SOLUTIONS OF SYSTEMS OF EQUATIONS VIA PUISEUX EXPANSIONS

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This paper is a survey on the article "Puiseux Power Series Solutions for Systems of Equations" [2] and presents an algorithm to compute the terms of a Puiseux series solution of an algebraic system. The idea of the algorithm is to make the calculations using points in the tropical variety of the ideal generated by the system.

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INTRODUCTION

Solving equations is an old goal of mathematics. Isaac Newton devised an algorithm to compute the terms of an y-root of a 2-variables equation $f(x, y) = 0$ up to any order. The y-root consists of a power series in the variable x with fractional exponents; such a series is called a Puiseux series. It is a classical result that the Puiseux series form an algebraically closed field.

Extending Newton's algorithm to systems of equations in larger dimensions proved rather difficult and was done as late as 1995 by J. McDonald in [10] and [11]. It was a partial extension as it worked only for "general" systems. One of the main difficulties was finding a generalization for the field of Puiseux power series for series of several variables. McDonald's idea was to consider series with exponents in positive cones with respect to a convenient monomial ordering (see Section 1).

In [2], Aroca, Ilardi and Lopez de Medrano give a different algorithm which applies to any system. While the classical Newton algorithm uses the Newton polygon, a geometrical object determined by the equation's exponents, and predictably, McDonald's algorithm uses Newton polyhedron, the algorithm of [2] uses the tropical variety of the system – it replaces the Newton polyhedra with their normal fans (for the definition of the normal fan see *e.g.* [5]).

Another partial extension of the Newton algorithm producing Puiseux series solutions for systems of equations of the form $f(x, y_1, \ldots, y_M) = 0$ (or parametrizations for curves in the zero set of a general algebraic system) is given in [8].

The present paper is a survey on [2] and is based on the author's master thesis at the University of Kaiserslautern. We present the generalised Newton algorithm as well as the necessary background material. Notably, we include, following [1], a constructive proof of the fact that the field of ω-positive Puiseux series is algebraically closed, (see Theorem 1.15).

We have "streamlined" the main algorithm of [2] by eliminating several technical complications in the original paper (*e.g.* ω-sets). Moreover, we have improved the algorithm's efficiency via a more careful control of the liftings (see Remark 3.11). Finally, in section 3.4, we have interpreted the proof of Theorem 1.15 (which states that the field of ω-positive Puiseux series is algebraically closed) via the link with the tropical variety − the conclusion is that the iterative construction in the proof of the theorem coincides with the construction given by the algorithm (see section 3.4).

The paper is divided into three Sections. In Section 1 we study the algebraical properties of $\mathcal{S}_\omega$, the field of ω-positive Puiseux series. Here $\omega \in \mathbb{R}^N$ is a vector whose components are linearly independent over $\mathbb{Q}$. An N-variables Puiseux series is said to be ω-positive if the exponents of its terms belong to a translate of an ω-positive convex cone. One checks easily that $\mathcal{S}_\omega$ is a field.

The main result of Section 1 is that $\mathcal{S}_\omega$ is algebraically closed (Theorem 1 in [1]). The proof is hard, the main tools being a variation of the Newton polygon, defined using the valuation map val_ω of $\mathcal{S}_\omega$, and an extension to the $(N+1)$-dimensional case of the notion of ω-positive cone, namely the ω-wedge. The proof presented here is essentially that of [1].

In Section 2 one studies the properties of the polynomial ring $\mathcal{S}_\omega[y]$, where $y = (y_1, \ldots, y_M)$, and specifically the monomial orders given by the vectors (ω, η) with $\eta \in (\mathbb{R} \cup \{\infty\})^M$. An order relation thus defined is not necessarily total and consequently, the initial form with respect to this order of a polynomial $f \in \mathcal{S}_\omega[y]$, $\mathrm{in}_\eta(f)$, might consist of more than one term.

We study the initial ideals relative to different "directions" η for ideals $I \subset \mathcal{S}_\omega[y]$ with finite zero sets. In particular, one checks that for any $\varphi \in V(I)$, the initial ideal $\mathrm{in}_{\mathrm{val}(\varphi)}(I)$ is monomial free (see Proposition 2.21). If I is the extension of an ideal $J \subset \mathbb{K}[x, y]$, the previous property means that (ω, η) belongs to the tropical variety of J.

We conclude the chapter by giving a procedure to construct inductively the terms of a zero of an ideal $I \subset \mathcal{S}_\omega[y]$ having a finite zero set. The construction assumes that one can find at each step i a "good direction" $\eta^{(i)} \in$

$(\mathbb{R}\cup\{\infty\})^M$ and a zero of the $\eta^{(i)}$-initial ideal of an inductively constructed ideal.

Section 3 is devoted to describing the main algorithm. Given a system of algebraic equations, or equivalently, an ideal $I \subset \mathbb{K}[x_1,\dots,x_{N+M}]$ we can suppose by Noether's Normalization Lemma, after renaming the variables that $I \subset \mathbb{K}[x,y]$ with $x=(x_1,\dots,x_N)$, $y=(y_1,\dots,y_M)$, and the natural mapping $\mathbb{K}[x] \to \mathbb{K}[x,y]/I$ is integral. This in turn implies that the extension of the ideal I to $\mathcal{S}_\omega[y]$ has a finite zero set. The algorithm consists in computing the terms of a zero of this extended ideal. In order to do so, one combines the construction at the end of Section 2 with determining points in the tropical variety of an ideal in $\mathbb{K}[x,y]$. The computation of points in the tropical variety can be done using the program Gfan (see [7]) or the computer algebra system Singular (see [4]).

I want to thank Professor Thomas Markwig for proposing me this theme and for his help during the preparation of this paper.

1. THE FIELD $\mathcal{S}_\omega$ OF ω-POSITIVE PUISEUX SERIES

The material in this section, with the exception of the examples, follows [1].

1.1. The ω-order

Let ω be a vector in $\mathbb{R}^N$. Throughout this paper, we shall assume that the components of ω are linearly independent over the field of rational numbers. For $\alpha \in \mathbb{R}^N$ we define the ω-order of α:

$$\mathrm{val}_\omega(\alpha) = \omega\cdot\alpha,$$

where $\omega\cdot\alpha$ represents the inner product, and for $x\in\mathbb{R}$, we define

$$H_\omega(x) = \left\{\alpha\in\mathbb{R}^N \;\middle|\; \mathrm{val}_\omega(\alpha)=x\right\}.$$

Since the components of ω are rationally independent, $\mathrm{val}_\omega\big|_{\mathbb{Q}^N}$ is an injective mapping and consequently, it induces a total ordering on $\mathbb{Q}^N$, and in turn it produces a monomial ordering in the case of the fractional exponents of Puiseux series.

Remark 1.1. Each hyperplane $H_\omega(x)$ contains at most one element in $\mathbb{Q}^N$. In particular, the origin is the only rational vector in $H_\omega(0) = ker(\mathrm{val}_\omega)$.

If ω is clear from the context, we shall simply write $\mathrm{val}(\cdot)$ or $H(\cdot)$ instead of $\mathrm{val}_\omega(\cdot)$ or $H_\omega(\cdot)$.

1.2. **Convex Rational Polyhedral Cones**

A convex rational polyhedral cone is a subset of $\mathbb{R}^N$ of the form

$$\mathcal{C} = \{\lambda_1 v_1 + \ldots + \lambda_n v_n \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0\},$$

where $v_1, \ldots, v_n \in \mathbb{Q}^N$.

A cone is called strongly convex if it contains no non-trivial linear spaces.

Let $\omega \in \mathbb{R}^N$ be a vector with rationally independent coordinates. A convex rational polyhedral cone, $\mathcal{C}$, is said to be ω-positive if $\mathrm{val}(\alpha) \geq 0$ for every α in $\mathcal{C}$.

For the sake of brevity, in the sequel, we shall shorten "convex rational polyhedral ω-positive cone" to "ω-positive cone".

We list now some properties of ω-positive cones that will be needed in the sequel. Since the proofs are elementary and straightforward, we omit them.

PROPOSITION 1.2. I. *Let $\mathcal{C}$ be an ω-positive cone and $\delta \in \mathbb{R}^N$. Then:*

(1) *$\mathcal{C}$ is strongly convex.*

(2) *If $x \in \mathbb{R}$ then $(\delta + \mathcal{C}) \cap H(x)$ is a compact set and moreover,*

$$(\delta + \mathcal{C}) \cap \left\{\alpha \in \mathbb{R}^N \;\middle|\; \mathrm{val}(\alpha) \leq x\right\}$$

is a compact set. Consequently, the intersection of the lattice $1/c\,\mathbb{Z}^N$, where $c \in \mathbb{N}$ with

$$(\delta + \mathcal{C}) \cap \left\{\alpha \in \mathbb{R}^N \,\middle|\, \mathrm{val}(\alpha) \leq x\right\}$$

is a finite set.

(3) *Let $c \in \mathbb{N}$. Then $\mathrm{val}\left(1/c\,\mathbb{Z}^N \cap (\delta + \mathcal{C})\right)$ is a bounded below discrete set. In particular, for any non-empty subset $A \subset 1/c\,\mathbb{Z}^N$, the set $\mathrm{val}\left(A \cap (\delta + \mathcal{C})\right)$ has a smallest element.*

(4) *Let $c \in \mathbb{N}$, $A \subset 1/c\,\mathbb{Z}^N \cap (\delta + \mathcal{C})$ and $\gamma \in \mathbb{R}^N$ such that $\mathrm{val}(\gamma) \leq \mathrm{val}(\alpha)$ for all $\alpha \in A$. Then there exists an ω-positive cone $\tilde{\mathcal{C}}$ such that $A \subseteq \gamma + \tilde{\mathcal{C}}$. In particular, if v_0 is the element of A of smallest ω-order, one can take $\gamma = v_0$.*

(5) *If $\mathcal{C}$ has a non-empty interior in $\mathbb{R}^N$, then for all $\delta_1, \delta_2 \in \mathbb{R}^N$ there exists $\delta \in \mathbb{R}^N$ such that*

$$(\delta_1 + \mathcal{C}) \cup (\delta_2 + \mathcal{C}) \subseteq \delta + \mathcal{C}.$$

II. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two ω-positive cones. Then:*

(1) *There exists an ω-positive cone $\mathcal{C}$ such that for all $\delta_1, \delta_2 \in \mathbb{R}^N$ there exists δ in $\mathbb{R}^N$ satisfying*

$$(\delta_1 + \mathcal{C}_1) \cup (\delta_2 + \mathcal{C}_2) \subseteq \delta + \mathcal{C}.$$

(2) *$\mathcal{C}_1 + \mathcal{C}_2$ is an ω-positive cone and for $\delta_1, \delta_2 \in \mathbb{R}^N$ one has*

$$(\delta_1 + \mathcal{C}_1) + (\delta_2 + \mathcal{C}_2) = (\delta_1 + \delta_2) + (\mathcal{C}_1 + \mathcal{C}_2).$$

Remark 1.3. Proposition 1.2 II.(1) implies that the set of translates of ω-positive cones is directed with respect to the inclusion.

1.3. **Puiseux Series**

Throughout this paper, $\mathbb{K}$ will denote a field of characteristic 0. We shall always assume that $\mathbb{K}$ is algebraically closed. Without this assumption, the field $\mathcal{S}_\omega$ of ω-positive Puiseux series (see Definition 1.5) is not algebraically closed (i.e. Theorem 1.15 is not true) and the algorithm to compute a Puiseux parametrization does not work. While the following definitions and many of the results may hold for general fields or even rings, the main result requires the full hypothesis.

Definition 1.4. An *N-variable fractional power series* φ over $\mathbb{K}$ is an expression of the form

$$\varphi = \sum_{\alpha \in \mathbb{Q}^N} a_\alpha x^\alpha$$

where $a_\alpha \in \mathbb{K}$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Q}^N$ and $x^\alpha = x_1^{\alpha_1} \ldots x_N^{\alpha_N}$. Denote by

$$\varepsilon(\varphi) := \left\{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^N \;\middle|\; a_\alpha \neq 0\right\}$$

the set of exponents of φ.

For $c \in \mathbb{N}$ let

$$\mathcal{S}_c = \left\{\varphi \text{ fractional power series over } \mathbb{K} \;\middle|\; \varepsilon(\varphi) \subseteq {}^1\!/_c\,\mathbb{Z}^N\right\}$$

be the set of formal *c-Puiseux series*. If c' is a multiple of c, then there is an obvious inclusion

$$\mathcal{S}_c \hookrightarrow \mathcal{S}_{c'}. \tag{1}$$

Let

$$\mathcal{S} = \varinjlim_{c \in \mathbb{N}} \mathcal{S}_c.$$

be the set of all Puiseux series.

Remark that $\mathcal{S}_c$ and $\mathcal{S}$ are additive groups with the usual addition operation. However, without supplementary conditions, multiplication may not always be defined since computing coefficients might imply infinite sums in $\mathbb{K}$.

For instance, if $N = 1$, and $\varphi \in \mathcal{S}_c$,

$$\varphi = \sum_{i \in \mathbb{Z}} x^{i/c},$$

then φ^2 does not make sense since, for example its term of degree 0 should be the infinite sum

$$1 + x^{1/c}x^{-1/c} + x^{2/c}x^{-2/c} + \cdots = 1 + 1 + \dots \ .$$

In order to define a multiplication and hence a ring structure, one has to introduce some kind of finiteness condition. McDonald [10] had the idea to construct Puiseux series with exponent sets included in ω-positive cones, where ω has rationally independent coordinates.

We would also like that our ring contain all monomials (we need the monomials with negative exponents in order that all monomials be invertible). For this we also need to consider Puiseux series with exponents in translates of ω-positive cones.

Definition 1.5. Let $\mathcal{C} \subset \mathbb{R}^N$ be an ω-positive cone and $c \in \mathbb{N}$. Define

$$\mathcal{S}_{\mathcal{C},c} = \left\{\varphi \in \mathcal{S}_c \;\middle|\; \text{there exists } \delta \in \mathbb{Q}^N \text{ such that } \varepsilon(\varphi) \subset \delta + \mathcal{C}\right\}$$

the set of c-Puiseux series with exponents in translates of $\mathcal{C}$.

The inclusion (1) in Definition 1.4 induces an inclusion

$$\mathcal{S}_{\mathcal{C},c} \hookrightarrow \mathcal{S}_{\mathcal{C},c'}$$

for all c' multiple of c and let

$$\mathcal{S}_{\mathcal{C}} = \varinjlim_{c \in \mathbb{N}} \mathcal{S}_{\mathcal{C},c}$$

be the set of Puiseux series with exponents in translates of $\mathcal{C}$.

If $\mathcal{C} \subset \mathcal{C}'$ are two ω-positive cones, then one has an obvious inclusion

$$\mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{S}_{\mathcal{C}'}$$

and denote

$$\mathcal{S}_{\omega} = \varinjlim_{\mathcal{C}} \mathcal{S}_{\mathcal{C}}$$

the set of Puiseux series with exponents in translates of ω-positive cones, where the inductive limit is taken over all ω-positive cones.

An element $\varphi \in \mathcal{S}_\omega$ will be called an *ω-positive Puiseux series.*

Thus, a series φ is in $\mathcal{S}_\omega$ iff there exists an ω-positive cone $\mathcal{C}$, $\delta \in \mathbb{Q}^N$ and $c \in \mathbb{N}$ such that $\varepsilon(\varphi) \subset (\delta + \mathcal{C}) \cap {}^1\!/_c\mathbb{Z}^N$.

Remark 1.6. Note that Proposition 1.2 I.(5) implies that if the cone $\mathcal{C}$ has non-empty interior, then the sum of two series in $\mathcal{S}_{\mathcal{C},c}$ is again in $\mathcal{S}_{\mathcal{C},c}$. Note also that Proposition 1.2 II.(1) implies that the inductive limit which defines $\mathcal{S}_\omega$ is taken on a directed set. Consequently, the addition of Puiseux series induces a structure of additive group on $\mathcal{S}_{\mathcal{C},c}$ respectively $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{S}_\omega$. Proposition 1.2 I.(2) ensures that the product of two ω-positive Puiseux series can be defined, while Proposition 1.2 II.(2) implies that the result is still in $\mathcal{S}_\omega$.

Thus, $\mathcal{S}_{\mathcal{C},c}$, $\mathcal{S}_{\mathcal{C}}$ and $\mathcal{S}_{\omega}$ have a commutative ring structure.

Remark 1.7. If $N = 1$ and ω is for instance the 1-dimensional vector (1), then there exists only one ω-positive cone $\mathcal{C}$, ($\mathcal{C} = \{\alpha \mid \alpha \geq 0\}$) and $\mathcal{S}_{\mathcal{C}} = \mathcal{S}_{\omega}$ consists of the series with exponents bounded from below.

Example 1.8. Let for instance ω be a vector in the first orthant (*i.e.* all the components of ω are > 0). Then any usual N-variable power series

$$\varphi = \sum_{\alpha \in \mathbb{N}^N} a_\alpha x^\alpha$$

is an ω-positive Puiseux series.

Similarly, if ω is a vector in the negative orthant (*i.e.* all the components of ω are < 0), then any Laurent series with negative exponents

$$\varphi = \sum_{-\alpha \in \mathbb{N}^N} a_\alpha x^\alpha$$

is an ω-positive Puiseux series.

Of course, many other situations can be treated as ω-positive Puiseux series, and so the definition is rather comprehensive.

Definition 1.9. If $m = a_\alpha x^\alpha \in \mathcal{S}_{\omega}$ is a one-term series, then the ω-order of m is

$$\mathrm{val}(m) = \mathrm{val}(\alpha).$$

For a general $\varphi \in \mathcal{S}_{\omega}$, $\varphi \neq 0$,

$$\varphi = \sum_{\alpha \in \varepsilon(\varphi)} a_\alpha x^\alpha,$$

one defines

$$\mathrm{val}(\varphi) = \min_{\alpha \in \varepsilon(\varphi)} \mathrm{val}(a_\alpha x^\alpha).$$

Note that by Proposition 1.2 I.(3), the minimum is attained in a unique $\alpha \in \varepsilon(\varphi)$. By convention, $val(0) = \infty$.

Denote by $\mathrm{in}(\varphi)$ the term of φ of minimal ω-order. By convention, $\mathrm{in}(0) = 0$.

Note that

$$\mathrm{val}(\mathrm{in}(\varphi)) = \mathrm{val}(\varphi).$$

It is easy to check the following properties:

PROPOSITION 1.10. *Let φ and φ' be two elements in $\mathcal{S}_{\omega}$. Then*

(1) $\mathrm{val}(\varphi+\varphi') \geq \min(\mathrm{val}\,\varphi, \mathrm{val}\,\varphi')$ *with strict inequality if and only if* $\mathrm{val}(\varphi) = \mathrm{val}(\varphi')$ *and* $\mathrm{in}(\varphi) + \mathrm{in}(\varphi') = 0$,

(2) $\operatorname{val}(\varphi \cdot \varphi') = \operatorname{val}(\varphi) + \operatorname{val}(\varphi')$ *and* $\operatorname{in}(\varphi \cdot \varphi') = \operatorname{in}(\varphi) \cdot \operatorname{in}(\varphi')$.

(3) $\operatorname{val}\big(\varphi(x_1^r, \dots, x_N^r)\big) = r \cdot \operatorname{val}\big(\varphi(x_1, \dots, x_N)\big)$ *for every* $r \in \mathbb{Q}$.

Thus, val is a valuation map on $\mathcal{S}_\omega$.

Remark 1.11. Note that $\varphi \in \mathcal{S}_\omega$ can be written uniquely in the form

$$\varphi = \sum_{i=1}^{\infty} \varphi_i, \tag{2}$$

where each $(\varphi_i)_i$ is a one-term series and, for all i, $\operatorname{val}(\varphi_i) < \operatorname{val}(\varphi_{i+1})$.

Remark 1.12. For $c \in \mathbb{N}$, let

$$\Delta(c) = \{0, 1, \dots, c-1\}.$$

Any exponent $\alpha \in {}^1\!/_c\mathbb{Z}^N$ can be written uniquely as

$$\alpha = \delta + \alpha',$$

where $\delta \in {}^1\!/_c\Delta(c)^N$ and $\alpha' \in \mathbb{Z}^N$. Hence, any $\varphi \in \mathcal{S}_\omega$ with $\varepsilon(\varphi) \subset {}^1\!/_c\mathbb{Z}^N$ can be written uniquely as

$$\varphi = \sum_{\delta \in ({}^1\!/_c\Delta(c))^N} x^\delta \varphi_\delta, \tag{3}$$

with each φ_δ a series with integer exponents. This means that each φ with $\varepsilon(\varphi) \subset {}^1\!/_c\mathbb{Z}^N$ is a linear combination of integer exponent series with "coefficients" fractional monomials with positive exponents < 1.

Note that the family $(\varepsilon(x^\delta\varphi_\delta))_\delta$ forms a partition of $\varepsilon(\varphi)$ and consequently

$$\operatorname{val}(\varphi) = \min_{\delta \in ({}^1\!/_c\Delta(c))^N} (\operatorname{val}(\delta) + \operatorname{val}(\varphi_\delta))$$

and the minimum is attained for a unique exponent δ.

