

A NOTE ON POWER BOUNDED ELEMENTS
OF VON NEUMANN ALGEBRAS

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Let A be a von Neumann algebra with predual A_* and the unit element 1 . An element a of A is said to be power bounded if $\sup_{n \geq 0} \|a^n\| < \infty$. In this note we show that, for any power bounded element a of A , $\theta = \sigma(A, A_*) - \lim_{n \rightarrow \infty} (\frac{1+a}{2})^n$ exists, θ is an idempotent and $\theta a = a\theta = \theta$.

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1. INTRODUCTION

Let A is a von Neumann algebra with the unit element 1 . By the weak* topology of A we mean the topology $\sigma(A, A_*)$, where A_* is the (unique) predual of A . In this note we prove that, for any power bounded element a of A , the weak* limit of the sequence $((\frac{1+a}{2})^n)_{n \geq 0}$ always exists, it is an idempotent, say θ , and is such that $\theta a = a\theta = \theta$. An important problem here (see Example 5 below) is to characterize the power bounded elements a of A for which the $weak^* - \lim_{n \rightarrow \infty} (\frac{1+a}{2})^n = \theta$ is different from zero. We do not know the answer of this question.

2. THE MAIN RESULT

Let A be a von Neumann algebra and 1 be its unit element. As stated above, our aim is to prove that, for any power bounded element a of A , the sequence $((\frac{1+a}{2})^n)_{n \geq 0}$ weak* converges to some idempotent element of A .

The main ingredient of the proofs below is the fact that the multiplication operator $\varphi : A \times A \rightarrow A$, $\varphi(a, b) = ab$, is weak*-weak* continuous in each variable when the other variable is kept fixed.

We need two preliminary results, Lemma 1 and Corollary 3 below. In connection with Lemma 1, we recall the following well-known result due to Katznelson and Tzafriri [4]:

For a power bounded linear operator T on a Banach space X ,

$$\|T^{n+1} - T^n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{iff } \sigma(T) \cap \Gamma \subseteq \{1\}.$$

Here $\sigma(T)$ is the spectrum of T and Γ is the unit circle. We shall not use this result; we cited it here because it explains when the hypothesis of the next lemma is satisfied.

LEMMA 1. *Let a be a power bounded element of A such that $\|a^{n+1} - a^n\| \rightarrow 0$, as $n \rightarrow \infty$. Then $\text{weak}^* - \lim_{n \rightarrow \infty} a^n = \theta$ exists, θ is an idempotent and $\theta a = a\theta = \theta$.*

Proof. To prove that the sequence $(a^n)_{n \geq 0}$ converges in the weak* topology of A , since this sequence is bounded, it is enough to prove that it has only one weak* cluster point. For a contradiction, let θ_1 and θ_2 be two weak* cluster points of the sequence $(a^n)_{n \geq 0}$. Then this sequence has two subnets $(a^{n_i})_{i \in I}$ and $(a^{m_j})_{j \in J}$ that converge weak*, respectively, to θ_1 and θ_2 . Since $\|a^{n+1} - a^n\| \rightarrow 0$, as $n \rightarrow \infty$,

$$\lim_i \|a^{n_i+1} - a^{n_i}\| = 0 \quad \text{and} \quad \lim_j \|a^{m_j+1} - a^{m_j}\| = 0 \quad \text{too.}$$

So, since the norm of A is weak* lower semi-continuous, from the first limit, we get

$$\|a\theta_1 - \theta_1\| \leq \liminf_i \|aa^{n_i} - a^{n_i}\| = \liminf_i \|a^{n_i+1} - a^{n_i}\| = 0.$$

Hence, $a\theta_1 = \theta_1$. In the preceding lines writing a^{n_i+1} as $a^{n_i+1} = a^{n_i}a$, we get $\theta_1 a = \theta_1$ too. Instead of the net $(a^{n_i})_{i \in I}$, if we use the other net $(a^{m_j})_{j \in J}$ we get that $a\theta_2 = \theta_2$ and $\theta_2 a = \theta_2$. Iterating, for instance equality $a\theta_1 = \theta_1$, we get $a^{n_i}\theta_1 = \theta_1$. Passing to the limit in the weak* topology of A , we see that θ_1 is an idempotent. Similarly we see that θ_2 is also an idempotent. Now iterating the equalities $a\theta_1 = \theta_1$ and $\theta_1 a = \theta_1$, we get that

$$a^{m_j}\theta_1 = \theta_1 \quad \text{and} \quad \theta_1 a^{m_j} = \theta_1.$$

Passing in these equalities to the limit in the weak* topology of A , we get

$$(1) \quad \theta_2\theta_1 = \theta_1\theta_2 = \theta_1.$$

This time iterating the equalities $a\theta_2 = \theta_2$ and $\theta_2 a = \theta_2$, we get the equalities

$$a^{n_i}\theta_2 = \theta_2 \quad \text{and} \quad \theta_2 a^{n_i} = \theta_2.$$

