtained exactly three nonisomorphic graphs. The adjacency matrices of these graphs are available online at:

http://www.math.uniri.hr/~mmaksimovic/53_z13.txt. $\square$

Let us now consider the number of the fixed vertices for the case $p = 2$.

LEMMA 3.4. *If an automorphism group of order two acts on a strongly regular graph $(53, 26, 12, 13)$, then the number of fixed vertices for such an action is not in $\{3, 25, 27, 29\}$.*

Proof. Let d_1 be the number of fixed vertices. For the orbit lengths distributions with $d_1 \in \{3, 25\}$, a prototype of a fixed row implies that, if such a graph exists, each fixed vertex is adjacent to zero or two other fixed vertices. This contradicts the parameters $\mu = 13$ and $\lambda = 12$, so that such graphs cannot exist. For the cases $d_1 \in \{27, 29\}$ there is no solution for a prototype of a nonfixed row. $\square$

For the case $p = 3$, we have the following result on the number of fixed vertices.

LEMMA 3.5. *If a $\text{SRG}(53, 26, 12, 13)$ admits Z_3 as an automorphism group, then the number of fixed vertices satisfies $d_1 \in \{5, 11\}$.*

Proof. Since $0 \leq d_1 \leq 30$, we have $d_1 \in \{2, 5, 8, 11, 14, 17, 20, 23, 26, 29\}$. After checking for possible prototypes, it turns out that in all cases with $d_1 > 11$ there is no solution for a prototype of a nonfixed row. In addition, there is no valid solution for a prototype of a fixed row for the case $d_1 = 2$. Prototypes exist for the case $d_1 = 8$, but there is no corresponding column orbit matrix.

Therefore, if Z_3 acts as an automorphism group on $\text{SRG}(53, 26, 12, 13)$, it fixes exactly 5 or 11 vertices. $\square$

The construction of orbit matrices for presumed action of an automorphism groups of order 2 and 3 is computationally very demanding, and we cannot complete these constructions. Since it is already known that at least one $\text{SRG}(53, 26, 12, 13)$ with Z_2 as an automorphism group exists, and the existence for Z_3 remains uncertain, we have turned to the next natural case: the investigation of the existence of a $\text{SRG}(53, 26, 12, 13)$ with the automorphism group Z_6 . In the next theorem, we classify all $\text{SRGs}(53, 26, 12, 13)$ with Z_6 as an automorphism group. The construction we used to obtain this result follows the method for constructing strongly regular graphs with composite order automorphism groups introduced in [9].

THEOREM 3.6. *Up to isomorphism, there are exactly 178 strongly regular graphs with parameters $(53, 26, 12, 13)$ with an automorphism group isomorphic to the cyclic group Z_6 . The full automorphism groups of these graphs are shown in Table 2.*

$\text{Aut}(\Gamma)$	$ \text{Aut}(\Gamma) $	$\#\text{SRGs}$
Z_6	6	170
Z_{12}	12	8

Table 2 – SRGs with parameters $(53, 26, 12, 13)$ having Z_6 as an automorphism group.

Proof. The construction was carried out in several steps. First, we determined all possible orbit lengths distributions for an action of the group Z_6 . An automorphism of order six, partitions the vertices into orbits of length 1, 2, 3 or 6. If d_i specifies the number of orbits of length i , for $i \in \{1, 2, 3, 6\}$, the corresponding orbit lengths distribution is denoted by (d_1, d_2, d_3, d_6) and we have

$$d_1 + 2d_2 + 3d_3 + 6d_6 = 53.$$

Taking into account the constraint imposed by the Z_3 action, which specifies either 5 or 11 fixed vertices as given in Lemma 3.5, we have the additional condition:

$$d_1 + 2d_2 \in \{5, 11\}.$$

Furthermore, by Lemma 3.4, we have

$$d_1 + 3d_3 \in \{2n + 1 : 0 \leq n \leq 11, n \neq 1\}.$$

These conditions lead to 31 orbit lengths distributions, for which we have to check the existence of corresponding orbit matrices in the next step of the

construction. The construction of the corresponding orbit matrices is possible only for the four cases given in the first column of Table 3. The number of orbit matrices for each case is given in the second column of Table 3.

Distribution	#OM- Z_6	#OM- Z_3	#SRGs
(1,2,4,6)	12	18	28
(1,2,0,8)	8	12	26
(5,0,4,6)	2	10	0
(5,0,0,8)	1	18	124

Table 3 – The number of orbit matrices and SRGs(53, 26, 12, 13) for the automorphism group Z_6 .