Example 1.13. For $N = 2$, $c = 5$ and $\varphi = x_1^{1/5}x_2^2 + x_1^{6/5}x_2^7 + x_1^{3/5}x_2^{7/5} + x_1^{8/5}x_2^9 + x_1^{13/5}x_2^3$, we can write φ as

$$x_1^{1/5}(x_2^2 + x_1x_2^7) + x_1^{3/5}x_2^{2/5}x_2 + x_1^{3/5}(x_1x_2^9 + x_1^2x_2^3).$$

We shall need a slight generalization of the formula (3).

Assume $l \in \mathbb{N}$ and c is a multiple of l. Then an exponent $\alpha \in {}^1\!/_c\mathbb{Z}^N$ can be written uniquely as

$$\alpha = \delta + \alpha',$$

where $\delta \in {}^l\!/_c\Delta(c/l)^N$ and $\alpha' \in {}^1\!/_l\,\mathbb{Z}^N$. If φ is a series with $\varepsilon(\varphi) \subset {}^1\!/_c\mathbb{Z}^N$ then φ can be written uniquely

$$\varphi = \sum_{\delta \in {}^l\!/_c\Delta(c/l)^N} x^\delta \varphi_\delta, \tag{4}$$

with $\varepsilon(\varphi_\delta) \subset {}^1\!/_l\,\mathbb{Z}^N$ for all δ. As above, the family $(\varepsilon(x^\delta\varphi_\delta))_\delta$ is a partition of $\varepsilon(\varphi)$ and

$$\operatorname{val}(\varphi) = \operatorname{val}(x^\delta\varphi_\delta)$$

for a unique $\delta \in {}^l\!/_c\Delta({}^c\!/_l)^N$.

PROPOSITION 1.14. *Let ω be a vector in $\mathbb{R}^N$ of rationally independent coordinates. Then $\mathcal{S}_\omega$ is a field.*

Proof. We have already seen that $\mathcal{S}_\omega$ is a ring. All we have left to show is that each $0 \neq \varphi \in \mathcal{S}_\omega$ is invertible.

First, we will show that for $\varphi \in \mathcal{S}_\omega$ with $\operatorname{val}(\varphi) > 0$, the sum

$$1 + \varphi + \varphi^2 + \dots$$

defines a series in $\mathcal{S}_\omega$.

Indeed, since $\varphi \in \mathcal{S}_\omega$, there exists a cone $\mathcal{C}$, such that $\varepsilon(\varphi) \subset \mathcal{C}$. Then, for all $n > 0$, $\varepsilon(\varphi^n) \subset \mathcal{C}$.

Since $\operatorname{val}(\varphi^n) = n\operatorname{val}(\varphi)$ it follows that each monomial appears in at most finitely many series φ^n, and consequently the infinite sum $1+\varphi+\varphi^2+\dots$ is a series in $\mathcal{S}_\omega$. This implies that if $\varphi \in \mathcal{S}_\omega$ with $\operatorname{val}(\varphi) > 0$ then the series $(1-\varphi)$ is invertible in $\mathcal{S}_\omega$.

For the general case, let $\varphi \in \mathcal{S}_\omega$ with $\varphi \neq 0$ and let $\operatorname{in}(\varphi) = c_\alpha x^\alpha$. Then

$$\varphi = c_\alpha x^\alpha(1-\psi),$$

with $\operatorname{val}\psi > 0$ which ends the proof. □

1.4. **The Newton Polygon**

The main result that we shall prove until the end of the chapter is ([1] Theorem 1):

THEOREM 1.15. *Let $\omega \in \mathbb{R}^N$ be a vector of rationally independent coordinates. Then $\mathcal{S}_\omega$ is an algebraically closed field.*

The main tools in the proof are a variation of the Newton polygon produced using the valuation $\operatorname{val}_\omega$ of $\mathcal{S}_\omega$ and an extension to the 2-dimensional case of the notion of ω-positive cone, the ω-wedge.

Throughout this section, f will be a polynomial in $\mathcal{S}_\omega[y]$

$$f = \sum_{i=0}^{d} \varphi_i(x)y^i,$$

with $\varphi_i(x) \in \mathcal{S}_\omega$. Since our purpose is to show that f has a root in $\mathcal{S}_\omega$, we will assume that $\varphi_0(x) \neq 0$ (*i.e.* $y = 0$ is not a root of f).

There exists an ω-positive cone $\mathcal{C}$, a vector $\delta \in \mathbb{Q}^N$ and $c \in \mathbb{N}$ such that $\varepsilon(\varphi_i) \subset \mathcal{I}(\mathcal{C}, \delta, c)$, for all i where

$$\mathcal{I}(\mathcal{C}, \delta, c) = (\delta + \mathcal{C}) \cap {}^1\!/_c\, \mathbb{Z}^N . \tag{5}$$

Then f can be written as the sum

$$f = \sum_{\substack{\alpha \in \mathcal{I}(\mathcal{C},\delta,c) \\ i=0,\ldots,d}} a_{\alpha,i} \cdot x^\alpha \cdot y^i,$$

where $a_{\alpha,i} \in \mathbb{K}$.

The exponent set of f is

$$\varepsilon(f) = \big\{(\alpha, i) \;\big|\; \alpha \in \mathcal{I}(\mathcal{C}, \delta, c),\ 0 \le i \le d,\ a_{\alpha,i} \neq 0\big\}.$$

The mapping $\mathrm{val}_\omega : \mathbb{R}^N \to \mathbb{R}$ (see Paragraph 1.1), induces the morphism

$$\begin{array}{rccc} \pi_\omega : & \mathbb{R}^N \times \mathbb{R} & \to & \mathbb{R}^2 \\ & (\alpha, i) & \mapsto & (\mathrm{val}(\alpha), i) \end{array} \tag{6}$$

Terminology 1.16. Let $v \in \mathbb{R}^N \times \mathbb{R}$.

(1) We denote by $P_{\mathrm{hor}}(v) \in \mathbb{R}^N$ the projection of v on the first factor, and call it the horizontal part of v.
(2) We denote by $P_{\mathrm{vert}}(v) \in \mathbb{R}$ the projection of v on the second factor, and call it the height of v.
(3) A subset $A \subset \mathbb{R}^{N+1}$ will be said to be horizontal if it has constant height.

Remark 1.17. The preimage of a point $P \in \mathbb{R}^2$, $P = (q, h)$ is the horizontal linear variety of codimension 2 $H_\omega(q) \times \{h\}$.

Proposition 1.18. *Let $\omega \in \mathbb{R}^N$ be a vector of rationally independent coordinates, let π_ω be the projection defined in* (6) *and let $P = (q, h)$ be a point in $\mathbb{R}^2$. Then the linear subvariety $\pi_\omega^{-1}(P)$ has at most one point in $\mathbb{Q}^N \times \mathbb{Q}$.*

Moreover,

$$\pi_\omega : \varepsilon(f) \to \pi_\omega\big(\varepsilon(f)\big)$$

is a bijection.

Proof. If $h \notin \mathbb{Q}$ then $\pi_\omega^{-1}(P)$ has obviously no point in $\mathbb{Q}^{N+1}$. If $h \in \mathbb{Q}$ then the statement follows from the properties of the mapping val, see Remark 1.1. □

Remark that the condition on the coefficients φ_i of f implies:

$$\pi_\omega(\varepsilon(f)) \subset \big\{(q, h) \in \mathbb{R}^2 \;\big|\; q \ge \mathrm{val}(\delta),\ 0 \le h \le d\big\}.$$

Definition 1.19. Let $f \in \mathcal{S}_\omega[y]$ be a polynomial. The ω-Newton polygon of f is the convex hull of the set $\pi_\omega\big(\varepsilon(f)\big) + (\mathbb{R}_{\geq 0} \times \{0\})$. In other words,

$$\mathrm{NP}_\omega(f) = \mathrm{conv}\left(\bigcup_{P \in \varepsilon(f)} \pi_\omega(P) + \big(\mathbb{R}_{\geq 0} \times \{0\}\big)\right).$$

Example 1.20. Let $f = (y - \varphi_1(x)) \cdot \ldots \cdot (y - \varphi_n(x))$, where $\varphi_1, \ldots, \varphi_n \in \mathcal{S}_\omega \setminus \{0\}$ and for each i,

$$\mathrm{in}(\varphi_i) = a_i x^{\alpha^{(i)}},$$

with $a_i \in \mathbb{K}$, and $\alpha^{(i)} \in \mathbb{Q}^N$. Then,

$$f = y^n + S_1(\varphi_1, \ldots, \varphi_n) y^{n-1} + \cdots + (-1)^n S_n(\varphi_1, \ldots, \varphi_n),$$

where S_i denotes the i-th elementary symmetric polynomial. Assume that $\mathrm{val}(\varphi_1) < \cdots < \mathrm{val}(\varphi_n)$. Our assumption implies that:

$$\mathrm{val}(S_i(\varphi_1, \ldots, \varphi_n)) = \mathrm{val}(\varphi_1) + \cdots + \mathrm{val}(\varphi_i).$$

The Newton polygon, $NP_\omega(f)$, for $n = 4$, can be seen in Fig. 1.

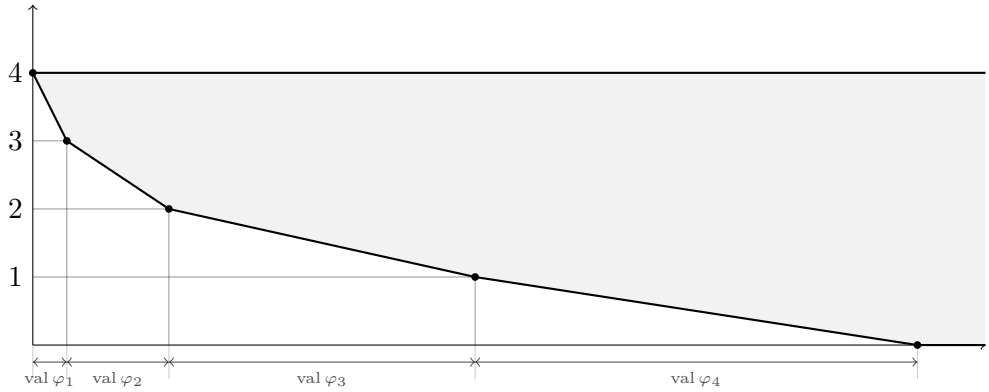

Fig. 1. $NP_\omega(f)$.

Remark 1.21. If

$$f = \sum_{i=0}^{d} \varphi_i(x) y^i$$

then for each height h such that $\varphi_h(x) \neq 0$, the ω-Newton polygon contains the half-line $(\mathrm{val}(\varphi_h), h) + (\mathbb{R}_{\geq 0} \times \{0\})$.

$NP_\omega(f)$ has a finite number of edges, the uppermost and the lowermost edges are horizontal half-lines, while the rest of the edges are bounded line segments.

In particular, the Newton polygon in Fig. 1, (see Example 1.20), has 5 vertices and 4 finite length edges.

We introduce for further use the following notations:

Notation 1.22. Let $f \in \mathcal{S}_\omega[y]$,

$$f = \sum_{i=1}^{d} \varphi_i(x) y^i$$

such that $\varphi_0(x) \neq 0$. Denote by:

- $v^{(0)}(f) \in \mathbb{R} \times \mathbb{N}$ denotes the vertex of height 0 of $NP_\omega(f)$.
- $v^{(1)}(f) \in \mathbb{R} \times \mathbb{N}$ denotes the vertex of lowest positive height of $NP_\omega(f)$.
- Let $q^{(0)}(f) \in \varepsilon(f)$ be the point such that $\pi_\omega\big(q^{(0)}(f)\big) = v^{(0)}(f)$.
- Let $q^{(1)}(f) \in \varepsilon(f)$ be the point such that $\pi_\omega\big(q^{(1)}(f)\big) = v^{(1)}(f)$.
- Let $ht(f) \in \mathbb{N}$ be the height of $q^{(1)}(f)$.
- The line segment in $\mathbb{R}^N \times \mathbb{R}$, $\operatorname{segm}(f) = [q^{(0)}(f), q^{(1)}(f)]$ will be called the ω-segment of f.
- Let $q^{(0)}(f)$ and $q^{(1)}(f)$, as above. Then

$$q^{(0)}(f) - q^{(1)}(f) = ht(f)\left(\frac{P_{\text{hor}}(q^{(0)}(f)) - P_{\text{hor}}(q^{(1)}(f))}{ht(f)}, -1\right). \tag{7}$$

The quotient

$$\text{slope } (f) = \frac{P_{\text{hor}}(q^{(0)}(f)) - P_{\text{hor}}(q^{(1)}(f))}{ht(f)}$$

will be called the ω-slope of f.

- Let $A \subset \varepsilon(f)$ and denote by $f\big|_A$ the sum of terms of f with exponents in A. The ω-characteristic polynomial of f is the polynomial $\chi(f) \in \mathbb{K}[y]$,

$$\chi(f)(y) = f\big|_{\text{segm}(f)}(1, \ldots, 1, y).$$

Example 1.23. In the context of Example 1.20,

$$v^{(0)}(f) = \left(\sum_{i=1}^{n} \text{val}(\varphi_i), 0\right)$$

$$v^{(1)}(f) = \left(\sum_{i=1}^{n-1} \text{val}(\varphi_i), 1\right)$$

$$q^{(0)}(f) = \left(\sum_{i=1}^{n} \alpha^{(i)}, 0\right)$$

$$q^{(1)}(f) = \left(\sum_{i=1}^{n-1} \alpha^{(i)}, 1\right)$$

$\operatorname{slope}(f) = \alpha^{(n)}$
$\chi(f) = (-1)^{n-1}a_1 \cdot \ldots \cdot a_{n-1}y + (-1)^n a_1 \cdot \ldots \cdot a_n = (-1)^{n-1}a_1 \cdot \ldots \cdot a_{n-1}(y - a_n).$

Remark 1.24. Note that the ω-characteristic polynomial of f, $\chi(f)(y)$ has degree equal to $ht(f)$, the height of $q^{(1)}(f)$, and furthermore, since $\varphi_0(x) \neq 0$ it follows that $y = 0$ is not a root of $\chi(f)(y)$. Moreover, since $\mathbb{K}$ is an algebraically closed field, $\chi(f)(y)$ has a root in $\mathbb{K}$. This is the only place in the proof where one uses the fact that $\mathbb{K}$ is an algebraically closed field.

LEMMA 1.25. *Let $f \in \mathcal{S}_\omega[y]$ such that $f(0) \neq 0$. The slope of the edge $[v^{(0)}(f), v^{(1)}(f)]$ of the Newton polygon $NP_\omega(f)$ is $-1/\operatorname{val}(\operatorname{slope}(f))$.*

Proof. Applying π_ω in (7), one gets

$$v^{(0)}(f) - v^{(1)}(f) = ht(f) \cdot (\operatorname{val}(\operatorname{slope}(f)), -1),$$

where $ht(f)$ is the height $v^{(1)}(f)$ and the Lemma is obvious. □

Remark 1.26. The proof of Lemma 1.25 shows that the vector (val(slope $(f)), -1)$ is parallel to the edge $[v^{(0)}(f), v^{(1)}(f)]$. In particular, in the case of Example 1.20, the edge $[v^{(0)}(f), v^{(1)}(f)]$ is parallel to $(\operatorname{val}(\varphi_n), -1)$ and its slope is $(-1)/\operatorname{val}(\varphi_n)$.

Remark 1.27. Since the Newton polygon $NP_\omega(f)$ is convex, it follows that for all $P = (q, h) \in NP_\omega(f)$, the line segment

$$\{P + \lambda(\eta, -1) \mid \eta = \operatorname{val}(\operatorname{slope}(f)),\ 0 \leq \lambda \leq h\}$$

(*i.e.* the line segment parallel to the edge $[v^{(0)}(f), v^{(1)}(f)]$ joining P to the abscissa) is contained in $NP_\omega(f)$.

1.5. **The Automorphisms T_φ**

In this section, we introduce a family of $\mathcal{S}_\omega$-algebra automorphisms of $\mathcal{S}_\omega[y]$ and study their effect on the exponent set and the Newton polygon. The proof of the Theorem 1.15 consists in iteratively applying the transformation to a polynomial of $\mathcal{S}_\omega[y]$.

For $\varphi \in \mathcal{S}_\omega$ let

$$T_\varphi : \begin{array}{ccc} \mathcal{S}_\omega[y] & \to & \mathcal{S}_\omega[y] \\ y & \mapsto & y + \varphi. \end{array} \tag{8}$$

Note that T_φ induces an action of $\mathcal{S}_\omega$ on $\mathcal{S}_\omega[y]$, and, in particular, if φ and $\psi \in \mathcal{S}_\omega$, one has that

$$T_\varphi \circ T_\psi = T_\psi \circ T_\varphi = T_{\varphi+\psi}. \tag{9}$$

Remark 1.28. If $f \in \mathcal{S}_\omega[y]$, then T_φ induces a one-to-one correspondence between the roots of f and the roots of $T_\varphi(f)$. More precisely, $\psi \in \mathcal{S}_\omega$ is a root of f iff $\psi - \varphi$ is a root of $T_\varphi(f)$.

In general, we shall use lower case Greek letters, φ, ψ to denote series in $\mathcal{S}_\omega$ and capital Greek letters, ϕ, Ψ to denote one-term series, ax^μ, with $a \in \mathbb{K}$ and $\mu \in \mathbb{Q}^N$.

In the sequel, we shall mostly consider transformations T_ϕ relative to one-term series. First, we estimate the exponent set of $T_\phi(f)$. The fundamental fact we use is:

LEMMA 1.29. *Let $\mu, \alpha \in \mathbb{Q}^N$, $j \in \mathbb{N}$, $a \in \mathbb{K}$ and $\phi = ax^\mu$. Then*

$$\varepsilon(T_\phi(x^\alpha y^j)) = \{(\alpha, j) + k(\mu, -1) \mid k \in \mathbb{N} \text{ and } 0 \le k \le j\}.$$

Proof. We have

$$T_\phi(x^\alpha y^j) = x^\alpha (y + ax^\mu)^j = x^\alpha \sum_{k=0}^{j} \binom{j}{k} a^k x^{k\mu} y^{j-k}.$$

Thus

$$\varepsilon(T_\phi(f)) = \{(\alpha, j) + k(\mu, -1) \mid k = 0, \dots, j\}. \quad \square$$

Lemma 1.29 implies that the exponents in $\varepsilon(T_\phi(x^\alpha y^j))$ consist of all the points of integer positive heights in the line segment starting at (α, j), (the only point in $\varepsilon(x^\alpha y^j)$), parallel to the vector $(\mu, -1)$ and ending at $(\alpha + j\mu, 0)$.

COROLLARY 1.30. *Let $\phi = cx^\mu$ be as in Lemma 1.29 and $f \in \mathcal{S}_\omega[y]$. Then:*

$$\varepsilon(T_\phi(f)) \subseteq \bigcup_{(\alpha,h) \in \varepsilon(f)} \big\{(\alpha, h) + k(\mu, -1) \;\big|\; 0 \le k \le h\big\}.$$

Notation 1.31. Let P be an "origin point" and $v \in \mathbb{R}^{N+1}$, $v \neq 0$ a vector. Let:

$$D^-(P, v) = \{P + \lambda v \mid \lambda \le 0\}$$

and

$$D^+(P, v) = \{P + \lambda v \mid \lambda \ge 0\}$$

denote the half-lines starting at P, parallel to v in the negative, respectively positive direction relative to v.

Lemma 1.29 immediately provides sufficient conditions for $P \in \mathbb{Q}^N \times \mathbb{N}$ to be or not to be an exponent of $T_\phi(f)$:

COROLLARY 1.32. *Let $\phi = ax^\mu$ be as in Lemma 1.29, $P \in \mathbb{Q}^N \times \mathbb{N}$ and $f \in \mathcal{S}_\omega[y]$. Then:*

(1) *If $\varepsilon(f) \cap D^-(P, (\mu, -1)) = \emptyset$, then $P \notin \varepsilon(T_\phi(f))$.*

(2) *If* $\varepsilon(f) \cap D^-(P,(\mu,-1))$ *consists of one point, then* $P \in \varepsilon(T_\phi(f))$.

For the proof of Theorem 1.15, we use a reformulation of the above Corollary:

COROLLARY 1.33. *Let* $\phi = ax^\mu$ *be as in Lemma 1.29,* $P \in \mathbb{Q}^N \times \mathbb{N}$ *and* $f \in \mathcal{S}_\omega[y]$. *Then:*

(1) *If* $P \in \varepsilon(T_\phi(f))$, *then* $\varepsilon(f) \cap D^-(P,(\mu,-1)) \neq \emptyset$.

(2) *If* $P \notin \varepsilon(T_\phi(f))$ *then the cardinality of* $\varepsilon(f) \cap D^-(P,(\mu,-1))$ *is either* 0 *or* ≥ 2.

