COMMUTATIVITY OF PRIME RINGS WITH GENERALIZED SKEW DERIVATIONS

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In the present paper, we investigate the action of generalized skew derivation $\mathfrak{F}$ in a prime ring $\mathcal{R}$ which satisfies differential identities involving anti-commutator for a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$. In fact, we obtain the characterisation of the map $\mathfrak{F}$ and the commutativity of a prime ring $\mathcal{R}$ which satisfies either of the identities $\mathfrak{F}(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$ or $\mathfrak{F}(x) \circ \mathfrak{F}(y) = x \circ y$ for all $x, y \in \mathcal{I}$.

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1. INTRODUCTION

In what follows $\mathcal{R}$ is a semi(-prime) ring with the center $\mathcal{Z}(\mathcal{R})$, $\mathcal{Q}$ is the Martindale quotient ring of $\mathcal{R}$ and $\mathcal{U}$ is the Utumi quotient ring of $\mathcal{R}$. The center of $\mathcal{U}$, denoted by $\mathcal{C}$ is called the extended centroid of $\mathcal{R}$ (we refer the reader to [2], for the definitions and related properties of these objects). An additive subgroup $\mathcal{L}$ of $\mathcal{R}$ is said to be a Lie ideal of $\mathcal{R}$ if for any $u \in \mathcal{L}$ and $x \in \mathcal{R}$, $ux - xu \in \mathcal{L}$ and it is said to be square closed if $u^2 \in \mathcal{L}$ for all $u \in \mathcal{L}$. For any $x, y \in \mathcal{R}$, the symbol $x \circ y$ stands for the anti-commutator $xy + yx$.

Recall that a ring $\mathcal{R}$ is prime if $x\mathcal{R}y = (0)$ implies either $x = 0$ or $y = 0$ and $\mathcal{R}$ is semiprime if $x\mathcal{R}x = (0)$ implies $x = 0$. An additive mapping $\mathfrak{d} : \mathcal{R} \to \mathcal{R}$ is called a derivation on $\mathcal{R}$ if $\mathfrak{d}(xy) = \mathfrak{d}(x)y + x\mathfrak{d}(y)$ holds for all $x, y \in \mathcal{R}$. An additive mapping $\mathfrak{G} : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation with an associated derivation $\mathfrak{d} : \mathcal{R} \to \mathcal{R}$, if $\mathfrak{G}(xy) = \mathfrak{G}(x)y + x\mathfrak{d}(y)$ holds for all $x \in \mathcal{R}$. An additive mapping $\mathfrak{D} : \mathcal{R} \to \mathcal{R}$ is called a skew derivation if $\mathfrak{D}(xy) = \mathfrak{D}(x)y + \alpha(x)\mathfrak{D}(y)$ for all $x, y \in \mathcal{R}$, where α is called an associated automorphism of $\mathfrak{D}$. Also, an additive mapping $\mathfrak{F} : \mathcal{R} \to \mathcal{R}$ is called a generalized skew derivation, if there exists a skew derivation $\mathfrak{D}$ of $\mathcal{R}$ with an associated automorphism α such that $\mathfrak{F}(xy) = \mathfrak{F}(x)y + \alpha(x)\mathfrak{D}(y)$. Moreover there exists $\mathfrak{1} = a \in \mathcal{Q}$ such that $\mathfrak{F}(x) = ax + \mathfrak{d}(x)$ for all $x \in \mathcal{R}$. Let us mention that a skew derivation $\mathfrak{D} : \mathcal{R} \to \mathcal{R}$ is called X-inner if there exists an