Passing to the limits in the weak* topology, we get

$$(2) \quad \theta_1\theta_2 = \theta_2\theta_1 = \theta_2.$$

From (1) and (2), we see that $\theta_1 = \theta_2$ so that the bounded sequence $(a^n)_{n \geq 0}$ has only one weak* cluster point, say θ . Thus, $\theta = \text{weak}^* - \lim_{n \rightarrow \infty} a^n$ exists, $a\theta = \theta a = \theta$ and θ is an idempotent. \square

The second result we need is the following result, known as Ishikawa Iteration Theorem. This result can be found in Ishikawa's paper ([3], p. 69) and also in the book ([2], Lemma 9.4) by K. Goebel and Kirk,

THEOREM 2. *Let X be a Banach space and $T : X \rightarrow X$ a (not necessarily linear) nonexpansive mapping (i.e., for $x, y \in X$, $\|T(x) - T(y)\| \leq \|x - y\|$). Then, for $0 < \lambda < 1$ and $x \in X$,*

$$\lim_{n \rightarrow \infty} \|(\lambda I + (1 - \lambda)T)^{n+1}(x) - (\lambda I + (1 - \lambda)T)^n(x)\| = 0.$$

If $T : X \rightarrow X$ is a bounded linear operator then, in the preceding theorem, instead of assuming that " T is nonexpansive" it is enough to assume that T is power bounded. Indeed, if $T : X \rightarrow X$ is a power bounded linear operator on X , for $x \in X$, $|x| = \text{Sup}_{n \geq 0} \|T^n(x)\|$ is a norm on X equivalent to the original norm of X . For this norm T is nonexpansive, so that, the norms being equivalent, the following corollary holds.

COROLLARY 3. *Let X be a Banach space and $T : X \rightarrow X$ a power bounded linear operator. Then, for any $x \in X$,*

$$\lim_{n \rightarrow \infty} \|(\frac{I+T}{2})^{n+1}(x) - (\frac{I+T}{2})^n(x)\| = 0.$$

At this point we would like to include the following remarks of the referee. The referee has kindly informed us that, from ([1], Lemma 2.1), one can get the inequality $\|(\frac{I+T}{2})^{n+1} - (\frac{I+T}{2})^n\| \leq c \cdot \frac{1}{\sqrt{n}}$ for some constant $c > 0$. From this inequality one gets of course that $\|(\frac{I+T}{2})^{n+1} - (\frac{I+T}{2})^n\| \rightarrow 0$. He has also indicated that this latter result can also be deduced from the above mentioned theorem of Katznelson and Tzafriri. Since for our purpose the above weaker result is sufficient, we decided to keep the above corollary as it is.

The main result of this note is the following theorem.

THEOREM 4. *Let A be a von Neumann algebra. Then, for any power bounded element a of A , the limit $\theta = \text{weak}^* - \lim_{n \rightarrow \infty} (\frac{1+a}{2})^n$ exists, θ is an idempotent and $\theta a = a\theta = \theta$.*

Proof. Let $a \in A$ be a power bounded element of A . To a we associate the linear operator $L_a : A \rightarrow A$ defined by $L_a(b) = ab$. As $L_a^n(b) = (L_a \circ \dots \circ L_a)(b) = a^n b$, we see that L_a is a power bounded linear operator on the Banach space A . Hence, by the preceding Corollary, for each $b \in A$,

$$\lim_{n \rightarrow \infty} \|(\frac{I+L_a}{2})^{n+1}(b) - (\frac{I+L_a}{2})^n(b)\| = 0.$$

For $b = 1$, the unit element of A , the preceding limit becomes

$$(3) \quad \lim_{n \rightarrow \infty} \left\| \left(\frac{1+a}{2} \right)^{n+1} - \left(\frac{1+a}{2} \right)^n \right\| = 0.$$

Hence, by Lemma 1, $\theta = \text{weak}^* - \lim_{n \rightarrow \infty} \left(\frac{1+a}{2} \right)$ exists, θ is an idempotent and

$$\theta \left(\frac{1+a}{2} \right) = \left(\frac{1+a}{2} \right) \theta = \theta.$$

Equivalently, $\theta a = a\theta = \theta$. This completes the proof. \square

The next example explains one of the reasons why understanding the power bounded elements a of A for which the $\text{weak}^* - \lim_{n \rightarrow \infty} \left(\frac{1+a}{2} \right)$ is different from zero is important.

Example 5. Let H be a separable Hilbert space and $A = B(H)$ the operator algebra of the bounded linear operators on H . The predual of $B(H)$ is the space of the trace class operators on H . By Theorem 4 above, for any power bounded $T \in B(H)$,

$$P = \text{weak}^* - \lim_{n \rightarrow \infty} \left(\frac{I+T}{2} \right)^n$$

exists, P is a projection and $P \circ T = T \circ P = P$. So, in the case where $I \neq P \neq 0$, the closed subspace $M = P(H)$ of H is proper and it is an invariant subspace for T .

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