Applying π_ω to Corollary 1.32, one obtains:

COROLLARY 1.34. *Let* $\phi = ax^\mu$ *be as in Lemma 1.29,* $Q \in \mathbb{Q}^N \times \mathbb{N}$ *and* $f \in \mathcal{S}_\omega[y]$. *Denote by* $P = \pi_\omega(Q)$ *and* $\eta = \mathrm{val}(\mu)$. *Then:*

(1) *If* $\pi_\omega(\varepsilon(f)) \cap D^-(P,(\eta,-1)) = \emptyset$, *then* $P \notin \pi_\omega(T_\phi(f))$.

(2) *If* $\pi_\omega(f) \cap D^-(P,(\mu,-1))$ *consists of one element, then* $P \in \pi_\omega(T_\phi(f))$.

Proof. Since π_ω induces a bijection between the lines

$$P + \lambda(\eta,-1) \text{ and } Q + \lambda(\mu,-1),$$

the result follows. □

Notation 1.35. Let $f \in \mathcal{S}_\omega[y]$ such that $y = 0$ is not a zero of f. Let

$$\phi(f,a) = ax^\mu,$$

where $\mu = \mathrm{slope}(f)$ and a is a root of the ω-characteristic polynomial of f.

Until the end of this paragraph, we shall restrict ourselves to the transformation $T_{\phi(f,a)}$.

Example 1.36. Let $f = y^2 + x_2^2 y - x_1^4$ and $\omega = (1,\sqrt{2})$. Then :
$\varepsilon(f) = \{(0,0,2),(0,2,1),(4,0,0)\}$.
$q^{(0)}(f) = (4,0,0)$, $q^{(1)}(f) = (0,0,2)$, $ht(f) = 2$ and $\mathrm{slope}(f) = (2,0)$.
$\chi(f) = y^2 - 1$.
$v^{(0)}(f) = (4,0)$ and $v^{(1)}(f) = (0,2)$.
The zeros of $\chi(f)$ are 1 and -1. Hence, we can either consider $\phi(f,1) = x_1^2$ or $\phi(f,-1) = -x_1^2$. And

$$T_{\phi(f,1)}(f) = y^2 + 2x_1^2 y + x_2^2 y + x_1^2 x_2^2$$

$$T_{\phi(f,-1)}(f) = y^2 - 2x_1^2 y + x_2^2 y - x_1^2 x_2^2.$$

PROPOSITION 1.37. *The lowest height vertex of the* ω*-segment of* f, $q^{(0)}(f)$ *is never an exponent of* $T_{\phi(f,a)}(f)$.

Proof. By Lemma 1.29 only the terms of f corresponding to the exponents on $\mathrm{segm}(f)$ can have a "contribution" to the term of $T_{\phi(f,a)}(f)$ with exponent $q^{(0)}(f)$. Moreover, one has:

$$T_{\phi(f,a)}\Big(f\big|_{\mathrm{segm}(f)}\Big) = T_{\phi(f,a)}(f)\big|_{\mathrm{segm}(f)}.$$

Recall that $f\big|_{\mathrm{segm}(f)}$ represents the sum of all the terms of f with the exponents on $\mathrm{segm}(f)$. Note that the terms of $T_{\phi(f,a)}$ of exponent $q^{(0)}(f)$ coincides with $T_{\phi(f,a)}(f)\big|_{\mathrm{segm}(f)}(x,0)$.

$$T_{\phi(f,a)}(f)\big|_{\mathrm{segm}(f)}(x,0) = T_{\phi(f,a)}(f\big|_{\mathrm{segm}(f)})(x,0) = f\big|_{\mathrm{segm}(f)}(x,ax^{\mu}) \tag{10}$$

Let $\beta = P_{\mathrm{hor}}(q^{(0)}(f))$ and $h = ht(f)$. By the proof of Lemma 1.29, all the terms in the right term side of (10) have the same degree in x and we have

$$f\big|_{\mathrm{segm}(f)}(x,ax^{\mu}) = x^{\beta} f\big|_{\mathrm{segm}(f)}(1,...,1,a) = 0.$$

Thus, by choice of a,

$$T_{\phi(f,a)}(f)\big|_{\mathrm{segm}(f)}(x,0) = 0,$$

which ends the proof. □

Example 1.38. Returning to Example 1.36, let

$$g = T_{\phi(f,1)}(f) = y^2 + 2x_1^2 y + x_2^2 y + x_1^2 x_2^2.$$

Then

$$\varepsilon(g) = \{(0,0,2),(2,0,1),(0,2,1),(2,2,0)\}.$$

Remark that $q^{(0)}(f) = (0,0,4)$ is no longer an exponent of g.

We shall study the effect of the transformation $T_{\phi(f,a)}$ on the ω-Newton polygon of f.

Proposition 1.39. *Let $f \in \mathcal{S}_{\omega}[y]$ be such that $y = 0$ is not a zero of f.*

(1) *All the positive height vertices of $NP_{\omega}(f)$ are vertices of $NP_{\omega}(T_{\phi(f,a)}(f))$.*
(2) *$NP_{\omega}(T_{\phi(f,a)}(f)) \subseteq NP_{\omega}(f)$.*
(3) *$v^{(0)}(f)$ does not belong to $NP_{\omega}(T_{\phi(f,a)}(f))$.*
(4) *If $y = 0$ is not a root of $T_{\phi(f,a)}(f)$, then*

$$\mathrm{val}(P_{\mathrm{hor}}(q^{(0)}(f))) < \mathrm{val}(P_{\mathrm{hor}}(q^{(0)}(T_{\phi(f,a)}(f)))).$$

Proof. We set $\eta = \mathrm{val}(\mathrm{slope}(f))$.

(1) Each vertex of $NP_{\omega}(f)$ belongs to $\pi_{\omega}(\varepsilon(f))$. Moreover, a vertex of $NP_{\omega}(f)$ of positive height, P, is the only point in $\varepsilon(f) \cap D^{-}(P,(\eta,-1))$. Then (1) follows from Corollary 1.34 (2).

(2) Applying π_ω to Corollary 1.30, one obtains

$$(11)\qquad \begin{aligned} \pi_\omega(\varepsilon(T_{\phi(f,a)}(f))) &\subseteq \bigcup_{(q,h)\in\pi_\omega(\varepsilon(f))} \big\{(q,h)+\lambda(\eta,-1) \;\big|\; 0\le\lambda\le h\big\} \\ \pi_\omega(\varepsilon(T_{\phi(f,a)}(f))) &\subseteq \bigcup_{(q,h)\in NP_\omega(f)} \big\{(q,h)+\lambda(\mu,-1) \;\big|\; 0\le\lambda\le h\big\} \end{aligned}$$

According to Remark 1.27, the right hand side term in (11) is $NP_\omega(f)$.

Since $NP_\omega(f)$ is convex and contains $P+\mathbb{R}_{\ge 0}\times\{0\}$, whenever $P\in NP_\omega(f)$, it follows that $NP_\omega(T_{\phi(f,a)}(f))\subseteq NP_\omega(f)$.

(3) If $T_{\phi(f,a)}(f)$ has $y=0$ as a root, then $NP_\omega(T_{\phi(f,a)}(f))$ has no point on the abscissa. If not,

$$v^{(0)}(T_{\phi(f,a)}(f)) = \pi_\omega(q^{(0)}(T_{\phi(f,a)}(f))) \in [v^{(0)}(f),\infty)\times\{0\}$$

and by Proposition 1.37 and (2),

$$v^{(0)}(T_{\phi(f,a)}(f)) \neq v^{(0)}(f)$$

which ends the proof.

(4) Is obvious from (3). □

Example 1.40. Returning to Examples 1.36 and 1.38
$q^{(0)}(g)=(2,2,0)$, $q^{(1)}(g)=(2,0,1)$, $ht(g)=1$ and $\mathrm{slope}(g)=(0,2)$.
$v^{(0)}(g)=(2+2\sqrt{2},0)$ and $v^{(1)}(g)=(2,1)$.
Note that $(0,0,2)\in\varepsilon(g)$ and hence $(0,2)$ remains a vertex in $NP_\omega(g)$.

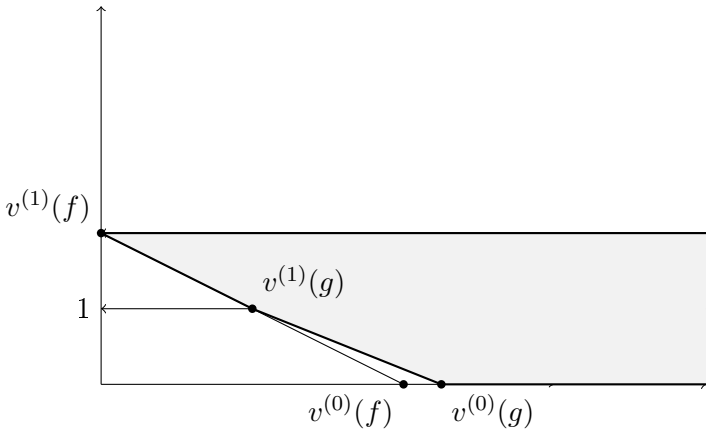

Fig. 2. $NP_\omega(f)$ and $NP_\omega(g)$.

In Fig. 2, one sees that $NP_\omega(g)\subset NP_\omega(f)$ and $v^{(0)}(f)\notin NP_\omega(g)$. Moreover, we have that

$$\mathrm{val}(P_{\mathrm{hor}}(q^{(0)}(f))) = 4 < \mathrm{val}(P_{\mathrm{hor}}(q^{(0)}(g) = 2+2\sqrt{2}.$$

Proposition 1.39 has several immediate consequences:

COROLLARY 1.41. *Assume that $y = 0$ is not a zero of $T_{\phi(f,a)}(f)$. Then:*

(1) *Either $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)}(f))$ or the height of $q^{(1)}(T_{\phi(f,a)}(f))$ is strictly smaller than the height of $q^{(1)}(f)$.*

(2) *The slope of the line $[v^{(1)}(f), v^{(0)}(f)]$ is strictly smaller than the slope of the segment $[v^{(1)}(T_{\phi(f,a)}(f)), v^{(0)}(T_{\phi(f,a)}(f))]$, i.e.*

$$-\frac{1}{\mathrm{val}(\mathrm{slope}(f))} < -\frac{1}{\mathrm{val}(\mathrm{slope}(T_{\phi(f,a)}(f)))},$$

or equivalently,

$$\mathrm{val}(\mathrm{slope}(f)) < \mathrm{val}(\mathrm{slope}\, T_{\phi(f,a)}(f)).$$

Proof. (1) Since π_ω preserves the height, it suffices to work on the corresponding vertices on the Newton polygon.

Above the height of $v^{(1)}(f)$, $NP_\omega(f)$ and $NP_\omega(T_{\phi(f,a)}(f))$ are identical (see Proposition 1.39 (1)).

Below the height of $v^{(1)}(f)$, one has the configuration in Fig. 3, (see Proposition 1.39 (2) and (4)).

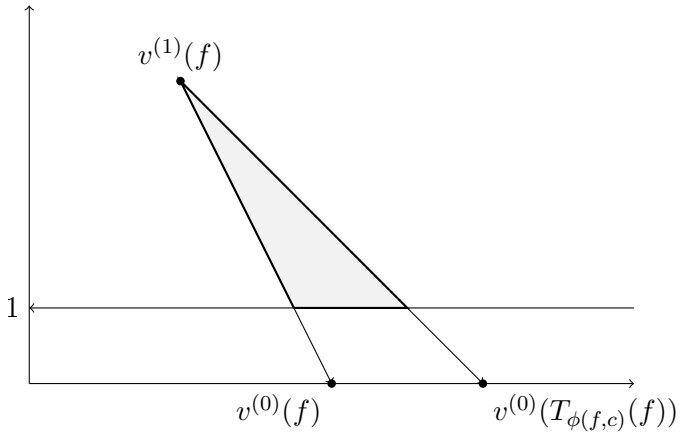

Fig. 3.

Thus, if $v^{(1)}(T_{\phi(f,a)}(f)) \neq v^{(1)}(f)$, then $v^{(1)}(T_{\phi(f,a)}(f))$ can only find itself in the emphasized triangle in Fig. 3.

(2) In both situations, of (1), $v^{(1)}(T_{\phi(f,a)}(f))$ belongs to the triangle in Fig. 3 and the statement is clear. □

Example 1.42. Let $f = y^n + \varphi(x)$, where $\varphi \in \mathcal{S}_\omega$ has more than one term. In the Figs. 4 and 5 are represented the Newton polygons of f and $T_{\phi(f,a)}(f)$.

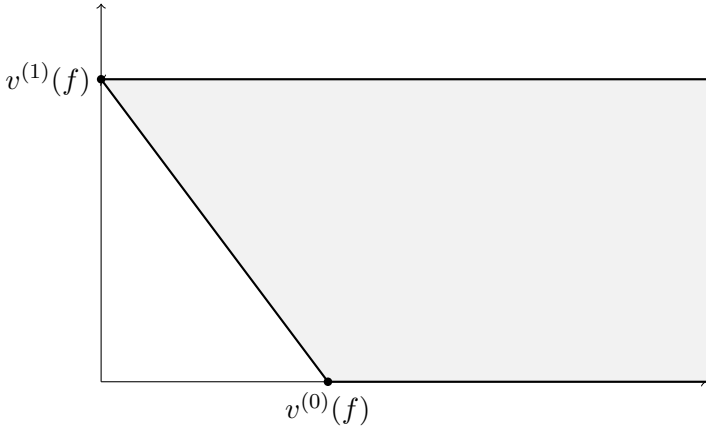

Fig. 4. $NP_\omega(f)$.

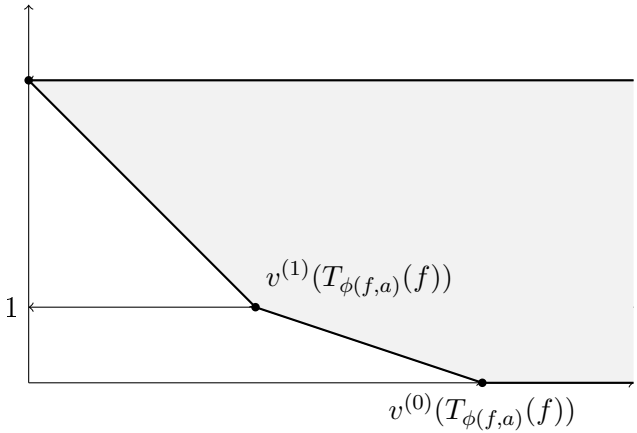

Fig. 5. $NP_\omega(T_{\phi(f,a)})$.

Note that by Proposition 1.39 and Corollary 1.41 it follows that $v^{(1)}(f)$ has height n and $v^{(1)}(T_{\phi(f,a)}(f))$ has height 1. Moreover, let $g = T_{\phi(f,a)}(f)$ and a' be a zero of $\chi(g)$, the characteristic polynomial of g. Then, using Proposition 1.39 again, one gets that $v^{(1)}(T_{\phi(g,a')}(g)) = v^{(1)}(g)$ (see also Remark 1.43 below).

In particular, let $f = y^4 - x_1^4 - 4x_1^2x_2^2$ and $\omega = (1, \sqrt{2})$. Then $ht(f) = 4$, $\text{slope}(f) = (1, 0)$ and $a = 1$ is a root of the ω-characteristic polynomial. Hence, $\phi(f, 1) = x_1$ and

$$g = T_{\phi(f,1)}(f) = y^4 + 4x_1y^3 + 6x_1^2y^2 + 4x_1^3y - 4x_1^2x_2^2 \text{ and } ht(g) = 1.$$

Furthermore, $\text{slope}(g) = (-1, 1)$ and $a = 1$ is a root of $\chi(f)$. Hence, $\phi(g, 1) =$

$x_1^{-1}x_2$ and

$T_{\phi(g,1)}(g) = y^4+4x_1^{-1}x_2y^3+4x_1y^3+6x_1^2x_2^2y^2+12x_2y^2+6x_1^2y+4x_1^{-3}x_2^3y+4x_1^3y+$
$12x_1^3x_2^2y + 12x_1x_2y + x_1^4x_2^4 + 4x_1^{-2}x_2^3 + 6x_1^4x_2^2$ and $ht(T_{\phi(1,g)})(g) = 1...$ etc .

Remark 1.43. In the context of Corollary 1.41 if $v^{(1)}(f)$ has height 1, then

$$v^{(1)}(T_{\phi(f,a)}(f)) = v^{(1)}(f),$$

for in this case, the emphasized triangle in Fig. 3 is reduced to a point.

COROLLARY 1.44. *Let $f \in \mathcal{S}_\omega[y]$ be such that $y = 0$ is not a zero of f or of $T_{\phi(f,a)}(f)$. Let h be the height of $q^{(1)}(f)$ and let $P \in [q^{(1)}(f), q^{(0)}(f)]$ be the point of height $h-1$. If $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)}(f))$, then $P \in \varepsilon(f)$ and $P \notin \varepsilon(T_{\phi(f,a)}(f))$.*

Proof. According to Proposition 1.39, the lowest finite edges of $NP_\omega(f)$ and $NP_\omega(T_{\phi(f,a)}(f))$ have only the point $v^{(1)}(f)$ in common, and moreover, the line segment $[v^{(1)}(f), v^{(0)}(f)]$ is outside $NP_\omega(T_{\phi(f,a)}(f))$. Hence, $\pi_\omega(P) \notin \pi_\omega(\varepsilon(T_{\phi(f,a)}(f)))$ or equivalently, $P \notin \varepsilon(T_{\phi(f,a)}(f))$. Note that

$$\varepsilon(f) \cap D^-(P, (\text{slope}(f), -1)) = \{q^{(1)}(f), P\}.$$

Indeed, the inclusion " $\subset$ " is obvious, and according to Corollary 1.33 (2) the left hand side term has at least 2 points. Thus, $P \in \varepsilon(f)$. $\square$

Finally, we want to establish a link between the lattice containing $\varepsilon(f)$ and the lattice containing $\varepsilon(T_{\phi(f,a)}(f))$.

PROPOSITION 1.45. *Let $f \in \mathcal{S}_\omega[y]$ such that $y = 0$ is not a zero of f or $T_{\phi(f,a)}(f)$.*

(1) *If $\varepsilon(f) \subset {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$ and $\text{slope}(f) \in {}^1\!/_c\mathbb{Z}^N$, then $\varepsilon(T_{\phi(f,a)}(f)) \subset {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$.*

(2) *If $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)})$ and $\varepsilon(f) \subset {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$, then $\text{slope}(f) \in {}^1\!/_c\mathbb{Z}^N$.*

Proof. (1) Let $P \in \varepsilon(T_{\phi(f,a)}(f))$. By Corollary 1.30 there existe $Q \in \varepsilon(f)$ such that

$$P \in D^+(Q, (\mu, -1)),$$

where $\mu = \text{slope}(f)$.

This means that

$$P = Q + k(\mu, -1),$$

for some $k \in \mathbb{N}$. Hence, $P \in {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$.

(2) Denote by P the point on $\text{segm}(f)$ at height $h-1$, where h is the height of $q^{(1)}(f)$. By Corollary 1.44, $P \in \varepsilon(f)$. Since $P = q^{(0)}(f) + (\text{slope}(f), -1) \in {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$, it follows that $\text{slope}(f) \in {}^1\!/_c\mathbb{Z}^N$. $\square$

Combining (1) and (2) from the above Proposition, one obtains:

COROLLARY 1.46. *Let $f \in \mathcal{S}_\omega[y]$ such that $y = 0$ is not a zero of f and $T_{(\phi(f,a)}(f)$ and assume that $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)})$. If $\varepsilon(f) \subset {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}$, then*

$$\varepsilon(T_{\phi(f,a)}(f)) \subset {}^1\!/_c\mathbb{Z}^N \times \mathbb{N}\,.$$

1.6. **Barrier Wedges**

A barrier wedge is a convex "container" of the exponent sets of the coefficients of a polynomial $f \in \mathcal{S}_\omega[y]$. More importantly, it provides the link between the ω-positive cones that contain the exponents of f and the "dynamic" of f in the privileged direction slope(f).

Definition 1.47. Let $\mathcal{C} \subset \mathbb{R}^n$ be an ω-positive cone and $l \subset \mathbb{R}^N \times \mathbb{R}$ be a non-horizontal line (*i.e.* a line non-parallel to the first factor, see Notation 1.16). The wedge of opening $\mathcal{C}$ and of edge l is the set

$$W(\mathcal{C}, l) = \bigcup_{P \in l} \{P + (\beta, 0) \mid \beta \in \mathcal{C}\}.$$

Thus the horizontal sections of $W(\mathcal{C}, l)$ are cones "starting" in a point on the line l.

Remark 1.48. Any point $P \in W(\mathcal{C}, l)$ can be written uniquely

$$P = P_1 + v,$$

where $P_1 \in l$ and $v \in \mathcal{C} \times \{0\}$, (see Fig. 6). Moreover, P and P_1 have the same height.

Remark 1.49. If $v_l \in \mathbb{R}^N \times \mathbb{R}$ is a vector parallel to l, then $P \in W(\mathcal{C}, l)$ iff $P + v_l \in W(\mathcal{C}, l)$.

If $v \in \mathcal{C} \times \{0\}$ and $P \in W(\mathcal{C}, l)$, then $P + v \in W(\mathcal{C}, l)$ (see Fig. 7.)