element $b \in \mathcal{Q}$ and an automorphism α of $\mathcal{R}$ such that $\mathfrak{D}(x) = bx - \alpha(x)b$ for all $x \in \mathcal{R}$. If a skew derivation $\mathfrak{D}$ is not X-inner, then it is called X-outer. Similarly a generalized skew derivation $\mathfrak{F} : \mathcal{R} \to \mathcal{R}$ is called X-inner, if there exists $a, b \in \mathcal{Q}$ and an automorphism α of $\mathcal{R}$ such that $\mathfrak{F} = ax - \alpha(x)b$ for all $x \in \mathcal{R}$. If a generalized skew derivation $\mathfrak{F}$ of $\mathcal{R}$ is not inner, then it is called X-outer. Thus the concept of generalized skew derivations can be regarded as the unified notion of both skew derivations and generalized derivations.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [1], where further references can be found). In [11] Herstein proved that if $\mathcal{R}$ is a prime ring with characteristic different from 2 and $\mathcal{R}$ admits a nonzero derivation $\mathfrak{d}$ such that $[\mathfrak{d}(x), \mathfrak{d}(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In [1], Ashraf and Rehman showed that if $\mathcal{R}$ is a prime ring, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and $\mathfrak{d}$ is a nonzero derivation of $\mathcal{R}$ such that $\mathfrak{d}(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. In [17] Quadri *et al.* discussed the commutativity of prime rings with generalized derivations. More precisely, they proved that if $\mathcal{R}$ is a prime ring, $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$ and $\mathfrak{G}$ a generalized derivation of $\mathcal{R}$, with an associated non-zero derivation $\mathfrak{d}$ such that $\mathfrak{G}(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. In 2007, Huang [9] proved that if $\mathcal{R}$ is a prime ring with $char(\mathcal{R}) \neq 2$, U a square closed Lie ideal of $\mathcal{R}$ and $\mathfrak{G}$ a generalized derivation associated with a derivation $\mathfrak{d}$ of $\mathcal{R}$ such that $\mathfrak{d}(x) \circ \mathfrak{G}(y) = x \circ y$ for all $x, y \in U$, then $\mathfrak{d} = 0$ or $\mathcal{R}$ is commutative. In [4] Bell and Rehman considered the similar situation where the derivation is replaced by generalized derivation. More precisely, they studied the identity $\mathfrak{G}(x) \circ \mathfrak{G}(y) = x \circ y$. Recently, Rehman and Raza [18] discussed the similar differential identities involving skew derivations. Inspired by the above mentioned work, we continue our line of investigation by examining the identities involving generalized skew derivation of $\mathcal{R}$ namely (i) $\mathfrak{F}(x \circ y) = x \circ y$ and (ii) $\mathfrak{F}(x) \circ \mathfrak{F}(y) = x \circ y$ for all x, y in an ideal $\mathcal{I}$ of $\mathcal{R}$ and obtain the commutativity of $\mathcal{R}$.

2. THE RESULTS IN PRIME RINGS

We begin our discussion with the following well known facts which are very vital in providing the proof of our main results.

FACT 2.1 ([3, Lemma 7.1]). *Let ${}_{\mathcal{D}}\mathcal{M}$ be a left vector space over a division ring $\mathcal{D}$ with $dim_{\mathcal{D}}\mathcal{M} \geq 2$ and $\mathcal{P} \in End(\mathcal{M})$. If x and $\mathcal{P}x$ are $\mathcal{D}$-dependent for every $x \in \mathcal{M}$, then there exists $\mu \in \mathcal{D}$ such that $\mathcal{P}x = \mu x$ for all $x \in \mathcal{M}$.*

We also recall some well known results that will be helpful in order to prove our results.

FACT 2.2 ([6]). *Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a two sided ideal of $\mathcal{R}$. Then $\mathcal{R}, \mathcal{I}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities with coefficients in $\mathcal{Q}$*

FACT 2.3 ([7]). *If $\mathfrak{D}$ is s nonzero skew derivation of $\mathcal{R}$, then one of the following statements holds:*