Orbit matrices for Z_6 were further refined using the composition series

$$1 \trianglelefteq Z_3 \trianglelefteq Z_6.$$

That means that we first refined the orbit matrices obtained for the presumed action of Z_6 to the orbit matrices for the subgroup $Z_3 \triangleleft Z_6$ (see [9]). The number of orbit matrices for Z_3 is given in the third column of Table 3. Finally, from these orbit matrices we constructed adjacency matrices for SRGs(53, 26, 12, 13). The number of constructed graphs, up to isomorphism, is given in the fourth column of Table 3. After performing isomorphism checks and analysing the full automorphism groups of all constructed graphs, we obtained the results shown in Table 2. The adjacency matrices of the constructed SRGs are available online at:

http://www.math.uniri.hr/~mmaksimovic/53_z6.txt. $\square$

Remark 3.7. Note that Theorem 3.6 implies the existence of strongly regular graphs (53, 26, 12, 13) with Z_3 as an automorphism group, which shows that all prime numbers from Theorem 3.2 occur.

4. REGULAR TWO-GRAPHS ON 54 VERTICES

A two-graph is a pair $(\mathcal{V}, \Delta)$ of a finite set of vertices $\mathcal{V}$ and a set Δ of unordered triples chosen from $\mathcal{V}$ such that every 4-subset of $\mathcal{V}$ contains an even number of triples of Δ . A regular two-graph is a two-graph in which every pair of vertices lies in the same number of triples of the two-graph. The complement of the two-graph $(\mathcal{V}, \Delta)$ is the two-graph $(\mathcal{V}, \overline{\Delta})$, where $\overline{\Delta}$ is the complement of Δ in the set of all 3-subsets of $\mathcal{V}$. A two-graph is called self-complementary if it is isomorphic to its complement.

From a two-graph $\Phi = (\mathcal{V}, \Delta)$ and any fixed vertex $x \in \mathcal{V}$ we can construct the graph Γ , which is called a descendant of Φ and has a vertex set $\mathcal{V} \setminus \{x\}$, by considering any two vertices y, z in Γ as neighbouring if $\{x, y, z\} \in \Delta$. The two-graph $(\mathcal{V}, \Delta)$ is regular if and only if each descendant is strongly regular with parameters $(v-1, k, \lambda, \mu)$, where $\mu = k/2$. There is a one-to-one correspondence between two-graphs and switching classes of graphs (see [5]), and we know that descendants of a regular two-graph $(\mathcal{V}, \Delta)$ are SRGs $(v-1, k, \lambda, k/2)$ that are in the same switching class. For further reading, we refer the reader to [5, 16].

Many authors have tried to classify regular two-graphs with at most 50 vertices (see [7, 14, 15, 18, 19]). In this section, we consider the next case: regular two-graphs with 54 vertices. As far as we know, only one regular two-graph with 54 vertices is known, namely the one connected to the Paley graph with 53 vertices $P(53)$. We provide here a complete enumeration of regular two-graphs with 54 vertices that have as descendant at least one SRG $(53, 26, 12, 13)$ with Z_6 or Z_{13} as an automorphism group. Our work yielded 53 regular two-graphs: 50 from the SRGs $(53, 26, 12, 13)$ with Z_6 as an automorphism group and 3 from the SRGs $(53, 26, 12, 13)$ with Z_{13} as an automorphism group. To the best of our knowledge, all constructed two-graphs, except the one associated with $P(53)$, are new.

4.1. Regular two-graphs having at least one descendant with Z_6 as an automorphism group

According to Theorem 3.6, there are 178 strongly regular graphs with parameters $(53, 26, 12, 13)$ that have Z_6 as an automorphism group. We constructed all corresponding two-graphs and obtained the following result.

THEOREM 4.1. *Up to isomorphism, there are exactly 50 regular two-graphs on 54 vertices that have at least one descendant with Z_6 as an automorphism group, among which 14 are self-complementary. These two-graphs have 634 nonisomorphic strongly regular graphs with the parameters $(53, 26, 12, 13)$ as descendants.*