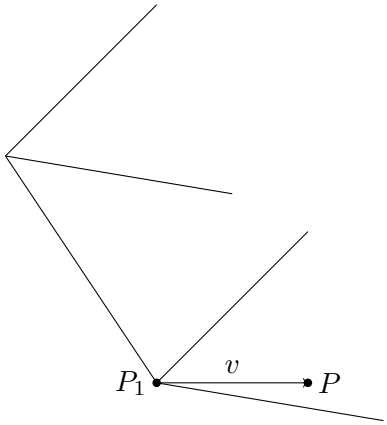

Fig. 6.

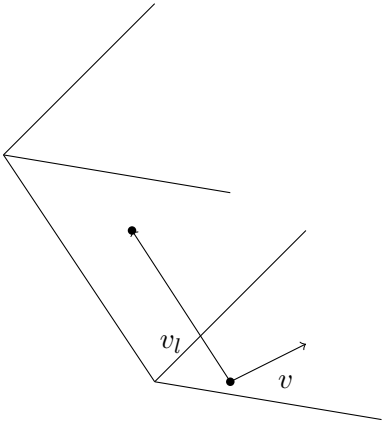

Fig. 7.

We shall need a linear algebra result:

LEMMA 1.50. *Let $W(\mathcal{C}, l) \subset \mathbb{R}^N \times \mathbb{R}$ be the wedge of opening $\mathcal{C}$ and edge l.*

(1) *If $P + v \in W(\mathcal{C}, l)$ with $P \in l$ and v a vector in $\mathbb{R}^N \times \mathbb{R}$, then for $Q \in W(\mathcal{C}, l)$, the half-line $\{Q + \lambda v \mid \lambda \in \mathbb{R}_{\geq 0}\}$ is contained in $W(\mathcal{C}, l)$.*

(2) *Let $P \in l$ and $(\delta, -1) \in \mathbb{R}^N \times \mathbb{R}$ parallel to the edge l then*

$$\delta + \mathcal{C} = \left\{\mu \in \mathbb{R}^N \;\middle|\; P + (\mu, -1) \in W(\mathcal{C}, l)\right\}.$$

Proof. (1) According to Remark 1.48

$$P + v = P_1 + w,$$

with $P_1 \in l$ and $w \in \mathcal{C} \times \{0\}$. Hence

$$v = (P_1 - P) + w = v_l + w,$$

where $v_l = P_1 - P$ is a vector parallel to l.

Thus, for $Q \in W(\mathcal{C}, l)$ and $\lambda \geq 0$,

$$Q + \lambda v = Q + \lambda v_l + \lambda w.$$

Since λv_l is a vector parallel to l and $\lambda w \in \mathcal{C}$, we are done by Remark 1.48.

(2) Let $P_0 \in l$ be the point of height 0 and $P_{-1} \in l$ be the point of height -1. Then,

$$P_{-1} = P_0 + (\delta, -1).$$

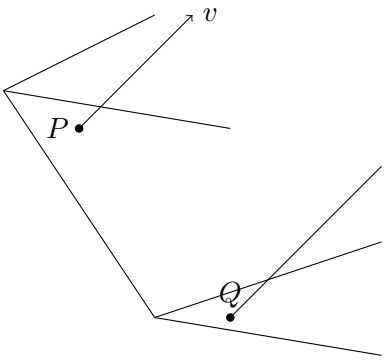

Fig. 8. Lemma 1.50 (1).

By Remark 1.48, one can take $P = P_0$ and:

$$P_0 + (\mu, -1) \in W(\mathcal{C}, l) \text{ iff there exists } v \in \mathcal{C} \times \{0\} \text{ such that}$$

$$P_0 + (\mu, -1) = P_{-1} + v = P_0 + (\delta, -1) + v,$$

which ends the proof.

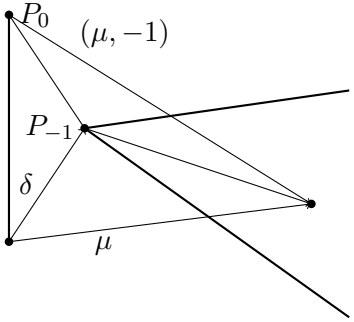

Fig. 9. Lemma 1.50 (2). □

Definition 1.51. Let $f \in \mathcal{S}_\omega[y]$. A wedge $W(\mathcal{C}, l)$ is called an ω-barrier wedge for f if:

(1) $\mathcal{C}$ is an ω-positive rational cone

(2) $\varepsilon(f) \subset W(\mathcal{C}, l)$.

(3) The edge l contains a point in $\varepsilon(f)$

Example 1.52. Let $f = y^n + \varphi(x)$ with $\varphi(x) \in \mathcal{S}_\omega$. There exists $\delta \in \mathbb{Q}^N$ and $\mathcal{C} \subset \mathbb{R}^N$, an ω-positive cone such that $\varepsilon(\varphi) \subset \delta + \mathcal{C}$. Let l be the line determined by $(\underline{0}, n)$ and $(\delta, 0)$, where $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^N$. Then $W(\mathcal{C}, l)$ is an ω-barrier wedge for f (see Fig. 10).

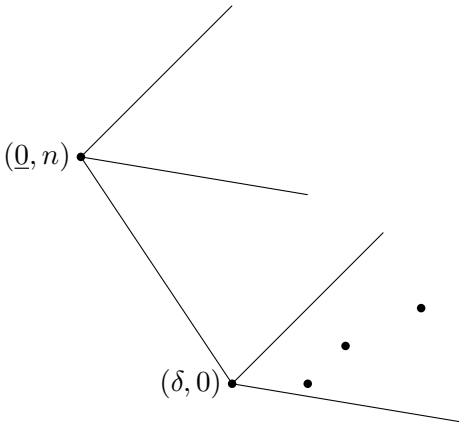

Fig. 10.

PROPOSITION 1.53. *Let $f \in \mathcal{S}_\omega[y]$ and let Q be a point in $\varepsilon(f)$ such that $\pi_\omega(Q)$ is a vertex in $NP_\omega(f)$. Then there exists an ω-barrier wedge of f containing Q in its edge.*

Proof. Let

$$f = \sum_{i=1}^{d} \varphi_i(x) y^i$$

and let l be a supporting line of $NP_\omega(f)$ passing through $\pi_\omega(Q)$. One checks easily that there exists a line $\tilde{l}$ through $Q \in \mathbb{R}^N \times \mathbb{R}$ such that $\pi_\omega(\tilde{l}) = l$, the correspondence being one-to-one, and denote by Q_i the point of height i on $\tilde{l}$ with $1 \leq i \leq d$.

Since

$$NP_\omega(f) \subset \pi_\omega(l) + (\mathbb{R}_{\geq 0} \times \{0\})$$

it follows that

$$\operatorname{val}(P_{\mathrm{hor}}(Q_i)) \leq \operatorname{val}(\varphi_i) \text{ for } i = 0, 1, \ldots, d.$$

According to Proposition 1.2 I.(4) for each i there exists an ω-positive rational cone $\mathcal{C}_i$ such that

$$\varepsilon(\varphi_i) \subset P_{\mathrm{hor}}(Q_i) + \mathcal{C}_i.$$

If $\mathcal{C}$ is an ω-positive cone such that $\mathcal{C}_i \subset \mathcal{C}$ for all i, then one checks easily that $W(\mathcal{C}, l)$ is an ω-barrier wedge for f. $\square$

Finally, we want to study the relation between ω-barrier wedges and the transformation $T_{\phi(f,a)}(f)$.

PROPOSITION 1.54. *Let $f \in \mathcal{S}_\omega[y]$ such that $y = 0$ is not a zero of f. Then there exists a non-horizontal line in $\mathbb{R}^N \times \mathbb{R}$ through $q^{(1)}(f)$ and an ω-positive cone $\mathcal{C} \subset \mathbb{R}^N$ such that $W(\mathcal{C}, l)$ is an ω-barrier wedge for f. Moreover, if the vector $(\delta, -1)$ is parallel to the line l, then:*

(1) $\operatorname{slope}(f) \in \delta + \mathcal{C}$.

(2) $\varepsilon(T_{\phi(f,a)}(f)) \subset W(\mathcal{C}, l)$.

(3) *If* $y = 0$ *is not a zero of* $T_{\phi(f,a)}(f)$ *and* $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)}(f))$, *then* $W(\mathcal{C}, l)$ *is also an* ω*-barrier wedge for* $T_{\phi(f,a)}(f)$.

Proof. The existence of the ω-barrier wedge $W(\mathcal{C}, l)$ follows from Proposition 1.53.

(1) Since $W(\mathcal{C}, l)$ is a convex set and $q^{(1)}(f), q^{(0)}(f) \in W(\mathcal{C}, l)$ it follows that $q^{(1)}(f) + (\operatorname{slope}(f), -1) \in W(\mathcal{C}, l)$. By Lemma 1.50 (2), it follows that $\operatorname{slope}(f) \in \delta + \mathcal{C}$.

(2) According to Corollary 1.32 each exponent of $T_{\phi(f,a)}(f)$ is on the half-line starting at an exponent parallel to $(\operatorname{slope}(f), -1)$ in the positive direction. Since $q^{(1)}(f) + (\operatorname{slope}(f), -1) \in W(\mathcal{C}, l)$, we get the result by Lemma 1.50 (1).

(3) Obvious since $q^{(1)}(f) = q^{(1)}(T_{\phi(f,a)}(f)) \in l$. □

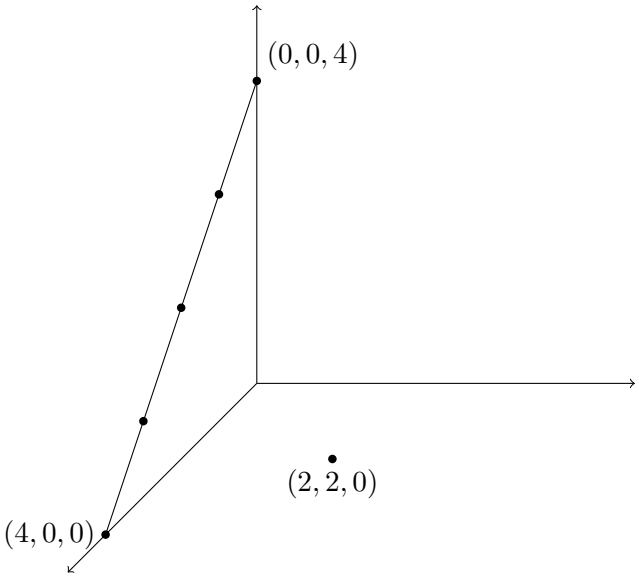

Fig. 11.

Example 1.55. Returning to Example 1.42 let $f = y^4 - x_1^4 - 4x_1^2x_2^2$ and $\omega = (1, \sqrt{2})$. Then:

$\varepsilon(f) = \{(0, 0, 4), (4, 0, 0), (2, 2, 0)\}$

$\operatorname{slope}(f) = (1, 0)$

$g = T_{\phi(f,1)}(f) = y^4 + 4x_1y^3 + 6x_1^2y^2 + 4x_1^3y - 4x_1^2x_2^2$

$\varepsilon(g) = \{(0, 0, 4), (1, 0, 3), (2, 0, 2), (3, 0, 1), (2, 2, 0)\}$

Let l be a line of direction $(\delta, -1)$ where $\delta \in \mathbb{R}^2$, $\delta = (0,0)$ and $\mathcal{C} \subset \mathbb{R}^2$, be the ω-positive cone $\mathcal{C} = \{x_1 \geq 0, x_2 \geq 0\}$.

One checks immediately that $\varepsilon(f) \subset W(\mathcal{C}, l)$ and since $(0,0,4) \in l$, it follows that $W(\mathcal{C}, l)$ is an ω-barrier wedge for f. Moreover, $\text{slope}(f) = (1,0) \in \delta + \mathcal{C} = \mathcal{C}$.

One can also check that $\varepsilon(g) \subset W(\mathcal{C}, l)$, and even though $q^{(1)}(f) \neq q^{(1)}(g)$, since $(0,0,4) \in l \cap \varepsilon(g)$ and $(0,0,4)$ gives a vertex in $NP_\omega(g)$, $W(\mathcal{C}, l)$ is an ω-barrier wedge for g.

1.7. **The Proof of the Theorem**

Let $f \in \mathcal{S}_\omega[y]$ be a polynomial of degree ≥ 1. We want to show that f has a root in $\mathcal{S}_\omega$. If $y = 0$ is a root of f, then we are done so from now on we shall suppose that $f(0) \neq 0$.

Define inductively the following sequence of polynomials using the automorphisms T_ϕ:

- $f^{(0)} = f$
- $f^{(i+1)} = T_{\phi(f^{(i)}, a^{(i)})}(f^{(i)})$,

if $f^{(i)}(0) \neq 0$, where $a^{(i)}$ is any root of the characteristic polynomial of $f^{(i)}$. If $f^{(i)} = 0$, then we have found a root for f (see Step 1 below).

Recall that $T_{\phi(f^{(i)}, a^{(i)})}(f^{(i)}) = f^{(i)}(x, y + a^{(i)} x^{\text{slope}(f^{(i)})})$, see Notation 1.35.

Step 1: *If $f^{(d)}(0) = 0$, then*

$$\psi(x) = \sum_{i=0}^{d-1} a^{(i)} x^{\text{slope}(f^{(i)})}$$

is a root of f.

Indeed, since

$$f^{(d)} = T_{\phi(f^{(d-1)}, a^{(d-1)})}(f^{(d-1)}) = \cdots = T_\psi(f^{(0)})(y) = T_\psi(f)(y),$$

if $y = 0$ is a root of $f^{(d)}$, then $y + \psi$ is a root of f according to Remark 1.28. Note that in this case, f has a root that is a polynomial in a ring $\mathbb{K}[x^{1/n}, x^{-1/n}]$ for some $n \in \mathbb{N}$.

From now on, we suppose that $f^{(i)} \neq 0$ for all $i \in \mathbb{N}$ and set

$$\varphi = \sum_{i=0}^{\infty} a^{(i)} x^{\text{slope}(f^{(i)})}.$$

Since $\text{val}(\text{slope}(f^{(i)}))_{i \in \mathbb{N}}$ is a strictly increasing sequence, see Corollary 1.41 (2), it follows that φ is a well defined fractional power series.

We shall check that φ is a series in $\mathcal{S}_\omega$ and a root of f.

Step 2: *The sequence of points* $(q^{(1)}(f^{(i)}))_{i\in\mathbb{N}}$ *is constant for* i *large enough.*

By Corollary 1.41 (1), the heights of $(q^{(1)}(f^{(i)}))_{i\in\mathbb{N}}$ form a decreasing sequence of natural numbers, which yields the result.

Definition 1.56. The constant value of the sequence $(q^{(1)}(f^{(i)}))_{i\in\mathbb{N}}$ for i large enough will be called the pivot of f; it will be denoted $\mathrm{pivot}(f)$.

Remark that if $q^{(1)}(f^{(k)}) = \mathrm{pivot}(f)$ for a $k \in \mathbb{N}$, then $q^{(1)}(f^{(i)}) = \mathrm{pivot}(f)$ for all $i \geq k$.

Step 3: *Let* $k \in \mathbb{N}$ *be such that* $q^{(1)}(f^{(k)}) = \mathrm{pivot}(f)$. *There exists an* ω*-positive cone* $\mathcal{C} \in \mathbb{R}^N$ *and a line* $l \in \mathbb{R}^N \times \mathbb{R}$ *with* $\mathrm{pivot}(f) \in l$ *such that* $W(\mathcal{C}, l)$ *is an* ω*-barrier wedge of* $f^{(i)}$ *for all* $i \geq k$.

By Proposition 1.53, there exists an ω-barrier wedge $W(\mathcal{C}, l)$ for $f^{(k)}$ with $\mathrm{pivot}(f) \in l$ and by Proposition 1.54 (3), $W(\mathcal{C}, l)$ is an ω-barrier wedge for $f^{(i)}$ for all $i \geq k$.

In the proof of Step 4, we shall need the following:

Corollary 1.57. *Let* $k \in \mathbb{N}$ *be such that* $q^{(1)}(f^{(k)}) = \mathrm{pivot}(f)$. *Then there exists an* ω*-positive cone* $\mathcal{C} \subset \mathbb{R}^N$ *and* $\delta \in \mathbb{Q}^N$ *such that the sequence* $(q^{(0)}(f^{(i)}))_{i\geq k} \subset \delta + \mathcal{C}$.

Proof. Indeed, if $W(\mathcal{C}, l)$ is the ω-barrier wedge in Step 3, such that

$$\varepsilon(f^{(i)}) \subset W(\mathcal{C}, l)$$

for all $i \geq k$, then if $(\delta, 0)$ is the point of height 0 of the line l, then $q^{(0)}(f^{(i)}) \subset \delta + \mathcal{C}$ for all $i \geq k$. $\square$

Step 4: *There exists an* ω*-positive cone* $\mathcal{C} \subset \mathbb{R}^N$ *and a vector* $\delta \in \mathbb{Q}^N$ *such that*

$$\varepsilon(\varphi) \subset \delta + \mathcal{C}.$$

Note that

$$\varepsilon(\varphi) = \{(\mathrm{slope}(f^{(i)}))_{i\in\mathbb{N}}\}.$$

Let $k \in \mathbb{N}$ such that $q^{(1)}(f^{(k)}) = \mathrm{pivot}(f)$. By Step 3, there exists an ω-barrier wedge $W(\mathcal{C}, l)$ for $f^{(i)}$ for all $i \geq k$ such that $\mathrm{pivot}(f^{(i)}) \in l$. Consider the vector $(\delta, -1) \in \mathbb{R}^N \times \mathbb{R}$ parallel to l. By Proposition 1.54 (1), $\mathrm{slope}(f^{(i)}) \in \delta + \mathcal{C}$ for all $i \geq k$.

Consequently, possibly modifying the vector $\delta \in \mathbb{Q}^N$, one can ensure that the $\mathrm{slope}(f^{(i)}) \in \delta + \mathcal{C}$ for all $i \in \mathbb{N}$, hence

$$\varepsilon(\varphi) \subset \delta + \mathcal{C}.$$

Step 5: *There exists* $c \in \mathbb{N}$ *such that*

$$\varepsilon(\varphi) \subset {}^1\!/_c\,\mathbb{Z}^n \text{ and } \varepsilon(f^{(i)}) \subset {}^1\!/_c\,\mathbb{Z}^n \times \mathbb{N} \text{ for all } i \in \mathbb{N}. \tag{12}$$

Indeed, let $k \in \mathbb{N}$ such that $q^{(1)}(f^{(k)}) = \text{pivot}(f)$ and let $c_1 \in \mathbb{N}$ be such that

$$\varepsilon(f^{(k)}) \subset {}^1\!/_{c_1}\,\mathbb{Z}^n \times \mathbb{N}.$$

By Corollary 1.46,

$$\varepsilon(f^{(i)}) \subset {}^1\!/_{c_1}\,\mathbb{Z}^n \times \mathbb{N}$$

for all $i \geq k$. Moreover, Proposition 1.45 (2),

$$\text{slope}(f^{(i)}) \in {}^1\!/_{c_1} \in \mathbb{Z}^N$$

for all $i \geq k$. Now for a convenient multiple c of c_1, (12) is obvious.

Step 4 and Step 5 imply that $\varphi \in \mathcal{S}_\omega$.

Step 6: $\lim_i(\text{val}(f^{(i)}(x,0)) = \infty$.

Note that $\text{val}(f^{(i)}(x,0))$ represents the abscissa of $p^{(0)}(f^{(i)}) = \pi_\omega(q^{(0)}(f^{(i)}))$. By Proposition 1.39 (4), the sequence $\text{val}(f^{(i)}(x,0))$ is strictly increasing. Let $k \in \mathbb{N}$ such that $q^{(1)}(f^{(k)}) = \text{pivot}(f)$. By Corollary 1.57 and Step 5, there exists an ω-positive cone $\mathcal{C} \subset \mathbb{R}^N$, a vector $\delta \in \mathbb{Q}^N$ and $c \in \mathbb{N}$, such that

$$\{(q^{(0)}(f^{(i)}))_{i>k}\} \subset (\delta + \mathcal{C}) \cap {}^1\!/_c\,\mathbb{Z}^N$$

and according to Proposition 1.2 (2),

$$\text{val}((\delta + \mathcal{C}) \cap {}^1\!/_c\,\mathbb{Z}^N)$$

is an infinite discrete set.

Step 7: φ *is a root of* f.

We split φ as

$$\varphi = T_k(\varphi) + R_k(\varphi),$$

where

$$T_k(\varphi) = \sum_{i=0}^{k-1} a^{(i)} x^{\text{slope}(f^{(i)})}$$

and

$$R_k(\varphi) = \sum_{i=k}^{\infty} a^{(i)} x^{\text{slope}(f^{(i)})}.$$

Then

$$f(x,\varphi(x)) = f(x, T_k(\varphi)(x) + R_k(\varphi)(x)) = f(x, T_k(\varphi)(x)) + \psi(x).$$

Let $n \in \mathbb{N}$. We shall show that for k large enough, $\text{val}(f(x, T_k(\varphi)(x)))$ and $\text{val}(\psi(x))$ are both $\geq n$, and this implies that

$$\text{val}(f(x,\varphi(x))) = \infty \text{ equivalently } f(x,\varphi(x)) = 0.$$

Write f as

$$f(x,y)=\sum_{i=0}^{d}\varphi_i(x)y^i,$$

with $\varphi_i\in\mathcal{S}_\omega$.