(i) *Either $\mathfrak{D}$ is an inner skew derivation.*

(ii) *$\mathcal{R}$ satisfies the generalized polynomial identity*

$$\Psi(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

FACT 2.4 ([7]). *Let $\mathcal{R}$ be a prime ring with an automorphism α. Suppose that $\mathfrak{D}$ is an $\mathcal{Q}$-outer skew derivation of $\mathcal{R}$ with an associated automorphism α. Then any generalized polynomial identity of $\mathcal{R}$ of the form $\Psi(x_i,\ \mathfrak{D}(x_i)) = 0$ gives rise to a generalized polynomial identity $\Psi(x_i,\ y_i) = 0$ of $\mathcal{R}$, where $x_i,\ y_i$ are distinct indeterminates.*

FACT 2.5 ([13]). *Let $\mathcal{R}$ be a domain and α be an outer automorphism. If $\Psi(x_i, \alpha(x_i))$ is a generalized polynomial identity of $\mathcal{R}$, then $\mathcal{R}$ also satisfies the non trivial generalized polynomial identity $\Psi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.*

FACT 2.6 ([16, Proposition]). *Let $\mathcal{R}$ be a prime algebra over an infinite field $\mathcal{K}$ and suppose $\mathcal{L}$ be the extension field of $\mathcal{K}$. Then $\mathcal{R}$ and $\mathcal{R} \otimes_{\mathcal{K}} \mathcal{L}$ satisfy the same generalized polynomial identities.*

FACT 2.7 ([5, Theorem 3]). *Suppose that $\mathcal{R}$ is a prime ring and $\mathcal{A}$ an independent subset of $\mathcal{G}$ modulo $\mathcal{H}_i$. Let $\phi = \Phi(s_i^{a_j}) = 0$ be a generalized identity with automorphisms of $\mathcal{R}$ reduced with respect to $\mathcal{A}$. If for every $s_i \in X$, $a_j \in \mathcal{G}$, the $s_i^{a_j}$-word degree of $\phi = \Phi(s_i^{a_j})$ is strictly less than $char(\mathcal{R})$ when $char(\mathcal{R}) \neq 0$, then $\Phi(z_{ij}) = 0$ is also a GPI of $\mathcal{R}$.*

Now we are in a position to prove our main result.

THEOREM 2.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$. Suppose that $\mathfrak{F}$ be a generalized skew derivation with an associated skew derivation $\mathfrak{D}$ such that $\mathfrak{F}(x \circ y) = x \circ y$ for all $x,\ y \in \mathcal{I}$, then the following statements hold:*

(i) *There exists some $\mathfrak{h} \in \mathcal{R}$ such that $\mathfrak{F}(x) = \mathfrak{h}x$ for all $x \in \mathcal{R}$.*

(ii) *$\mathcal{R}$ is commutative.*

Proof. As remarked in the introduction, we can write $\mathfrak{F}(x) = ax + \mathfrak{D}(x)$, for all $x \in \mathcal{R}$, where $\mathfrak{D}$ is an associated skew derivation. With out loss of generality let α be an non-identity automorphism associated with $\mathfrak{D}$ and $\mathfrak{F}$. Thus by our hypothesis we can write

$$a(xy + yx) + \mathfrak{D}(x)y + \alpha(x)\mathfrak{D}(y) + \mathfrak{D}(y)x + \alpha(y)\mathfrak{D}(x) = xy + yx. \tag{2.1}$$

By using the Kharchenko's theory [14], we have the following cases:

Case 1. If $\mathfrak{D}$ is $\mathcal{Q}$-inner skew derivation, then $\mathfrak{D}(x) = \alpha(x)q - qx$ for all $x \in \mathcal{R}$, where $q \in \mathcal{Q}$. Thus from the relation (2.1) we obtain

$$\begin{aligned} a(xy + yx) + (\alpha(x)q - qx)y + \alpha(x)(\alpha(y)q - qy) + \\ (\alpha(y)q - qy)x + \alpha(y)(\alpha(x)q - qx) = xy + yx. \end{aligned} \tag{2.2}$$