Proof. From 178 SRGs $(53, 26, 12, 13)$ from Theorem 3.6, we constructed corresponding two-graphs and eliminated the isomorphic ones. We obtained 50 regular two-graphs, labelled as $\Phi_i, i \in \{1, 2, \dots, 50\}$. The summary of these results is shown in Table 4, where each complementary pair of two-graphs is represented by only one member of the pair. The structures of the full automorphism groups of their descendants are given in the third column of the table, where “E” stands for the trivial automorphism group. In total, these

two-graphs generate 634 mutually nonisomorphic descendants. Of the 50 two-graphs, 14 are self-complementary, and there are 18 pairs of complementary two-graphs. This property is indicated in the last column of Table 4, where “Yes” stands for self-complementary graphs.

i	$ \text{Aut}(\Phi_i) $	Descendants of Φ_i	“S”
1-2	36	$[4 \times Z_3, 1 \times Z_6]$	Yes
3-4	12	$[4 \times E, 2 \times Z_6, 2 \times Z_{12}]$	No
5-11	6	$[6 \times E, 4 \times Z_2, 2 \times Z_3, 2 \times Z_6]$	No
12-14	6	$[8 \times E, 6 \times Z_6]$	No
15-26	6	$[8 \times E, 6 \times Z_6]$	Yes
27-32	6	$[8 \times E, 2 \times Z_3, 2 \times Z_6]$	No

Table 4 – Descendants of the regular two-graphs on 54 vertices that have at least one descendant with Z_6 as an automorphism group,

For each two-graph from Table 4, the adjacency matrix of one of its descendants is available online at:

http://www.math.uniri.hr/~mmaksimovic/54_descendantsz6.txt.

The structures of the full automorphism groups of 634 descendants are given in Table 5 and the list of their adjacency matrices can be found online at:

http://www.math.uniri.hr/~mmaksimovic/54_all_descendants_z6.txt.

$\text{Aut}(\Gamma)$	$ \text{Aut}(\Gamma) $	#SRGs
E	1	340
Z_2	2	56
Z_3	3	60
Z_6	6	170
Z_{12}	12	8

Table 5 – SRGs(53, 26, 12, 13) obtained as descendants of the regular two-graphs on 54 vertices that have at least one descendant with Z_6 as an automorphism group.

The proof is now complete. $\square$

4.2. Regular two-graphs having at least one descendant with an automorphism group of order 13

By Theorem 3.3, there are exactly three strongly regular graphs with parameters (53, 26, 12, 13) that have Z_{13} as an automorphism group. For the corresponding two-graphs, we have the following theorem.

THEOREM 4.2. *Up to isomorphism, there are exactly three regular two-graphs on 54 vertices that have at least one descendant with Z_{13} as an automorphism group, one of which is self-complementary. These two-graphs lead to nine nonisomorphic strongly regular graphs with parameters $(53, 26, 12, 13)$.*

Proof. From three strongly regular graphs $(53, 26, 12, 13)$ with Z_{13} as an automorphism group, we constructed corresponding two-graphs and eliminated the isomorphic ones. This led to three nonisomorphic regular two-graphs, which are labelled Φ_i for $i \in \{1, 2, 3\}$ in Table 6. Among these three two-graphs, there is one self-complementary two-graph and one pair of complementary two-graphs which is represented in the table by only one two-graph.

i	$ \text{Aut}(\Phi_i) $	Descendants of Φ_i	“S”
1	74412	$[1 \times Z_{53} : Z_{26}]$	Yes
2	52	$[1 \times Z_2, 2 \times Z_4, 1 \times D_{26}]$	No

Table 6 – Descendants of the regular two-graphs on 54 vertices that have at least one descendant with Z_{13} as an automorphism group.

For each two-graph from Table 6, the adjacency matrix of one of its descendants is available online at:

http://www.math.uniri.hr/~mmaksimovic/54_descendantsz13.txt.
 Two-graphs from Table 6 have nine nonisomorphic descendants, whose automorphism groups are given in Table 7 and a list of their adjacency matrices can be found online at:
http://www.math.uniri.hr/~mmaksimovic/54_all_descendants_z13.txt.

$\text{Aut}(\Gamma)$	$ \text{Aut}(\Gamma) $	#SRGs
Z_2	2	2
Z_4	4	4
D_{26}	26	2
$Z_{53} : Z_{26}$	1378	1

Table 7 – SRGs with parameters $(53, 26, 12, 13)$ obtained as descendants of the constructed regular two-graphs on 54 vertices that have at least one descendant with Z_{13} as an automorphism group.

The proof is now complete. □

Note that the SRGs from Theorem 4.1 and Theorem 4.2 are not isomorphic to each other. Overall, we have proved the existence of 53 regular two-graphs on 54 vertices and 643 mutually nonisomorphic SRGs $(53, 26, 12, 13)$.

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