As in Step 1,

$$f(x,T_k(\varphi)(x))=f^{(k)}(x,0)$$

and by step 6, for k large enough, $\mathrm{val}(f(x,T_k(\varphi)(x)))>n$.

On the other hand, the terms of $\psi(x)$ are of the form

$$m=a\varphi_i(x)x^{\alpha'}x^{\alpha''},$$

where $a\in\mathbb{K}$, $x^{\alpha'}$ is a monomial coming from $T_k(\varphi)$ and $x^{\alpha''}$ is a monomial coming from $R_k(\varphi)$, *i.e.* α', $\alpha''\in\varepsilon(\varphi)$ and:

$$\mathrm{val}(\alpha')<\mathrm{val}(\mathrm{slope}(f^{(k)})),$$

$$\mathrm{val}(\alpha'')\geq\mathrm{val}(\mathrm{slope}(f^{(k)})).$$

Thus

$$\mathrm{val}(m)\geq d_1+d_2+\mathrm{val}(\mathrm{slope}(f^{(k)})),$$

where

$$d_1=\min_{0\leq i\leq d}\mathrm{val}(\varphi_i)$$

and

$$d_2=\mathrm{val}(\mathrm{slope}(f^{(0)})).$$

Hence, if $\mathrm{val}(\mathrm{slope}(f^{(k)}))>n-d_1-d_2$, then for any term of m, one has that $\mathrm{val}(m)>n$.

Consequently, for k large enough,

$$\mathrm{val}(\psi)>n$$

and we are done.

Remark 1.58. The construction of the polynomials $(f^{(i)})_{i\in\mathbb{N}}$ depends at every step i on the choice of a root of the ω-characteristic polynomial of $f^{(i-1)}$, and each choice produces a different root of f. Since f has finitely many roots, it follows that for i large enough, the ω-characteristic polynomial of $f^{(i)}$ has only one root with multiplicity equal to the degree of $\chi(f^{(i)})$.

2. THE RING $\mathcal{S}_\omega[y]$

Most of the material in Section 2 and Section 3 comes from [2]. The examples, the second proof of Lemma 2.18, the proof of Proposition 3.1 (Proposition 9.1 in [2]) and section 3.4 are original. The algorithm in section 3.3 has been reformulated and modified to improve efficiency.

2.1. The η-order in $\mathcal{S}_\omega[y]$

Let $y = (y_1, \ldots, y_M)$ be a family of variables. In this Section, we study some of the algebraic properties of the M-variable polynomial ring $\mathcal{S}_\omega[y_1, \ldots, y_M]$.

Note that $\mathcal{S}_\omega[y]$ contains the polynomial ring $\mathbb{K}[x, y]$, where $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_M)$. It also contains the polynomial rings $\mathbb{K}[x, x^{-1}, y]$ and even $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ for all $l \in \mathbb{N}$.

Let

$$f = \sum_{\beta \in B} \varphi_\beta(x) y^\beta$$

be an element of $\mathcal{S}_\omega[y]$, where $B \subset \mathbb{N}^M$ is a finite set. Then, there exists an ω-positive convex cone, $\mathcal{C} \subset \mathbb{R}^N$, a vector $\delta \in \mathbb{Q}^N$ and $c \in \mathbb{N}$ such that $\varepsilon(\varphi_\beta(x)) \subset \mathcal{I}(\mathcal{C}, \delta, c)$ for all $\beta \in B$, (for the definition of $\mathcal{I}(\mathcal{C}, \delta, c)$ see 5) and f can be written:

$$f = \sum_{\substack{\beta \in B \\ \alpha \in \mathcal{I}(\mathcal{C}, \delta, c)}} a_{\alpha,\beta} x^\alpha y^\beta,$$

where each $a_{\alpha,\beta} \in \mathbb{K}$.

As in the previous Section, the exponent set of f is

$$\varepsilon(f) = \big\{(\alpha, \beta) \in \mathbb{Q}^N \times \mathbb{N}^M \;\big|\; \alpha \in \mathcal{I}(\mathcal{C}, \delta, c),\ \beta \in B,\ a_{\alpha,\beta} \neq 0\big\}.$$

We are interested in monomial orderings on the set of (x, y)-monomials in $\mathcal{S}_\omega[y]$ that coincide with the ω-weighted ordering on the x-monomials of $\mathcal{S}_\omega$.

For this, we shall "complete" the vector $\omega \in \mathbb{R}^N$ with a vector $\eta = (\eta_1, \ldots, \eta_M)$ and define the (ω, η)-order for a monomial $m = x^\alpha y^\beta$ as:

$$\mathrm{val}_{(\omega,\eta)}(m) = \mathrm{val}(\alpha) + \eta \cdot \beta,$$

where $\eta \cdot \beta$ is the inner product.

The key point of the construction of the Puiseux series parametrization is that we do not fix an η, but choose it such that (ω, η) is a point in the tropical variety of certain ideals. The price to pay is that the ordering on the monomials of $\mathcal{S}_\omega[y]$ may no longer be a total ordering. In fact, we need vectors η such that the initial ideal in "direction" η (see definition below) is monomial free, which means that the ordering is essentially not total. Moreover, sometimes the variety that we want to parametrize is in a subvariety given by $\{y_{i_1} = \cdots = y_{i_k} = 0\}$. For technical reasons, we want to exclude these coordinates. In order to do that, we assign the value ∞ to $\eta_{i_1}, \ldots, \eta_{i_k}$, and set

$$\mathrm{Fin}(\eta) = \{i \mid \eta_i < \infty\}.$$

Thus, we can consider $\eta \in (\mathbb{R} \cup \{\infty\})^M$.

To make everything compatible, the following conventions will be needed:

$$\infty \cdot a = \infty,$$

for all $a \in \mathbb{R}_{>0}$ and

$$\infty \cdot 0 = 0.$$

There is an obvious order relation on $(\mathbb{R} \cup \{\infty\})^M$. If $\eta = (\eta_1, \ldots, \eta_M)$, $\eta' = (\eta'_1, \ldots, \eta'_M) \in (\mathbb{R} \cup \{\infty\})^M$, then $\eta \leq \eta'$ iff $\eta_i \leq \eta'_i$ for all i. We shall say that η is strictly smaller that η' iff $\eta \leq \eta'$ and $\eta_i < \eta'_i$ for all $i \in \mathrm{Fin}(\eta)$.

Definition 2.1. Let $\eta \in (\mathbb{R} \cup \{\infty\})^M$, and $f \in \mathcal{S}_\omega[y]$,

$$f = \sum_{\beta \in B} \varphi_\beta(x) y^\beta,$$

the η-order of f is

$$\mathrm{val}_{(\omega,\eta)}(f) = \inf_{\beta \in B} (\mathrm{val}(\varphi_\beta) + \eta \cdot \beta).$$

If $\lambda = \mathrm{val}_{(\omega,\eta)}(f)$, $\lambda < \infty$, the η-initial part of f is

$$\mathrm{in}_{(\omega,\eta)}(f) = \sum_{\{\beta |\, \mathrm{val}(\varphi_\beta) + \eta \cdot \beta = \lambda} \mathrm{in}(\varphi_\beta) y^\beta.$$

As usual, we will write val_η and in_η instead of $\mathrm{val}_{(\omega,\eta)}$ respectively $\mathrm{in}_{(\omega,\eta)}$ if no confusion is likely.

Example 2.2. Let $N = 1$, $M = 2$, $\omega = (1)$, $\eta = (1, 1)$ and $f = (1 + x^2)y_1 - (1 + x^{1/2})y_2$. Then $val_\eta(f) = 1$ and $\mathrm{in}_\eta(f) = y_1 - y_2$.

Remark 2.3. If $f \neq 0$, then $\mathrm{val}_\eta(f) = \infty$ iff f belongs to the ideal generated by $\{y_i \mid i \notin \mathrm{Fin}(\eta)\}$.

Remark 2.4. Since $\mathrm{in}(\varphi_\beta)$ is a one-term series depending on the variable x, $\mathrm{in}_\eta(f)$ is a polynomial in the ring $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ for a suitable $l \in \mathbb{N}$. Furthermore, $\mathrm{in}_\eta(f)$ is a partial sum of the sum of initial parts of all terms in f,

$$\sum_{\beta \in B} \mathrm{in}(\varphi_\beta) y^\beta.$$

In particular, we have

$$\mathrm{val}_\eta(f) = \mathrm{val}_\eta \left(\sum_{\beta \in B} \mathrm{in}\,(\varphi_\beta) y^\beta \right)$$

and

$$\mathrm{in}_\eta(f) = \mathrm{in}_\eta \left(\sum_{\beta \in B} \mathrm{in}\,(\varphi_\beta) y^\beta \right).$$

Remark 2.5. If $\eta_i = \infty$, then no term containing y_i appears in $\mathrm{in}_\eta(f)$.

In analogy with formula (2) in Remark 1.11, any $f \in \mathcal{S}_\omega[y]$ can be written

$$f = \sum_{\imath \in \mathbb{N}} f^{(i)}(x, y) + h, \tag{13}$$

where h belongs to the ideal $\langle \{y_i \mid i \notin \mathrm{Fin}(\eta)\}\rangle$, each $f^{(i)}$ is a polynomial in y whose coefficients are terms in $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$, for a suitable $l \in \mathbb{N}$, where terms have the same η-order and such that the sequence $(\mathrm{val}_\eta(f^{(i)}))_i$ is strictly increasing.

In analogy with Proposition 1.10 one shows easily that the η-order and the η-initial part have the following properties:

PROPOSITION 2.6. *Let f and g be two elements in $\mathcal{S}_\omega[y]$. Then*

(1) $\mathrm{val}_\eta(f+g) \geq \min(\mathrm{val}_\eta(f), \mathrm{val}_\eta(g))$
(2) $\mathrm{val}_\eta(f+g) \neq \min(\mathrm{val}_\eta(f),\ \mathrm{val}_\eta(g))$ *iff* $\mathrm{val}_\eta(f) = \mathrm{val}_\eta(g)$ *and* $\mathrm{in}_\eta(f) + \mathrm{in}_\eta(g) = 0$,
(3) $\mathrm{val}_\eta(f \cdot g) = \mathrm{val}_\eta(f) + \mathrm{val}_\eta(g)$
(4) $\mathrm{in}_\eta(f \cdot g) = \mathrm{in}_\eta(f) \cdot \mathrm{in}_\eta(g)$
(5) $\mathrm{in}_\eta(f) = \mathrm{in}_\eta(\mathrm{in}_\eta(f))$.

In the sequel, we shall have to consider M-tuples of series in $\mathcal{S}_\omega$. As in Section 1, we shall use lower case Greek letters like $\varphi = (\varphi_1, \ldots, \varphi_M)$ for general M-tuples and capital Greek letters like $\phi = (\phi_1, \ldots, \phi_M)$ to denote M-tuples where each component is either a one-term series or 0.

We can extend the ω-order and the ω-initial term for M-tuples of Puiseux series.

Definition 2.7. If $\varphi = (\varphi_1, \ldots, \varphi_M) \in \mathcal{S}_\omega$, then

$$\mathrm{val}(\varphi) = (\mathrm{val}(\varphi_1), \ldots, \mathrm{val}(\varphi_M))$$

and

$$\mathrm{in}(\varphi) = (\mathrm{in}(\varphi_1), \ldots, \mathrm{in}(\varphi_M)).$$

Consider the mapping $\mathcal{S}_\omega[y] \to \mathcal{S}_\omega$ given by substituting $y = (y_1, \ldots, y_m)$ with the M-tuple of series $\varphi = (\varphi_1, \ldots, \varphi_M)$. We study now the link between the order, respectively the initial term of $f(\varphi)$ and η-order, respectively the η-initial part of f, where $\eta = \mathrm{val}(\varphi)$.

PROPOSITION 2.8. *Let $f \in \mathcal{S}_\omega[y]$ and $\varphi = (\varphi_1, \ldots, \varphi_M)$. Then*

(1) $\mathrm{val}_{\mathrm{val}(\varphi)}(f) = \mathrm{val}(f(\varphi))$
(2) $\mathrm{in}_{\mathrm{val}(\varphi)}(f)(\mathrm{in}(\varphi)) = \mathrm{in}(f(\varphi))$.

Proof. Assume first that $f = \psi y^\beta$, with $\psi \in \mathcal{S}_\omega$. Then, one has immediatly that

$$\mathrm{val}_{\mathrm{val}(\varphi)}(\psi y^\beta) = \mathrm{val}(\psi) + \beta\,\mathrm{val}(\varphi))$$

and

$$\mathrm{val}(\psi\varphi^\beta) = \mathrm{val}(\psi) + \beta\,\mathrm{val}(\varphi)),$$

hence, the formula holds for f.

For f general, $f = \sum \psi_\beta y^\beta$, the equality holds term by term, and consequently, holds for f.

To prove the second formula, assume again that $f = \psi y^\beta$. Then:

$$\mathrm{in}_{\mathrm{val}(\varphi)}(\psi y^\beta)(\mathrm{in}(\varphi)) = (\mathrm{in}(\psi)y^\beta)(\mathrm{in}(\varphi)) = \mathrm{in}(\psi)\cdot(\mathrm{in}(\varphi))^\beta$$

and

$$\mathrm{in}(\psi\,\mathrm{in}(\varphi)^\beta) = \mathrm{in}(\psi)\cdot(\mathrm{in}(\varphi))^\beta$$

hence, the formula holds for f.

For f general, $f = \sum \psi_\beta y^\beta$, the results follows from the particular case and the fact that the terms that contribute to $\mathrm{in}(f(\varphi))$ and $\mathrm{in}_{\mathrm{val}(\varphi)}(f)$ are the same. □

COROLLARY 2.9. *Under the hypothesis of Proposition 2.8, if φ is a zero of f, then* $\mathrm{in}(\varphi)$ *is a zero of* $\mathrm{in}_{\mathrm{val}(\varphi)}(f)$.

Note that all the terms of the form $\mathrm{in}_{\mathrm{val}(\varphi)}(f)$ have the same $\mathrm{val}(\varphi)$-order. Consequently, all the terms that appear when we replace y with $\mathrm{in}(\varphi)$ have the same exponent in $x = (x_1, \ldots, x_N)$. Thus Proposition 2.8 implies :

COROLLARY 2.10. *Under the hypothesis of Proposition 2.8,*

$$\mathrm{in}_{\mathrm{val}(\varphi)}(f)(\mathrm{in}(\varphi)) = 0 \text{ iff } \mathrm{in}(\varphi)(\underline{1}) \text{ is a zero of } \mathrm{in}_{\mathrm{val}(\varphi)}(f(\underline{1}, y)),$$

where $\underline{1}$ *is the vector* $(1, \ldots, 1) \in \mathbb{R}^N$.

2.2. Initial Ideals in $\mathcal{S}_\omega[y]$

In this paragraph, we define the initial ideal of an ideal in $\mathcal{S}_\omega[y]$ relative to an η-order. The same definitions carry over to the subrings of $\mathcal{S}_\omega[y]$.

If $I \subset \mathcal{S}_\omega[y]$ is an ideal, then the η-initial ideal of I is the ideal in $\mathcal{S}_\omega[y]$:

$$\mathrm{in}_\eta(I) = \langle\{\mathrm{in}_\eta(f) \mid f \in I, \mathrm{val}_\eta(f) < \infty\} \cup \{y_i \mid y_i \notin \mathrm{Fin}(\eta)\}\rangle_{\mathcal{S}_\omega[y]}.$$

We define in the same way the η-initial ideal in $I \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ of an ideal $I \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$, for some $l \in \mathbb{N}$.

As an example, we compute the η-initial ideal of a maximal ideal of $\mathcal{S}_\omega[y]$ for all possible values $\eta \in (\mathbb{R}\cup\{\infty\})^M$. This example is a fundamental fact in the construction of the Puiseux parametrization and makes the link with the tropical variety. Note that since $\mathcal{S}_\omega$ is an algebraically closed field, any maximal ideal $I \subset \mathcal{S}_\omega[y]$ is of the form :

$$I_\varphi = \langle y_1 - \varphi_1, \ldots, y_M - \varphi_M\rangle,$$

where $\varphi = (\varphi_1, \ldots, \varphi_M) \in \mathcal{S}_\omega^M$.

Notation 2.11. Let $\varphi \in \mathcal{S}_\omega^M$ and $\eta \in (\mathbb{R}\cup\{\infty\})^M$ such that $\eta \leq \mathrm{val}(\varphi)$ componentwise, (recall that we use small case Greek letters to denote series or M-tuples of series and capital Greek letters to denote M-tuples of one-term series), we set $\phi(\eta,\varphi) = (\phi_1(\eta,\varphi), \ldots, \phi_M(\eta,\varphi))$, where

$$\phi_i(\eta,\varphi) = \begin{cases} \mathrm{in}\,\varphi_i & \text{if} \quad \mathrm{val}\,\varphi_i = \eta_i, \\ 0 & \text{if} \quad \mathrm{val}\,\varphi_i > \eta_i. \end{cases}$$

Proposition 2.12. *Let $\varphi = (\varphi_1, \ldots, \varphi_M) \in \mathcal{S}_\omega^M$, $\eta \in \left(\mathbb{R}\cup\{\infty\}\right)^M$ and let I_φ be the maximal ideal associated to φ. Then:*

(1) *If there exists $i \in \{1, \ldots, M\}$ such that $\mathrm{val}(\varphi_i) < \eta_i$ then $\mathrm{in}_\eta(I_\varphi) = \mathcal{S}_\omega[y]$.*
(2) *If $\eta \leq \mathrm{val}(\varphi)$ componentwise, then $\mathrm{in}_\eta(I_\varphi) = I_{\phi(\eta,\varphi)}$, the maximal ideal associated to $\phi(\eta,\varphi)$.*

Proof. Since

$$\mathrm{val}_\eta(y_i - \varphi_i) = \min\left(\eta_i, \mathrm{val}(\varphi_i)\right)$$

it follows that

$$\mathrm{in}_\eta(y_i - \varphi_i) \quad = \quad \begin{cases} y_i, & \text{if} \quad \eta_i < \mathrm{val}(\varphi_i), \\ y_i - \mathrm{in}(\varphi_i), & \text{if} \quad \eta_i = \mathrm{val}(\varphi_i), \\ \mathrm{in}\,\varphi_i, & \text{if} \quad \eta_i > \mathrm{val}\,\varphi_i. \end{cases}$$

It follows that if $\mathrm{val}(\varphi_i) < \eta_i$, then $\mathrm{in}_\eta(y_i - \varphi_i) = \mathrm{in}\,\varphi_i$ is an invertible element in $\mathcal{S}_\omega[y]$ which implies statement (1) of the Proposition.

Let $P \in I_\varphi$. Then P can be written as follows:

$$P = (y_1-\varphi_1)P_1(y_1, ..., y_M) + (y_2-\varphi_2)P_2(y_2, ..., y_M) + ... + (y_M-\varphi_M)P_M(y_M).$$

Hence, $\mathrm{in}_\eta(P)$ is a partial sum of

$$\mathrm{in}_\eta(P) = \sum_{i=1}^{M} \mathrm{in}_\eta(y_i - \varphi_i)\,\mathrm{in}_\eta\left(P_i(y_i, .., y_M)\right) = \sum_{i=1}^{M} \phi_i(\eta,\varphi)\,\mathrm{in}_\eta\left(P_i(y_i, .., y_M)\right),$$

and so, $\mathrm{in}_\eta(I_\varphi) \subset I_{\phi(\eta,\varphi)}$. Since the other inclusion is obvious, we are done. $\square$

Remark 2.13. Under the hypothesis in Proposition 2.12, if $\eta \leq \mathrm{val}(\varphi)$, then the η initial ideal of I_φ is the maximal ideal associated to the M-tuple of one-term series $\phi(\eta, \varphi)$. Moreover, $\eta = \mathrm{val}(\varphi)$, is the only value $\eta \in \left(\mathbb{R} \cup \{\infty\}\right)^M$ such that

$$\mathrm{in}_\eta(I_\varphi) = I_{\mathrm{in}(\varphi)}.$$

PROPOSITION 2.14. *Let I and J be two ideals in $\mathcal{S}_\omega[y]$. Then the following hold:*

(1) $\mathrm{in}_\eta(I \cap J) \subseteq \mathrm{in}_\eta(I) \cap \mathrm{in}_\eta(J)$,
(2) *if* $I \subseteq J$, *then* $\mathrm{in}_\eta(I) \subseteq \mathrm{in}_\eta(J)$,
(3) $\mathrm{in}_\eta(I \cdot J) \subseteq \mathrm{in}_\eta(I \cap J)$,
(4) $\mathrm{in}_\eta(I) \cdot \mathrm{in}_\eta(J) \subseteq \mathrm{in}_\eta(I \cdot J)$.