If α is an outer automorphism then by using the Fact 2.7, we obtain the following identity

$$\begin{aligned} a(xy + yx) + (uq - qx)y + u(vq - qy) + (vq - qy)x + \\ v(uq - qx) \quad = \quad xy + yx. \end{aligned}$$

In particular take $x = y = 0$, then $\mathcal{Q}$ satisfies the following generalized polynomial identity

$$uvq + vuq = 0$$

for all $u, v \in \mathcal{Q}$.By Chuang [7, Theorem 1 and Theorem 2], $\mathcal{Q}$ satisfies this polynomial identities as well as $\mathcal{R}$. Note that this is a polynomial identity and hence there exists a field $\mathbb{F}$ such that $\mathcal{R} \subseteq \mathcal{M}_K(\mathbb{F})$, the ring of $K \times K$ matrices over a field $\mathbb{F}$, where K is positive integer greater than 1. Moreover $\mathcal{R}$ and $\mathcal{M}_K(\mathbb{F})$ satisfy the same polynomial identity [6]. Thus

$$uvq + vuq = 0 \quad \text{for all } u, \quad v \in \mathcal{M}_K(\mathbb{F}). \tag{2.3}$$

Let e_{ij} denote the matrix unit with 1 in the (i, j)-th entry and zero elsewhere. By taking $u = e_{11}$, $v = e_{12}$ and $q = \sum q_{ij}e_{ij}$, where $q_{ij} \in \mathbb{F}$ in (2.3) and then post-multiplying by e_{12}, we obtain $q_{21} = 0$. Similarly we can get $q_{12} = 0$, showing that $q \in \mathcal{C}$, which is a contradiction, since we are dealing with the non-trivial generalized polynomial identity.

Now suppose that α is an $\mathcal{Q}$-inner automorphism, then there exists an invertible element $\mathfrak{T} \in \mathcal{Q}$ such that $\alpha(x) = \mathfrak{T}x\mathfrak{T}^{-1}$ for all $x \in \mathcal{R}$. Thus equation (2.2) implies that

$$\begin{aligned} a(xy + yx) + (\mathfrak{T}x\mathfrak{T}^{-1}q - qx)y + \mathfrak{T}x\mathfrak{T}^{-1}(\mathfrak{T}y\mathfrak{T}^{-1}q - qy) \\ +(\mathfrak{T}y\mathfrak{T}^{-1}q - qy)x + \mathfrak{T}y\mathfrak{T}^{-1}(\mathfrak{T}x\mathfrak{T}^{-1}q - qx) \quad = \quad xy + yx \end{aligned}$$

for all $x, y \in \mathcal{Q}$. Here we can see that if $\mathfrak{T}^{-1}q \in \mathcal{C}$, then $\mathfrak{F}(x) = ax + \mathfrak{T}x\mathfrak{T}^{-1}qx - qx$ which implies $\mathfrak{F} = ax$ and thus our identity reduces to $a(x \circ y) = x \circ y$.

As $\mathfrak{F} = ax$ is a particular case of generalized derivation, by putting $n = 1$ in Theorem A of [10], we get $\mathcal{R}$ is commutative. If $\mathfrak{T}^{-1}q \notin \mathcal{C}$. Thus we have that $\mathcal{Q}$ satisfies the following generalized polynomial identity

$$\begin{aligned}\Psi(x,y) = a(xy+yx) + (\mathfrak{T}x\mathfrak{T}^{-1}q - qx)y + \mathfrak{T}x\mathfrak{T}^{-1}(\mathfrak{T}y\mathfrak{T}^{-1} - qy) + \\ (\mathfrak{T}y\mathfrak{T}^{-1}q - qy)x + \mathfrak{T}y\mathfrak{T}^{-1}(\mathfrak{T}x\mathfrak{T}^{-1}q - qx) - (xy+yx).\end{aligned} \tag{2.4}$$