Proof. The first two properties are obvious. For the third one, we know that $I \cdot J \subset I \cap J$ and we apply (2). For the forth property we use that $\mathrm{in}_\eta(f \cdot g) = \mathrm{in}_\eta(f) \cdot \mathrm{in}_\eta(g)$, which proves our result. □

Proposition 2.12 describes in particular the zero set of a maximal ideal I_φ, namely,

$$V(in_\eta(I_\varphi)) = \begin{cases} \{\phi(\eta, \varphi)\}, & if \quad \mathrm{val}\,\varphi \leq \eta, \\ \emptyset, & otherwise. \end{cases}$$

Using Hilbert's Nullstellensatz, this result can be extended to ideals in $\mathcal{S}_\omega[y]$ with a finite number of zeros.

PROPOSITION 2.15. *Let I be an ideal in $\mathcal{S}_\omega[y]$, such that $V(I) \subseteq \mathcal{S}_\omega^M$ is a finite set. Then we have*

$$V(\mathrm{in}_\eta(I)) = \bigcup_{\varphi \in V(I)} V(\mathrm{in}_\eta(I_\varphi)),$$

where I_φ is the maximal ideal associated to $\varphi \in \mathcal{S}_\omega^M$.

Proof. Since $I \subseteq \bigcap_{\varphi \in V(I)} I_\varphi$ Hilbert's Nullstellensatz, implies that there exists $k \in \mathbb{N}$ such that

$$\left[\bigcap_{\varphi \in V(I)} I_\varphi\right]^k \subseteq I \subseteq \bigcap_{\varphi \in V(I)} I_\varphi.$$

By applying to this Proposition 2.14(1) and (2), one gets that

$$\left[\mathrm{in}_\eta\left(\bigcap_{\varphi \in V(I)} I_\varphi\right)\right]^k \subseteq \mathrm{in}_\eta\left(\left[\bigcap_{\varphi \in V(I)} I_\varphi\right]^k\right) \subseteq \mathrm{in}_\eta(I) \subseteq \mathrm{in}_\eta\left(\bigcap_{\varphi \in V(I)} I_\varphi\right)$$

and from Proposition 2.14(3) and (4) it follows that

$$\prod_{\varphi\in V(I)} \operatorname{in}_\eta(I_\varphi) \subseteq \operatorname{in}_\eta\left(\prod_{\varphi\in V(I)} I_\varphi\right) \subseteq \operatorname{in}_\eta\left(\bigcap_{\varphi\in V(I)} I_\varphi\right) \subseteq \bigcap_{\varphi\in V(I)} \operatorname{in}_\eta(I_\varphi).$$

Since

$$V\left(\prod_{\varphi\in V(I)} I_\varphi\right) = V\left(\bigcap_{\varphi\in V(I)} \operatorname{in}_\eta(I_\varphi)\right)$$

and these are the zero sets of the extreme terms of the above inclusions, hence it follows that all the zero sets of the ideals in the inclusions coincide. In particular,

$$V\left(\operatorname{in}_\eta\left(\bigcap_{\varphi\in V(I)} I_\varphi\right)\right) = V\left(\bigcap_{\varphi\in V(I)} \operatorname{in}_\eta(I_\varphi)\right).$$

But since

$$V\left(\bigcap_{\varphi\in V(I)} \operatorname{in}_\eta(I_\varphi)\right) = \bigcup_{\varphi\in V(I)} V(\operatorname{in}_\eta(I_\varphi)),$$

one gets

$$V(\operatorname{in}_\eta(I)) = \bigcup_{\varphi\in V(I)} V(\operatorname{in}_\eta(I_\varphi)). \quad \square$$

Remark 2.16. Proposition 2.15 implies in particular that all the zeros of $V(\operatorname{in}_\eta(I))$ are one-term series in $\mathcal{S}_\omega^M$.

PROPOSITION 2.17. *Let I be an ideal in $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and let $\eta \in \left(\mathbb{R}\cup\{\infty\}\right)^M$. Then*

$$\operatorname{in}_\eta(\operatorname{in}_\eta(I)) = \operatorname{in}_\eta(I).$$

Proof. Since $\operatorname{in}_\eta(\operatorname{in}_\eta(f)) = \operatorname{in}_\eta(f)$ for any $f \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ it follows that the inclusion $\operatorname{in}_\eta(\operatorname{in}_\eta(I)) \supseteq \operatorname{in}_\eta(I)$ holds.

The other inclusion, follows immediately from the Lemma below. $\square$

LEMMA 2.18. *Under the hypothesis of Proposition 2.17, if $f \in \operatorname{in}_\eta(I)$, then there exists $g \in I$ such that $\operatorname{in}_\eta(f) = \operatorname{in}_\eta(g)$.*

Proof. First, remark that if $P \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and $h \in I$, then $P\operatorname{in}_\eta(h)$ is a sum of η-initial parts of elements in I.

Indeed, if

$$P = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha y^\beta,$$

then

$$P\,\mathrm{in}_\eta(h) = \sum_{\alpha,\beta} a_{\alpha,\beta}x^\alpha y^\beta\,\mathrm{in}_\eta(h) = \sum_{\alpha,\beta}\mathrm{in}_\eta\left(a_{\alpha,\beta}x^\alpha y^\beta h\right).$$

Thus, if $f \in \mathrm{in}_\eta(I)$, then

$$f = \sum_i P_i\,\mathrm{in}_\eta(h_i),$$

with $P_i \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and $h_i \in I$. With the remark above,

$$f = \sum_{j\in J}\mathrm{in}_\eta(f_j),$$

where each $f_j \in I$.

One can obviously suppose, that for any subset J' in J, the partial sum

$$\sum_{j\in J'}\mathrm{in}_\eta(f_j) \neq 0.$$

This, in turn implies that $\mathrm{in}_\eta(f)$ is a partial sum of $\sum_{j\in J}\mathrm{in}_\eta(f_j)$ (the sum of η-initial parts $\mathrm{in}_\eta(f_j)$ with minimal η-order). Consequently,

$$g = \sum_{j\in J} f_j \in I$$

and $\mathrm{in}_\eta(f) = \mathrm{in}_\eta(g)$. □

We also give an alternative proof for Lemma 2.18.

Proof. Let

$$f = \sum_{i=1}^{r} g_i\,\mathrm{in}_\eta(h_i),$$

with $g_i \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and $h_i \in I$.

Let $d = \mathrm{val}_\eta(f)$ and $d_i = \mathrm{val}_\eta(h_i)$ and denote by $g_i^{(u)}$ the sum of the terms of g_i having the η-order u, if any.

Then

$$\mathrm{in}_\eta(f) = \sum_i g_i^{(d-d_i)}\,\mathrm{in}_\eta(h_i) = \sum_i \mathrm{in}_\eta(g_i^{(d-d_i)}h_i)$$

which ends the proof. □

Remark 2.19. The second proof also holds for ideals in $\mathcal{S}_\omega[y]$.

2.3. The constructive solution in $\mathcal{S}_\omega[y]$

In this paragraph, we construct inductively the terms of a zero of an ideal $I \subset \mathcal{S}_\omega[y]$ having a finite zero set. The construction is done modulo finding at each step of a good direction $\eta^{(i)} \in (\mathbb{R} \cup \{\infty\})^M$ and of a zero of the $\eta^{(i)}$-initial ideal of an inductively constructed ideal. In the next Section, we shall see that if I is the extension of a suitable ideal in the polynomial ring $\mathbb{K}[x, x^{-1}, y]$ this good direction is a point in the tropical variety of this polynomial ideal.

We return to the case of a maximal ideal.

PROPOSITION 2.20. *Let $I = I_\varphi$ be a maximal ideal in $\mathcal{S}_\omega[y]$ and $\eta \in \left(\mathbb{R} \cup \{\infty\}\right)^M$ such that $\mathrm{in}_\eta(I)$ is a proper ideal of $\mathcal{S}_\omega[y]$. The following statements are equivalent:*

(1) $\eta = \mathrm{val}(\phi(\eta, \varphi))$.
(2) $\phi(\eta, \varphi) = \mathrm{in}(\varphi)$,
(3) $\phi_i(\eta, \varphi) = 0$ *if and only if* $\eta_i = \infty$ *for all* $i \in \{1, \ldots, M\}$,
(4) $\mathrm{in}_\eta(I_\varphi) \cap \mathcal{S}_\omega\left[(y_i)_{i \in \mathrm{Fin}(\eta)}\right]$ *is monomial free.*

For the definition of $\phi(\eta, \varphi)$ see Notation 2.11.

Proof. The equivalence of the first three statements follows almost immediately from Proposition 2.12.

"1 $\Rightarrow$ 2" It can be easily deduced from Proposition 2.12 since

$$\phi_i(\eta, \varphi) = \begin{cases} \mathrm{in}(\varphi_i), & \text{if} \quad \mathrm{val}(\varphi_i) = \eta_i, \\ 0, & \text{if} \quad \mathrm{val}(\varphi_i) > \eta_i. \end{cases}$$

and if ϕ_i is 0 for some i, this would imply that $\eta_i = \mathrm{val}(\phi_i(\eta, I)) = \infty$.

"2 $\Rightarrow$ 1", "2 $\Rightarrow$ 3" and "3 $\Rightarrow$ 1" are direct consequences of Proposition 2.12.

"3 $\Rightarrow$ 4" Assume the opposite. Let $x^\alpha y^\beta$ be a monomial in $\mathrm{in}_\eta(I)$ such that $\eta_i = 0$ for all $i \notin \mathrm{Fin}(\eta)$. Since $\phi_i(\eta, \varphi) \neq 0$ for every $i \in \mathrm{Fin}(\eta)$, $x^\alpha(\phi(\eta, \varphi))^\beta \neq 0$ but $\phi(\eta, \varphi)$ is a solution for $\mathrm{in}_\eta(I_{\phi(\eta,\varphi)})$ which leads to a contradiction, hence we have that $\mathrm{in}_\eta(I_{\phi(\eta,\varphi)}) \cap \mathcal{S}_\omega\left[(y_i)_{i \in \mathrm{Fin}(\eta)}\right]$ is monomial free.

"4 $\Rightarrow$ 3" Since $\mathrm{in}_\eta(I) \neq \mathcal{S}_\omega[y]$, $\eta \leq \mathrm{val}(\varphi)$. Assume (3) is false. Then there exists $i \in \mathrm{Fin}(\eta)$ such that $\phi_i(\eta, \varphi) = 0$. Let $v \in \mathbb{R}^M$ the vector,

$$v_i = \begin{cases} 0, & i \notin \mathrm{Fin}(\eta), \\ 1, & i \in \mathrm{Fin}(\eta). \end{cases}$$

Then $\phi(\eta, \varphi)$ is a zero of y^v. Since I is a maximal ideal, $y^v \in \mathrm{in}_\eta(I)$, which contradicts our hypothesis that we have no monomials in $\mathrm{in}_\eta(I) \cap \mathcal{S}_\omega\left[(y_i)_{i \in \mathrm{Fin}(\eta)}\right]$ is monomial free. □

Proposition 2.20 implies in the case of an ideal with finitely many zeros the following Proposition:

PROPOSITION 2.21. *Let I be an ideal in $\mathcal{S}_\omega[y]$ such that $V(I)$ is a finite set and $\eta \in (\mathbb{R}\cup\{\infty\})^M$. Then the following statements are equivalent:*

(1) *there exists $\phi \in V(\mathrm{in}_\eta(I))$ such that $\mathrm{val}(\phi) = \eta$,*
(2) *there exists $\varphi \in V(I)$ with $\eta \leq \mathrm{val}(\varphi)$ componentwise such that $\phi(\eta, \varphi) = \mathrm{in}(\varphi)$*
(3) *there exists $\phi \in V(\mathrm{in}_\eta(I))$ such that $\phi_i = 0$ iff $i \notin \mathrm{Fin}(\eta)$,*
(4) *$\mathrm{in}_\eta(I) \cap \mathcal{S}_\omega\left[(y_i)_{i\in \mathrm{Fin}(\eta)}\right]$ is monomial free.*

Proof.

"1 $\Leftrightarrow$ 2" and "2 $\Leftrightarrow$ 3" are clear from Proposition 2.20 and Proposition 2.15, since for $\phi \in V(\mathrm{in}_\eta(I))$, there exists $\varphi \in V(I)$ such that $\phi \in V(\mathrm{in}_\eta(I_\varphi))$.

"3 $\Rightarrow$ 4"Same as in the proof of Proposition 2.20.

"4 $\Rightarrow$ 3" We know that $\mathrm{in}_\eta(I) \neq \mathcal{S}_\omega[y]$. Assume (3) is false. Then any zero $\varphi \in V(\mathrm{in}_\eta(I))$ is not in the set $\{\mathrm{val}(\varphi) \mid \varphi \in V(I)\}$ and hence, there exists $i \in \mathrm{Fin}(\eta)$ such that $\phi_i \neq 0$. Consider the vector $v \in \mathbb{R}^M$,

$$v_i = \begin{cases} 0, & i \notin \mathrm{Fin}(\eta), \\ 1, & i \in \mathrm{Fin}(\eta). \end{cases}$$

Then ϕ is a zero for y^v and Nullstellensatz implies there exists $k \in \mathbb{N}$ such that $y^{kv} \in \mathrm{in}_\eta(I)$ which contradicts the hypothesis. □

Let $\varphi \in \mathcal{S}_\omega^M$ be an M-tuple of series. As in Paragraph 1.5, consider the $\mathcal{S}_\omega$-algebra morphism

$$\begin{aligned} T_\varphi : \qquad \mathcal{S}_\omega[y] &\longrightarrow \mathcal{S}_\omega[y] \\ y &\longmapsto y + \varphi \end{aligned}$$

Remark 2.22. Let $I \subset \mathcal{S}_\omega[y]$ be an ideal. As in Remark 1.28, T_φ induces a one-to-one correspondence between the zeros of I, $V(I)$ and the zeros of $T_\varphi(I)$, $V(T_\varphi(I))$. More precisely, $\psi \in V(I)$ iff $\psi - \varphi \in V(T_\varphi(I))$.

We have the following Lemma:

LEMMA 2.23. *Let $\varphi \in \mathcal{S}_\omega^M$ and ϕ an M-tuple of one-term series. Then:*

$$\mathrm{val}(\varphi - \phi) \begin{cases} > \mathrm{val}(\phi), & \text{if } \mathrm{in}\,\varphi = \phi, \\ \leq \mathrm{val}(\phi), & \text{otherwise.} \end{cases}$$

Example 2.24. Lemma 2.23 allows us to see how a transformation T_ϕ acts on $V(I)$, where I is an ideal in $\mathcal{S}_\omega[y]$. To get some intuition in what happens, let us suppose $M = 2$.

Let $\phi = (\phi_1, \phi_2) = \mathrm{in}(\psi)$ for some $\psi \in V(I)$ and assume $\phi_i \neq 0$ for $i = 1, 2$. Assume moreover that $\phi_i \neq \varphi_i$ for all $\varphi \in V(I)$, $i = 1, 2$. This means that no component of a zero φ is a one-term-series equal to ϕ_i. This assumption ensures that we do not get ∞ values in $\mathrm{val}(V(T_\phi(I)))$.

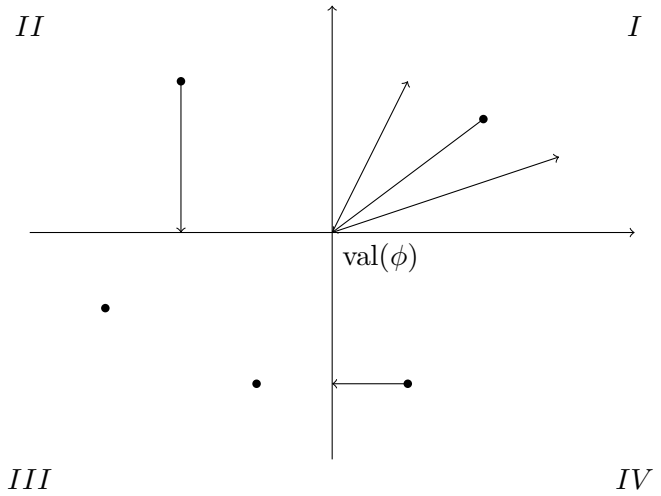

Fig. 12.

If $\mathrm{val}(\varphi) = \mathrm{val}(\phi)$ and $\mathrm{in}(\varphi_i) = \mathrm{in}(\phi_i)$, then $\mathrm{val}(T_\phi(\varphi))$ is in the interior of quadrant I.
If $\mathrm{val}(\varphi) = \mathrm{val}(\phi)$ and $\mathrm{in}(\varphi_i) \neq \mathrm{in}(\phi_i)$ for $i = 1, 2$, (meaning $\mathrm{in}(\varphi_i)$ and ϕ differ by a coefficient), then $\mathrm{val}(T_\phi(\varphi)) = \mathrm{val}(\varphi)$.
If $\mathrm{val}(\varphi)$ is in quadrant I, then $\mathrm{val}(T_\phi(\varphi)) = \mathrm{val}(\phi)$.
If $\mathrm{val}(\varphi)$ is in quadrant II, then $\mathrm{val}(T_\phi(\varphi))$ is the projection of $\mathrm{val}(\varphi)$ onto the abscissa.
If $\mathrm{val}(\varphi)$ is in quadrant III, then $\mathrm{val}(T_\phi(\varphi)) = \mathrm{val}(\phi)$.
If $\mathrm{val}(\varphi)$ is in quadrant IV, then $\mathrm{val}(T_\phi(\varphi))$ is the projection of $\mathrm{val}(\varphi)$ onto the ordinate.

In conclusion, $\mathrm{val}(T_\phi(\varphi))$ belongs to the interior of quadrant I iff $\mathrm{in}(\varphi) = \phi$.

Definition 2.25. Let $I \subset \mathcal{S}_\omega[y]$ be an ideal and $\left(\phi^{(i)}\right)_{i\in\mathbb{N}}$ a sequence of M-tuples of one-term series . Consider the following sequence of ideals

$I^{(0)} = I$
$I^{(i+1)} = T_{\phi^{(i)}}\left(I^{(i)}\right)$.

The sequence $\left(\phi^{(i)}\right)_{i\in\mathbb{N}}$ is called an ω-sequence for the ideal I if:

(1) The sequence $\mathrm{val}\left(\phi^{(i)}\right)$ is coordinatewise strictly increasing,
(2) $\phi^{(i)}$ is a zero for $\mathrm{in}_{\mathrm{val}\left(\phi^{(i)}\right)}\left(I^{(i)}\right)$.

A finite sequence of one-term series $\left(\phi^{(i)}\right)_{i=0,k}$ satisfying (1) and (2) above will be called a (k, ω)-sequence for I.

PROPOSITION 2.26. *Let I be an ideal in $\mathcal{S}_\omega[y]$ and $\varphi \in V(I)$. If*

$$\varphi = \sum_{i\in\mathbb{N}} \phi^{(i)}$$

where $\phi^{(i)} \in \mathcal{S}_\omega^M$ is an M-tuple of one-term series and $\operatorname{val}\left(\phi^{(i)}\right)$ is coordinate-wise strictly increasing, then $\phi^{(i)}$ is an ω-sequence for I.

The proof is obvious.

PROPOSITION 2.27. *Let $I \subset \mathcal{S}_\omega[y]$ be an ideal such that $V(I)$ is a finite set.*

(1) *If $\left(\phi^{(i)}\right)_{i\in\mathbb{N}}$ is an ω-sequence for I, then $\left(\phi^{(i)}\right)_{i\in\mathbb{N}}$ are the terms of zero of I.*
(2) *If $\left(\phi^{(i)}\right)_{i=1,k}$ is a (k,ω)-sequence for I, then there exists $\phi^{(k+1)}$, an M-tuple of one-term series such that $\left(\phi^{(i)}\right)_{i=1,k+1}$ is a $(k+1,\omega)$-sequence for I.*

Proof. (1) $\phi^{(0)}$ is a zero for $\operatorname{in}_{\operatorname{val}\left(\phi^{(0)}\right)}\left(I^{(0)}\right)$. According to Proposition 2.21, there exists $\varphi \in V(I)$ such that $\phi^{(0)} = \operatorname{in}\varphi$. The only zeros of $I^{(0)}$ with ω-order greater than $\operatorname{val}\left(\phi^{(0)}\right)$ are of the form $\varphi - \phi^{(0)}$, with $\varphi \in \{\psi \in V(I) \operatorname{in}(\psi) = \phi^{(0)}\}$, see Lemma 2.23 and Example 2.24.

Iterating this reasoning, since $V(I)$ is finite, it follows that $\left(\phi^{(i)}\right)$ are the terms of a zero of $\varphi \in V(I)$.

(2) By the proof of (1), there exists a zero $\varphi \in V(I)$ such that $\left(\phi^{(i)}\right)_{i=1,k}$ are the first $k+1$ (with respect to the ω-order) of φ. Hence,

$$\phi^{(k+1)} = \operatorname{in}\left(\varphi - \sum_{k}^{i=0} \phi^{(i)}\right)$$

completes $\left(\phi^{(i)}\right)_{i=1,k}$ to a $(k+1,\omega)$-series. □

Proposition 2.27 (1) immediately implies:

COROLLARY 2.28. *Let $(\phi^{(i)})_{i\in\mathbb{N}}$ be an ω-sequence for $I \subset \mathcal{S}_\omega[y]$. Then there exists an ω-positive cone $\mathcal{C} \subset \mathbb{R}^N$, a vector $\delta \in \mathbb{Q}^N$ and $c \in \mathbb{N}$ such that $\varepsilon(\phi_j^{(i)}) \subset (\delta + \mathcal{C}) \cap {}^1\!/_c\,\mathbb{Z}^N$, for all $i \in \mathbb{N}$ and $1 \le j \le M$.*

Remark 2.29. The proof of Proposition 2.27 describes an algorithm to construct the terms of a solution of I.

3. $\mathcal{S}_\omega$-SOLUTIONS TO POLYNOMIAL EQUATIONS

3.1. Extended Ideals

Let $P_1, \ldots, P_k$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ and I be the ideal generated by $P_1, \ldots, P_k$. If $N = \dim(I)$, denote $M = n - N$. Using Noether's Normalization Lemma, there exists a coordinate change in $\mathbb{K}[x_1, \ldots, x_n]$ such that after renaming the variables as (x, y), $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_M)$, the natural morphism

$$\mathbb{K}[x] \to \mathbb{K}[x, y]/I$$

is integral. This means that the natural projection morphism

$$\pi : V(I) \to \mathbb{K}^N$$

is dominant and of finite generic fiber.

Following [2], we shall say that with this choice of coordinates the ideal I is N-admissible.

Extending the scalars, to $\mathcal{S}_\omega[y]$, we get the commutative diagram

$$\begin{array}{ccc} \mathbb{K}[x] & \xrightarrow{f} & \mathbb{K}[x, y]/I \\ \downarrow & & \downarrow \\ \mathcal{S}_\omega & \xrightarrow{g} & \mathcal{S}_\omega[y]/I^e \end{array}$$

where, I^e denotes the extension of the ideal I to $\mathcal{S}_\omega[y]$. Since f is integral, it follows that g is integral and, hence, $V(I^e)$ is a finite set.

We shall produce parametrizations of the zeros of I by studying the finite set of zeros of the extended ideal I^e, which are M-tuples of Puiseux series in $\mathcal{S}_\omega$.

Since when constructing inductively the terms of an element of $V(I^e)$, one may obtain negative or fractional exponents, we shall work with extensions of the ideal I to the intermediary rings $\mathbb{K}[x, x^{-1}, y]$ or even $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ for some $l \in \mathbb{N}$.

An ideal $J \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ for $l \in \mathbb{N}$ is said to be N-admissible if $J \cap \mathbb{K}[x, y]$ is N-admissible.

As above, if $I \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ is an ideal, I^e denotes the extension of I to $\mathcal{S}_\omega[y]$. The extension of ideals commutes with taking the η-initial ideal.

PROPOSITION 3.1. *Let I be an ideal in $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and $\eta \in \left(\mathbb{R} \cup \{\infty\}\right)^M$. Then*

$$(\mathrm{in}_\eta(I))^e = \mathrm{in}_\eta(I^e).$$

Proof. The inclusion $(\mathrm{in}_\eta(I))^e \subseteq \mathrm{in}_\eta(I^e)$ is clear since every $f \in \mathrm{in}_\eta(f)$ also belongs to $\mathrm{in}_\eta(I^e)$.

To prove $\mathrm{in}_\eta(I^e) \subseteq (\mathrm{in}_\eta(I))^e$, consider $f \in I^e$, $f \neq 0$. We can assume that $\mathrm{val}_\eta(f) < \infty$. Indeed, if $\mathrm{val}_\eta(f) = \infty$, then f is in the ideal generated by $\{y_i \mid i \notin \mathrm{Fin}(\eta)\}$ (see Remark 2.3), and by definition, $\{y_i \mid i \notin \mathrm{Fin}(\eta)\} \subset \mathrm{in}_\eta(I)$.

Let

$$f = \sum_{i=1}^{r} g_i P_i,$$

where $g_i \in \mathcal{S}_\omega[y]$ and $P_i \in I$.

First assume that $\varepsilon(g_i) \subset {}^1\!/_l\,\mathbb{Z}$ for all i. Let m be a term of g_i. If

$$\mathrm{val}_\eta(m) > \mathrm{val}_\eta(f) - \mathrm{val}_\eta(P_i),$$

then

$$\mathrm{val}_\eta(mP_i) > \mathrm{val}_\eta(f).$$

Consequently, no term of mP_i appears in $\mathrm{in}_\eta(f)$ and we can truncate the series g_i such that the η-initial part of f remains the same.

Thus if

$$g_i = \sum_{j \in \mathbb{N}} g_i^{(j)}(x, y) + h_i,$$

with h_i in the ideal generated by $\{y_i \mid i \notin \mathrm{Fin}(\eta)\}$, see (13), we set

$$c_i^{(j)} = \mathrm{val}_\eta(g_i^{(j)}) < \infty,$$

and

$$c_i = \mathrm{val}_\eta(f) - \mathrm{val}_\eta(P_i).$$

Set

$$\tilde{g}_i = \sum_{\{j \mid c_i^{(j)} \leq c_i\}} g_i^{(j)}(x, y)$$

and

$$f' = \sum_i \tilde{g}_i P_i.$$

Since $\mathrm{in}_\eta(f') = \mathrm{in}_\eta(f)$ and $\tilde{g}_i \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ for all i, it follows that $f' \in I$ and consequently, $\mathrm{in}_\eta(f') \in \mathrm{in}_\eta(I)$.

Now we remove the supplementary assumption and let $c \in \mathbb{N}$ such that $\varepsilon(g_i) \subset {}^1\!/_c\,\mathbb{Z}^N$. We can assume that c is a multiple of l. Then

$$g_i = \sum_{\delta \in {}^l\!/_c \Delta (c/l)^N} x^\delta g_{i,\delta},$$

and $\varepsilon(g_{i,\delta}) \subset {}^1\!/_l\,\mathbb{Z}^N$. Consequently, f can be written

$$f = \sum_{\delta \in {}^l\!/_c \Delta (c/l)^N} x^\delta f_\delta,$$

where

$$f_\delta = \sum_{i=1}^{r} g_{i,\delta} P_i.$$

Note that if $\delta_1 \neq \delta_2$, then $\varepsilon(f_{\delta_1}) \cap \varepsilon(f_{\delta_2}) = \emptyset$ since the fractional exponents of x are different.

Hence, $\operatorname{in}_\eta(f)$ is a partial sum of

$$\sum_{\delta \in l/c\Delta(c/l)^N} \operatorname{in}_\eta(x^\delta f_\delta) = \sum x^\delta \operatorname{in}_\eta(f_\delta)$$

and since for each δ, $\operatorname{in}_\eta(f_\delta) \in \operatorname{in}_\eta(I)$, according to the particular case, we are done. □

3.2. **Link with the Tropical Variety**

PROPOSITION 3.2. *Let $l \in \mathbb{N}$ and $I \subseteq \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ an ideal and denote by I^e the extended ideal in $\mathcal{S}_\omega[y]$. If $\eta \in \left(\mathbb{R} \cup \{\infty\}\right)^M$, then the following statements are equivalent:*

(1) $\operatorname{in}_\eta(I) \cap \mathbb{K}\left[x^{1/l}, x^{-1/l}, (y_i)_{i \in \operatorname{Fin}(\eta)}\right]$ *is monomial free.*

(2) $\operatorname{in}_\eta(I^e) \cap \mathcal{S}_\omega\left[(y_i)_{i \in \operatorname{Fin}(\eta)}\right]$ *is monomial free.*

Proof.

"2 ⇒ 1" Is clear since $\operatorname{in}_\eta(I) \subseteq \operatorname{in}_\eta(I^e)$.

"1 ⇒ 2" Assume there exists a monomial $m \in \operatorname{in}_\eta(I^e) \cap \mathcal{S}_\omega[(y_i)_{i \in \operatorname{Fin}(\eta)}]$. Since x is invertible in $\mathcal{S}_\omega[(y_i)_{i \in \operatorname{Fin}(\eta)}]$, we may assume that $m = y^\beta$ with the property that if $i \notin \operatorname{Fin}(\eta)$ then $\beta_i = 0$.

Choose $f_1, \ldots, f_k \in I$ such that $\operatorname{in}_\eta(f_1), \ldots, \operatorname{in}_\eta(f_k)$ generate $\operatorname{in}_\eta(I) \cap \mathbb{K}[x^{1/l}, x^{-1/l}, (y_i)_{i \in \operatorname{Fin}(\eta)}]$ and let

$$d_i = \operatorname{val}_\eta(f_i), \text{ for } i = 1, \ldots, k, \text{ and } d = \operatorname{val}_\eta(x^\beta).$$

Since by Proposition 3.1

$$\operatorname{in}_\eta(I^e) = (\operatorname{in}_\eta(I))^e,$$

it follows that $\operatorname{in}_\eta(f_1), \ldots, \operatorname{in}_\eta(f_k)$ generate $\operatorname{in}_\eta(I^e) \cap \mathcal{S}_\omega[x^{1/l}, x^{-1/l}, (y_i)_{i \in \operatorname{Fin}(\eta)}]$.

Hence

$$y^\beta = \sum_{i=1}^{k} g_i \operatorname{in}_\eta(f_i), \tag{14}$$

with $g_i \in \mathcal{S}_\omega[y]$. Since on the left hand side of the above equality one has a monomial of η-order d, one can consider that $\operatorname{val}_\eta(g_i) = d - d_i$ for each i.

Let s be a multiple of l such that $g_1, \dots, g_k \in \mathbb{K}[x^{1/s}, x^{-1/s}, y]$. By formula (4) in Section 1.3, each g_i can be written uniquely as

$$g_i = \sum_{\delta \in l/s\Delta(s/l)^N} x^\delta g_{i\delta}$$

with $g_{i\delta} \in \mathbb{K}[x^{1/l}, x^{-1/l}, y]$. Thus formula (14) becomes

$$y^\beta = \sum_\delta x^\delta \sum_{i=1}^k g_{i\delta} \operatorname{in}_\eta(f_i). \tag{15}$$

Since x does not appear on the left hand side of formula (15), all the sums $\sum g_{i\delta} \operatorname{in}_\eta(f_i)$ with $\delta \neq (0, \dots, 0)$ must be null and hence

$$y^\beta = \sum_{i=1}^k g_{i(0,\dots,0)} \operatorname{in}_\eta(f_i) \in \operatorname{in}_\eta(I) \cap \mathbb{K}[x^{1/l}, x^{-1/l}, (y_i)_{i \in \operatorname{Fin}(\eta)}]$$

which contradicts statement 1.

We also include another argument for "$1 \Rightarrow 2$" that uses the version of Hilbert's Nullstellensatz stated in Matsumura, [9], Theorem 5.4. We are grateful to the reviewer for indicating us the reference.

With the above notations, assume that there exists

$$y^\beta \in \operatorname{in}_\eta(I^e) \cap \mathcal{S}_\omega[(y_i)_{i \in \operatorname{Fin}(\eta)}].$$

Let $\mathbb{K}(x^{1/l})$ be the field of fractions of $\mathbb{K}[x^{1/l}]$. Note that $\mathbb{K}(x^{1/l})$ is not, in general, algebraically closed, but, obviously, its algebraic closure belongs to $\mathcal{S}_\omega$. One has the inclusions

$$\mathbb{K}[x^{1/l}, x^{-1/l}, y] \subseteq \mathbb{K}(x^{1/l})[y] \subseteq \mathcal{S}_\omega[y].$$

Denote by $\tilde{I}$ the extension of the ideal $\operatorname{in}_\eta(I)$ to $\mathbb{K}(x^{1/l})[y]$. Then

$$V(\tilde{I}) = V(\operatorname{in}_\eta(I)) = V(\operatorname{in}_\eta(I^e)) = V\big((\operatorname{in}_\eta(I))^e\big)$$

and hence, the monomial m is null on $V(\tilde{I})$. By Nullstellensatz, there exists $k \in \mathbb{N}$ such that $m^k \in \operatorname{in}_\eta(\tilde{I})$. Consequently, there exist $f_1, \dots, f_n \in \operatorname{in}_\eta(I)$ and $h_1, \dots, h_n \in \mathbb{K}[x^{1/l}] \setminus \{0\}$ such that

$$m^k = \sum_{i=1}^n \frac{1}{h_i} f_i.$$

It follows that

$$\left(\prod_{i=1}^n h_i\right) \cdot m^k \subseteq \operatorname{in}_\eta(I).$$

Finally, since $\mathrm{in}_\eta\left(\mathrm{in}_\eta(I)\right) = \mathrm{in}_\eta(I)$ by Proposition 2.17, one has that

$$\mathrm{in}_\eta\left(\left(\prod_{i=1}^{n} h_i\right)\cdot m^k\right) = \prod_{i=1}^{n} \mathrm{in}_\eta\, h_i \cdot m^k \in \mathrm{in}_\eta(I)$$

which contradicts statement 1 of the Proposition. □

We need the following definition.

Definition 3.3. Let I be an ideal in $\mathbb{K}[x, x^{-1}, y]$, the tropical variety of I is defined as

$$\tau(I) = \left\{(\omega,\eta) \in \mathbb{R}^N \times \left(\mathbb{R}\cup\{\infty\}\right)^M \;\middle|\; \mathrm{in}_\eta(I) \text{ is monomial free}\right\}.$$

In particular, for $l = 1$, we have the following corollary.

COROLLARY 3.4. *Let $I \subseteq \mathbb{K}[x, x^{-1}, y]$ be an ideal and let I^e be the extended ideal in $\mathcal{S}_\omega[y]$. Then the following statements are equivalent:*

(1) $(\omega,\eta) \in \tau(I)$,
(2) $\mathrm{in}_\eta(I^e) \cap \mathcal{S}_\omega\left[(y_i)_{i\in \mathrm{Fin}(\eta)}\right]$ *is monomial free.*

For $l \in \mathbb{N}$, consider the $\mathbb{K}$-algebra isomorphism

$$\begin{aligned} u_l : \qquad \mathbb{K}[x^{1/l}, x^{-1/l}, y] &\longrightarrow \mathbb{K}[x, x^{-1}, y]\,. \\ x^{1/l} &\longmapsto x \end{aligned}$$

Remark that for each monomial m in $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$,

$$\mathrm{val}_{(\omega,\eta)}(m) = \mathrm{val}_{(\omega,l\eta)}(u_l(m)).$$

Thus, if $I \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ is an ideal, then

$$\mathrm{in}_{(\omega,\eta)}(I) = \mathrm{in}_{(\omega,l\eta)}(u_l(I)),$$

and moreover, one checks immediately:

PROPOSITION 3.5. *Let $I \subset \mathbb{K}[x^{1/l}, x^{-1/l}, y]$ be an ideal. Then the following statements are equivalent:*

(1) $\mathrm{in}_{(\omega,\eta)}(I) \cap \mathbb{K}[x^{1/l}, x^{-1/l}, (y_i)_{i\in\mathrm{Fin}(\eta)}]$ *is monomial free.*
(2) $(\omega, l\eta) \in \tau(u_l(I))$.

3.3. **The Algorithm**

Let $I \subset \mathbb{K}[x, x^{-1}, y]$ be an N-admissible ideal. We shall construct an ω-sequence $(\phi^{(i)})_{i\in\mathbb{N}}$ for the extended ideal $I^e \subseteq \mathcal{S}_\omega[y]$ using the tropical variety. The idea is to see each transformed ideal $(I^{(i)}) = T_{\phi^{(i)}}(I^{(i-1)})$ (see Definition 2.25) as an extended ideal of an ideal in a polynomial ring $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and to construct the next $\phi^{(i+1)}$ using a point in the tropical variety of this polynomial ideal.

Definition 3.6. If ϕ is an M-tuple of one term series, we set

$$\operatorname{den}(\phi) = \min\{k \in \mathbb{N} \mid \phi(x^k) \in \mathbb{K}[x, x^{-1}, y]\}.$$

PROPOSITION 3.7. *Let I be an ideal in $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$ and $\phi \in \mathcal{S}_\omega^M$ be an M-tuple of one-term series. If $s = \operatorname{lcm}\big(l, \operatorname{den}(\phi)\big)$, then $T_\phi(I^e)$ is the extension of an ideal in $\mathbb{K}[x^{1/s}, x^{-1/s}, y]$.*

Proof. Since the exponents of ϕ might not belong to $\frac{1}{l}\mathbb{Z}^N$, one cannot lift T_ϕ to $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$. It is easy to see that s is the smallest integer such that $\mathbb{K}[x^{1/s}, x^{-1/s}, y]$ contains the generators of I and all the components of ϕ.

Then one has the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{K}\left[x^{1/l}, x^{-1/l}, y\right] & & \\
\big\downarrow & & \\
\mathbb{K}\left[x^{1/s}, x^{-1/s}, y\right] & \xrightarrow{T_\phi} & \mathbb{K}\left[x^{1/s}, x^{-1/s}, y\right] \\
\big\downarrow & & \big\downarrow \\
\mathcal{S}_\omega[y] & \xrightarrow{T_\phi} & \mathcal{S}_\omega[y]
\end{array}$$

Let $\tilde{I}$ be the extension of I to $\mathbb{K}\left[x^{1/s}, x^{-1/s}, y\right]$. From this we get

$$\Big(T_\phi(\tilde{I})\Big)^e = T_\phi\Big((\tilde{I})^e\Big) = T_\phi(I^e). \quad \square$$

Description of the Algorithm. The algorithm unfolds on three levels:

Level 1: The level of the polynomial ring $\mathbb{K}[x, x^{-1}, y]$; this is the level where the computations in the tropical variety take place. Ideals at this level will be denoted $I^{(0)}, I^{(1)}, \ldots$.

Level 2: The level of polynomial rings $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$. It is at this level that one lifts the automorphisms T_ϕ defined at Level 3, (see Proposition 3.7). The ideals at this level will be denoted $\tilde{I}^{(0)}, \tilde{I}^{(1)}, \ldots$.

One goes from Level 2 to Level 1 and vice-versa via the isomorphisms u_l, $u_{1/l}$, see Corollary 3.4.

Lebel 3: The level of $\mathcal{S}_\omega[y]$; at this level "live" the zeros we want to approximate. Also, at this level, we define initially the transformations T_ϕ. The ideals at this level are extended ideals of the ideals at level 2. They will be denoted by $J^{(0)}, J^{(1)}, \ldots$, with

$$J^{(i)} = (\tilde{I}^{(i)})^e.$$

One starts with an ideal $I = I^{(0)} \subset \mathbb{K}[x, x^{-1}, y]$ and computes a point $\eta^{(0)}$ in the tropical variety. Using this point, one determines the first term $\phi^{(0)}$ of a zero of $J^{(0)} = I^e$, where I^e denotes the extension of I to $\mathcal{S}_\omega[y]$.

Consider the transformation $T_{\phi^{(0)}}$ on $\mathcal{S}_\omega[y]$ and lift it to Level 2, (rings of polynomials with fractional exponents); move to Level 1 via an isomorphism u_l and find a point in the tropical variety of $I^{(1)}$ such that $\eta^{(1)} > \eta^{(0)}$, and one iterates the reasoning.

If the point in the tropical variety has components equal to ∞, then this implies that the corresponding components of the solution consist of finitely many terms and we have already found all these terms; we simply eliminate the y variables corresponding to the ∞ values and continue to work with polynomials with less variables.

$$\begin{array}{ccccc}
I^{(i)} \subset & \longrightarrow & \mathbb{K}\left[x, x^{-1}, y\right] & \longleftarrow & \supset I^{(i+1)} \\
 & & \downarrow{\scriptstyle u_{1/l^{(i)}}} & & \uparrow{\scriptstyle u_{l^{(i)}}} \\
\tilde{I}^{(i)} \subset & \longrightarrow & \mathbb{K}\left[x^{1/l^{(i)}}, x^{-1/l^{(i)}}, y\right] & \xrightarrow{T_{\phi^{(i)}}} \mathbb{K}\left[x^{1/l^{(i)}}, x^{-1/l^{(i)}}, y\right] & \longleftarrow \supset \tilde{I}^{(i+1)} \\
 & & \downarrow & & \downarrow \\
J^{(i)} \subset & \longrightarrow & \mathcal{S}_\omega[y] & \xrightarrow{T_{\phi^{(i)}}} \mathcal{S}_\omega[y] & \longleftarrow \supset J^{(i+1)}
\end{array}$$

The details of the algorithm

Input data: $P_1^{(0)}(x,y), \ldots, P_k^{(0)}(x,y) \in \mathbb{K}[x, x^{-1}, y]$ a system of generators for the N-admissible ideal $I = I^{(0)}$.

Inductive construction:

Assume we have constructed:

- $(\phi^{(i)})_{0 \le i \le n-1}$ an $(n-1, \omega)$-sequence for $J^{(0)} = I^e$.
- The families of polynomials $(P_1^{(i)}(x,y), \ldots, P_k^{(i)}(x,y))_{0 \le i \le n} \subset \mathbb{K}[x, x^{-1}, y]$.
- The integers $(l^{(i)})_{i=0,n}$ such that $l^{(i)} \mid l^{(i+1)}$ for $0 \le i \le n-1$ and $l^{(0)} = 1$.