Clearly $\Psi(x,y)$ is a non-trivial generalized polynomial identity of $\mathcal{Q}$. Denote by $\mathbb{F}$ either the algebraic closure of $\mathcal{C}$ or $\mathcal{C}$ according as $\mathcal{C}$ is either infinite or finite. By using the Fact 2.6, $\Psi(x,y)$ is also satisfied by $\mathcal{Q} \otimes_{\mathcal{C}} \mathbb{F}$. Also by [8, Theorem 3.5], $\mathcal{Q} \otimes_{\mathcal{C}} \mathbb{F}$ is a prime ring with $\mathbb{F}$ as its extended centroid. Hence $\mathcal{Q} \otimes_{\mathcal{C}} \mathbb{F}$ also satisfies the generalized polynomial identity and its extended centroid $\mathbb{F}$ is either an algebraically closed field or a finite field. Since both $\mathcal{Q}$ and $\mathcal{Q} \otimes_{\mathcal{C}} \mathbb{F}$ are prime and centrally closed we can replace $\mathcal{R}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_{\mathcal{C}} \mathbb{F}$. Thus we may assume that $\mathcal{R}$ is centrally closed and the field $\mathbb{F}$ which is either algebraically closed of finite and $\mathcal{R}$ satisfies the generalized polynomial identity (2.4). By Martindale's theorem [2, Corollary 6.1.7], $\mathcal{R}$ is a primitive ring with nonzero socle with $\mathcal{D}$ as its associated division ring. Moreover by Jacobson theorem [12, p.75] $\mathcal{R}$ is isomorphic to the dense subring of ring of linear transformations on a vector space $\mathcal{V}$ over $\mathcal{D}$, containing nonzero linear transformations of finite rank. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_K(\mathcal{D})$, where $K = dim_{\mathcal{D}}\mathcal{V}$. Suppose that $dim(\mathcal{V}_{\mathcal{D}} \geq 2)$.

Firstly our aim is to show that w and $\mathfrak{T}^{-1}qw$ are linearly $\mathcal{D}$-dependent for all $w \in \mathcal{V}$. If $\mathfrak{T}^{-1}qw = 0$, then w and $\mathfrak{T}^{-1}qw$ are linearly $\mathcal{D}$-dependent. Suppose not, then w and $\mathfrak{T}^{-1}qw$ are linearly $\mathcal{D}$-independent for some $w_0 \in \mathcal{D}$. If $\mathfrak{T}^{-1}w \notin span_{\mathcal{D}}\{w, \mathfrak{T}^{-1}qw\}$, then $\{w, \mathfrak{T}^{-1}qw, \mathfrak{T}^{-1}w\}$ is linearly $\mathcal{D}$-independent. By density of $\mathcal{R}$, there exists $x, y \in \mathcal{R}$ such that

$$\begin{array}{lll} x\mathfrak{T}^{-1}w = 0 & x\mathfrak{T}^{-1}qw = \mathfrak{T}^{-1}w & xw = 0 \\ y\mathfrak{T}^{-1}w = \mathfrak{T}^{-1}w & y\mathfrak{T}^{-1}qw = 0 & yw = w. \end{array}$$

Thus from equation (2.4), we obtain

$$\begin{aligned}0 = \Big(a(xy+yx) + (\mathfrak{T}x\mathfrak{T}^{-1}q - qx)y + \mathfrak{T}x\mathfrak{T}^{-1}(\mathfrak{T}y\mathfrak{T}^{-1}q - qy) \\ + (\mathfrak{T}y\mathfrak{T}^{-1}q - qy)x + \mathfrak{T}y\mathfrak{T}^{-1}(\mathfrak{T}x\mathfrak{T}^{-1}q - qx) - (xy+yx)\Big)w = w \neq 0,\end{aligned}$$

which is a contradiction.