Let $I^{(i)} \subset \mathbb{K}[x, x^{-1}, y]$ be the ideal generated by $P_1^{(i)}(x,y), \ldots, P_k^{(i)}(x,y) \in \mathbb{K}[x, x^{-1}, y]$ and $\tilde{I}^{(i)} = u_{1/l^{(i)}}(I^{(i)}) \subset \mathbb{K}[x^{1/l^{(i)}}, x^{-1/l^{(i)}}, y]$, *i.e.* the ideal generated by $u_{1/l^{(i)}}(P_1^{(i)}), \ldots, u_{1/l^{(i)}}(P_k^{(i)})$.

Then

$$(J)^{(i)} = (\tilde{I}^{(i)})^e.$$

Recall that the hypothesis on $\left(\phi^{(i)}\right)_i$ means that:

- $\phi^{(i)} \in V\left(\mathrm{in}_{\mathrm{val}\left(\phi^{(i)}\right)}(J^{(i)})\right)$
- $\mathrm{val}(\phi^{(i)}) < \mathrm{val}(\phi^{(i+1)})$ coordinatewise for all $0 \le i \le n-2$.

Step 1. *Find* $\eta^{(n)} \in (\mathbb{R} \cup \{\infty\})^M$ *such that* $(\omega, l^{(n)}\eta^{(n)}) \in \tau(I^{(n)})$ *and* $\eta^{(n)} > \eta^{(n-1)}$.

If $n = 0$ there is no order condition on $\eta^{(0)}$; one just computes a point in the tropical variety $\tau(I)$. Note that by Proposition 2.27 (2) there exists an

M-tuple of one-term series $\phi^{(n)}$ such that

$$\operatorname{val}(\phi^{(n)}) > \operatorname{val}(\phi^{(n-1)}) \text{ componentwise and}$$

$$\phi^{(n)} \in V\Big(\operatorname{in}_{\operatorname{val}\left(\phi^{(n)}\right)}(J^{(n)})\Big) = V\Big(\operatorname{in}_{\operatorname{val}\left(\phi^{(n)}\right)}((\tilde{I})^{(n)})\Big).$$

Hence by Proposition 2.21 (4), 3.2 and 3.5, a solution $\eta^{(n)}$ exists.

Step 2. *Find the unique vectors* $\alpha^{(i)} \in \mathbb{Q}^N$ *such that* $\operatorname{val}(\alpha^{(i)}) = \eta_i^{(n)}$ *for* $i \in \operatorname{Fin}(\eta^{(n)})$.

The argument that ensures the existence of $\eta^{(n)}$ (see Step 1), shows that its components "lift" to rational coordinate vectors.

Step 3. *Compute a standard basis* $Q_1(x,y), \ldots, Q_s(x,y)$ *for* $I^{(n)}$ *relative to the monomial order* $(\omega, l^{(n)}\eta^{(n)})$.

Step 4. *Find a common zero* $c \in \mathbb{K}^M$ *of* $(\operatorname{in}_{l^{(n)}\eta^{(n)}}(Q_i(\underline{1},y)))_{1\le i\le s}$, *where* $\underline{1} = (1,\ldots,1)$ *(recall that* $\mathbb{K}$ *is algebraically closed).*

Set

$$\phi_i^{(n)} = \begin{cases} c_i x^{\alpha^{(i)}}, & \text{if } i \in \operatorname{Fin}(\eta^{(n)}), \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 2.8 and Corollary 2.10, we have that

$$\phi^{(n)} \in V\Big(\operatorname{in}_{\operatorname{val}\left(\phi^{(n)}\right)}(J^{(n)})\Big).$$

Hence, $\left(\phi^{(i)}\right)_{1\le i\le n}$ is an (n,ω)-sequence for J.

Note that the ideal generated by $(\operatorname{in}_{l^{(n)}\eta^{(n)}}(Q_i(\underline{1},y)))_{1\le i\le s}$ is similar to the ideal t-$\operatorname{in}(I^{(n)})$ in [8].

Step 5. *Let* $l^{(n+1)} = \operatorname{lcm}(l^{(n)}, \operatorname{den}(\phi^{(n)}))$ *and*

$$P_i^{(n+1)} = P_i^{(n)}(x^{l^{(n+1)}/l^{(n)}}, y + \phi^{(n)}(x^{(l^{(n+1)})}))$$

for $i = 1,\ldots,k$.

By choice of $l^{(n+1)}$ it is clear that $P_i^{(n+1)} \in \mathbb{K}[x,x^{-1},y]$ and Proposition 3.7 ensures that

$$\left(\tilde{I}^{(n+1)}\right)^e = (J)^{(n+1)},$$

and thus we can restart Step 1 and compute the term $\phi^{(n+1)}$.

Example 3.8. Let $M = N = 1, \omega = (1)$ and $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$. Let $I \subset \mathbb{C}[x,x^{-1},y]$ be the ideal generated by f. Set:

$$I^{(0)} = I, \text{ and } l^{(0)} = 1.$$

We can choose $\eta^{(0)} = 3/2 \in \tau(I^{(0)})$, and hence

$$\alpha^{(1)} = 3/2 \text{ and } \operatorname{in}_{3/2}(Q_1) = y^4 - 2x^3y^2 + x^6.$$

Thus, we have $c = 1$ and $c = -1$ the zeros of $\operatorname{in}_{3/2}(Q_1(\underline{1},y))$. Let us choose $c = 1$. Thus, we get the first term of the parametrization $\phi^{(0)} = x^{3/2}$.

By iterating, we obtain the ideal

$$I^{(1)} = \langle -x^{14} - 4x^{13} - 4x^{10}y + 4x^6y^2 + 4x^3y^3 + y^4 \rangle \text{ and } l^{(1)} = 2.$$

We choose the point $7/2 \in \tau(I^{(1)})$, and hence, $\eta^{(1)} = 7/4 = \alpha^{(1)}$. Note that $7/4 > 3/2$ and $\text{in}_{7/4}(Q_1) = -4x^{13} + 4x^6y^2$. Again 1 is a zero of $\text{in}_{7/2}(Q_1(\underline{1}, y))$ and hence, we have $\phi^{(1)} = x^{7/4}$.

Remark that $\varphi = x^{3/2} + x^{7/4}$ is already a parametrization.

Example 3.9. Let $M = N = 2$, $\omega = (1, \sqrt{7})$ and $f = y_1^2y_2 + x_1^3y_2^2 + x_2^2y_1 + x_1^4x_2^3 - 2x_1^5$. Let $I \subset \mathbb{C}[x, x^{-1}, y]$ be the ideal generated by f. Set:

$$I^{(0)} = I, \text{ and } l^{(0)} = 1.$$

We can choose $\eta^{(0)} = (2, 1) \in \tau(I^{(0)})$, and hence

$$\alpha^{(1)} = (2, 0),\ \alpha^{(2)} = (1, 0),\ \text{in}_{(2,1)}(Q_1) = y_1^2y_2 + 2x_1^3y_2^2 - 2x^5,$$

and $c = (1, 1)$ is a zero of $\text{in}_{(2,1)}(Q_1(\underline{1}, y))$. Thus, we get the first term of the parametrization $\phi^{(0)} = (x_1^2, x_1)$.

Remark 3.10. The algorithm allows for the computation of the terms of any zero of I^e by making different choices in Step 1 and Step 4.

Remark 3.11. According to Corollary 2.28 the sequence $l^{(i)}$ is constant for i large enough. Hence, for i large enough, $T_{\phi^{(i)}}$ lifts to the same fractional exponent polynomial ring $\mathbb{K}[x^{1/l}, x^{-1/l}, y]$. The "economical" approach presented here (*i.e.* considering at each step the lcm between the denominators of the previous exponents of the step and the exponents of the latest term $\phi^{(i)}$) is equivalent to the approach in [2] Section 15 formula (15.1). However, our approach is more advantageous for a computer implementation than the huge product in [2].

3.4. The Case $M = 1$

There is a striking resemblance between the construction of a root in the proof of Theorem 1.15 (see Section 1.7) and the algorithm in Section 3.3. In this section we look at the algorithm in the particular case of polynomials in one variable y (*i.e.* $M = 1$).

First, we link the ω-Newton polygon defined in Section 1 with the results in Sections 2 and 3.

Let $\eta \in \mathbb{R}$ and consider the linear mapping

$$\begin{aligned} \lambda_\eta : \quad & \mathbb{R}^2 \longrightarrow \mathbb{R} \\ & (x, y) \longmapsto x + \eta y \end{aligned} \quad .$$

LEMMA 3.12. *The restriction of λ_η to any parallel to the abscissa is a strictly increasing function.*

The proof of the Lemma is obvious.

Let f be a one variable polynomial in $\mathcal{S}_\omega[y]$. Lemma 3.12 immediately implies that the minimum of λ_η on the set $\pi_\omega(\varepsilon(f))$ is attained on the vertices of the Newton polygon $NP_\omega(f)$ (see Fig. 13).

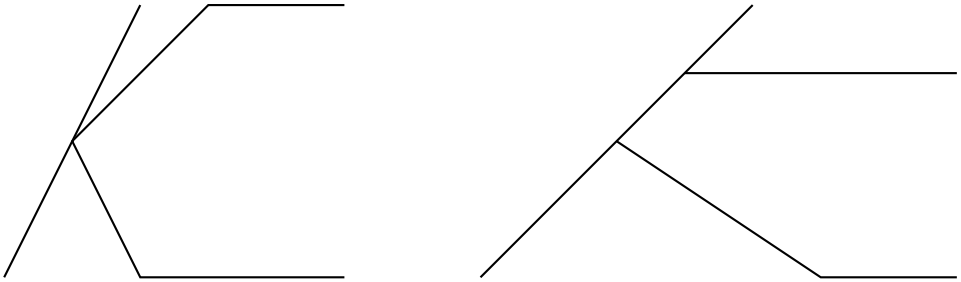

Fig. 13.

LEMMA 3.13. *Let $f \in \mathcal{S}_\omega[y]$, $(\alpha, i) \in \varepsilon(f)$ and $\eta \in \mathbb{R} \cup \{\infty\}$. Then*

$$\mathrm{val}_{(\omega,\eta)}(\alpha, i) = \lambda_\eta \circ \pi_\omega(\alpha, i).$$

Proof. Both terms evaluate to $\omega \cdot \alpha + \eta \cdot i$. $\square$

Let $f \in \mathcal{S}_\omega[y]$. Using the Newton Polygon, $NP_\omega(f)$, one can compute the form $\mathrm{in}_\eta(f)$ for all $\eta \in \mathbb{R}$. The following completes Notation 1.22:

Notation 3.14. Let $f \in \mathcal{S}_\omega[y]$ such that $\deg(f) \geq 1$.

- Let $L_1(f), \ldots, L_k(f)$ be the bounded edges of $NP_\omega(f)$. The edges are numbered from the abscissa upward so that
$$L_1(f) = \pi_\omega(\mathrm{segm}(f)).$$
- Let $v^{(0)}(f), \ldots, v^{(k)}(f)$ be the vertices of $NP_\omega(f)$; thus
$$L_i(f) = [v^{(i-1)}(f), v^{(i)}(f)], \text{ for } 1 \leq i \leq k.$$
- Let $q^{(0)}(f), \ldots, q^{(k)}(f) \in \varepsilon(f)$ be the exponents that correspond to $v^{(0)}(f), \ldots, v^{(k)}(f)$.
- Let $s_1(f), \ldots, s_k(f)$ be the slopes of the line segments $L_1(f), \ldots, L_k(f)$. Consider the usual tangent function restricted to the interval $[0, \pi)$:
$$\mathrm{tg} : [0, \pi) \longrightarrow \mathbb{R} \cup \{\infty\}$$

with $\mathrm{tg}(\pi/2) = \infty$. Note that it is a one-to one correspondence.

- Let $t_1(f), \ldots, t_k(f) \in [0, \pi)$ be such that $s_i(f) = \mathrm{tg}(t_i(f))$ for $1 \leq i \leq k$, and set $t_0(f) = \pi$ and $t_{k+1}(f) = 0$.

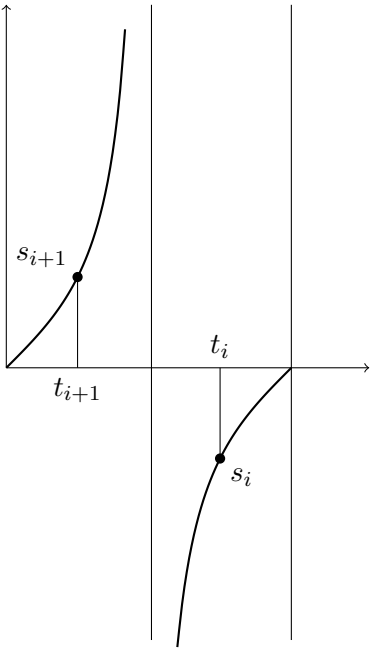

Fig. 14.

- Let $\eta(t) = {}^{-1}/_{\mathrm{tg}(t)}$ for $t \in (0, \infty)$ and set

$$\eta_i(f) = \eta(t_i(f)) = {}^{-1}/_{\mathrm{tg}(t_i(f))} = {}^{-1}/_{s_i(f)}$$

for $1 \leq i \leq k$.

Note that the function $\eta(t) : (0, \pi) \longrightarrow \mathbb{R} \cup \{\infty\}$ is one-to-one and that

$$\eta_1(f) > \eta_2(f) > \cdots > \eta_k(f).$$

We shall write L_i, $v^{(i)}$, $q^{(i)}$, s_i, t_i instead of $L_i(f)$, $v^{(i)}(f)$, $q^{(i)}(f)$, $s_i(f)$, $t_i(f)$ whenever f is clear from the context.

LEMMA 3.15. *Let $f \in \mathcal{S}_\omega[y]$ of degree ≥ 1. With the above notations, we have:*

(1) *If $t \in (t_{i+1}, t_i)$ with $0 \leq i \leq k$, then*

$$\mathrm{in}_{\eta(t)}(f) = f\big|_{q^{(i+1)}},$$

(i.e. the term of f corresponding to the uppermost vertex of L_i).

(2) *If $t = t_i$, with $1 \leq i \leq k$ then*

$$\mathrm{in}_{\eta_i}(f) = f\big|_{[q^{(i-1)}, q^{(i)}]}.$$

Proof. Recall that $f\big|_A$ with $A \subseteq \varepsilon(f)$ is the sum of terms of f with exponents in A (see Notation 1.22).

(1) If for instance $i = k$, then for any $t \in (t_{k+1}, t_k)$, the direction of the slope $\mathrm{tg}(t)$ is "between" the slope $\mathrm{tg}(t_{k+1}) = 0$ and the slope of L_k, $\mathrm{tg}(t_k) = s_k$ (here *between* means $\mathrm{tg}(t) \in (\mathrm{tg}(t_{k+1}), \mathrm{tg}(t_k))$ – see Figs. 14 and 15).

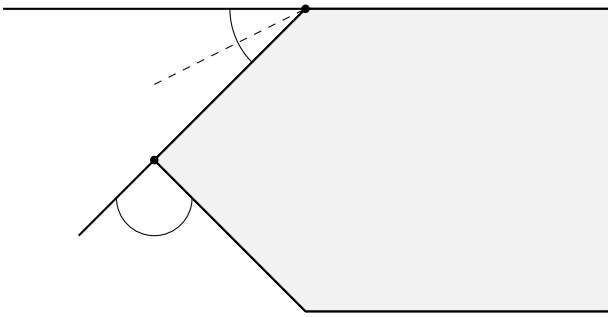

Fig. 15. $NP_\omega(f)$.

One sees immediately that $\lambda_{\eta(t)}$ has its minimum on $\pi_\omega(\varepsilon(f))$ at $v^{(k+1)}$. Hence $\mathrm{in}_{\eta(t)}(f)$ is the term of f with exponent $q^{(k+1)}$.

Note that $\chi(f)$, the ω-characteristic polynomial of f, coincides with $\mathrm{in}_{\eta(t)}(f)$ (for the definition of $\chi(f)$ see Notation 1.22).

(2) Let $t = t_i$, where $1 \leq i \leq k$. The line $\lambda_{\eta(t)} = 0$ is parallel to the edge L_i. Consequently, $\lambda_{\eta_i} = \lambda_{\eta(t)}$ has its minimum on $\pi_\omega(\varepsilon(f))$ at all the points on L_i. By Lemma 3.13

$$\mathrm{in}_{\eta_i}(f) = f\big|_{[q^{(i+1)}, q^{(i)}]}. \quad \square$$

Lemma 3.15 immediately implies:

COROLLARY 3.16. *Let $f \in \mathcal{S}_\omega[y]$ be a polynomial of degree ≥ 1 and let $I \subset \mathcal{S}_\omega[y]$ be the ideal generated by f. The following statements are equivalent:*

(1) $\mathrm{in}_\eta(I)$ *is monomial free.*

(2) $-1/\eta \in \{s_1, \ldots, s_k\}$, *where s_i is the slope of L_i, the i-th bounded edge of $NP_\omega(f)$.*

Proof. Since I is a principal ideal, $\mathrm{in}_\eta(f)$ is monomial free iff $\mathrm{in}_\eta(f)$ consists of more than one term and the equivalence follows from Lemma 3.15. $\square$

COROLLARY 3.17. *Let $f \in \mathbb{K}[x, y]$ be a polynomial of degree ≥ 1 in y and let $I \subset \mathbb{K}[x, y]$ be the ideal generated by f. The following statements are equivalent:*

(1) $(\omega, \eta) \in \tau(I)$.

(2) $-1/\eta \in \{s_1, \ldots, s_k\}$, *where* s_i *is the slope of* L_i, *the* i*-th bounded edge of* $NP_\omega(f)$.

Proof. Follows immediately from Corollary 3.16. □

Thus for a fixed $\omega \in \mathbb{R}^N$, the set:

$$\{\eta \in \mathbb{R}^N \mid (\omega, \eta) \in \tau(i)\}$$

is in a one-to-one correspondence with the set of bounded edges of $NP_\omega(f)$.

Note also that since $M = 1$, the coordinatewise order relation between "directions" η coincides with the usual order relation on $\mathbb{R}$.

The iterative construction of a zero of the polynomial $f \in \mathcal{S}_\omega[y]$ in the proof of Theorem 1.15 (see Section 1.7), coincides with the construction given by the algorithm in Section 3.3. Namely, in Step 1 of the algorithm we always choose $\eta_1(f^{(i)})$, the direction corresponding to the edge $\pi_\omega(\mathrm{segm}(f^{(i)}))$ *i.e.* the largest η such that $(\omega, \eta) \in \tau(I^{(i)})$, where $I^{(i)}$ denotes the ideal generated by $f^{(i)}$.

Remark 3.18. One checks easily the following small variation of the construction in the proof of Theorem 1.15, allowing one to compute any root of the polynomial $f \in \mathcal{S}_\omega[y]$ (again assume $f(0) \neq 0$).

- We initially choose any bounded edge $L_{j_0}(f^{(0)})$ of $NP_\omega(f) = NP_\omega(f^{(0)})$. By Corollary 3.16 and Proposition 2.20 the ω-orders of the roots of f are in one-to-one correspondence with the bounded edges of $NP_\omega(f^{(0)})$. Thus we start computing the terms of one of the roots with ω-order equal to $\eta_{j_0}(f^{(0)})$.
- Assume that at iteration i we have chosen the bounded edge $L_{j_i}(f^{(i)})$. Then at iteration $i+1$ we may choose any edge $L_{j_{i+1}}(f^{(i+1)})$ below the vertex $v^{(j_i)}(f^{(i)})$.
  Of course, for i large enough, $j_i = 1$ and the only edge below $v^{(j_i)}(f^{(i)}) = v^{(1)}(f^{(i)})$ is $L_1(f^{(i+1)})$.

REFERENCES

[1] F. Aroca and G. Ilardi, *A family of algebraically closed fields containing polynomials in several variables.* Comm. Algebra **37** (2009), 1284–1296.

[2] F. Aroca, G. Ilardi and L. Lopez de Medrano, *Puiseux power series solutions for systems of equations.* Internat. J. Math. **21** (2010), 1439–1456.

[3] M. Atiyah and J. McDonald, *Introduction to commutative algebra.* Addison-Wesley, 1969.

[4] W. Deker, G.-M. Greuel, G. Pfister and H. Schönemann, *Singular 3.1.6. A Computer algebra system for polynomial computations.* 2013, `http://www.singular.uni-kl.de`.

[5] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminants, resultants and multidimensional determinants.* Birkhäuser, 1994.

[6] G.-M. Greuel and G. Pfister, *A singular introduction to commutative algebra.* Second edition, Springer, 2007.

[7] A. Jensen, *Gfan, a software for Gröbner fans.* `http://www.math.tuberlin.de/jensen/software/gfan/gfan.html`.

[8] A.N. Jensen, H. Markwig and T. Markwig, *An algorithm for lifting points in a tropical variety.* Collect. Math. **59** (2008), *2*, 129–165.

[9] H. Matsumura, *Commutative ring theory.* Cambridge University Press, 1986.

[10] J. McDonald, *Fiber polytopes and fractional power series.* J. Pure Appl. Algebra **104** (1995), *2*, 213–233.

[11] J. McDonald, *Fractional power series solutions for systems of equations.* Discrete Comput. Geom. **27** (2002), *4*, 501–529.

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