If $\mathfrak{T}^{-1}w \in Span_{\mathcal{D}}\{w, \mathfrak{T}^{-1}qw\}$, then $\mathfrak{T}^{-1}w = \beta w + \gamma\mathfrak{T}^{-1}qw$ for some $0 \neq \beta,\ \gamma \in \mathcal{D}$. Since the $\{w,\ \mathfrak{T}^{-1}qw\}$ is linearly $\mathcal{D}$-independent. Thus by invoking the density theorem we have, there exists $x,\ y \in \mathcal{R}$ such that

$$x\mathfrak{T}^{-1}qw = \beta w + \gamma\mathfrak{T}^{-1}qw, \quad xw = 0$$
$$y\mathfrak{T}^{-1}qw = 0, \qquad yw = w.$$

Further by the relation (2.4), we have

$$0 = \Big(a(xy+yx) + (\mathfrak{T}x\mathfrak{T}^{-1}q - qx)y + \mathfrak{T}x\mathfrak{T}^{-1}(\mathfrak{T}y\mathfrak{T}^{-1}q - qy)$$
$$+ (\mathfrak{T}y\mathfrak{T}^{-1}q - qy)x + \mathfrak{T}y\mathfrak{T}^{-1}(\mathfrak{T}x\mathfrak{T}^{-1}q - qx) - (xy+yx)\Big)w = \mathfrak{T}\beta w \neq 0,$$

which is a contradiction.

Hence we conclude that w and $\mathfrak{T}^{-1}qw$ are linearly $\mathcal{D}$-dependent, for all $w \in \mathcal{V}$. Thus by the Fact 2.1 there exists $\lambda \in \mathcal{D}$ such that $\mathfrak{T}^{-1}qw = \lambda w$ for all $w \in \mathcal{V}$. Thus for $z \in \mathcal{R}$ and $w \in \mathcal{V}$, we can write $\mathfrak{T}^{-1}qw = \lambda w$, $z(\mathfrak{T}^{-1}qw) = z(w\lambda)$ and also $\mathfrak{T}^{-1}q(zw) = (zw)\lambda$. Thus $0 = [\mathfrak{T}^{-1}q,\ z]w$ for any $w \in \mathcal{V}$, i.e., $[\mathfrak{T}^{-1}q,\ z]\mathcal{V} = 0$. Since $\mathcal{V}$ is left faithful irreducible $\mathcal{R}$ module, we have $[\mathfrak{T}^{-1}q, z] = 0$ for every $z \in \mathcal{R}$, i.e., $\mathfrak{T}^{-1}q \in \mathcal{C}$, a contradiction. Hence $dim(\mathcal{V}_{\mathcal{D}}) = 1$, which implies that $\mathcal{R}$ is commutative.

Case 2. Let $\mathfrak{D}$ be $\mathcal{Q}$-outer Skew derivation, then $\mathcal{I}$ satisfies the following generalized polynomial identity

(2.5) $a(xy+yx) + sy + \alpha(x)t + tx + \alpha(y)s = xy + yx$ for all $x,\ y,\ s,\ t\ \in \mathcal{I}$.

Here in its first place we assume that α is not $\mathcal{Q}$-inner, thus by using Fact 2.7, we have

$$a(xy+yx) + sy + ut + tx + vs = xy + yx \quad \text{for all } x,\ y,\ s,\ t,\ u,\ v\ \in \mathcal{I}.$$

Now take $x = y = 0$ and $v = t,\ u = s$, we have $\mathcal{I}$ satisfies the blended component $st + ts$ for all $s,\ t \in \mathcal{I}$. Replacing t by tm and using $st = -ts$, we get $t[m,\ s] = 0$ for all $m,\ s,\ t \in \mathcal{I}$. Since primeness of $\mathcal{R}$ implies the primeness of $\mathcal{I}$. Thus we have $[m,\ s] = 0$ for all $m,\ s \in \mathcal{I}$, which further implies the commutativity of $\mathcal{R}$.

Now consider, α as $\mathcal{Q}$-inner automorphism, then there exists an invertible element $\mathfrak{T} \in \mathcal{Q}$ such that $\alpha(x) = \mathfrak{T}x\mathfrak{T}^{-1}$ for all $x \in \mathcal{R}$. Hence by the relation (2.5), we have

$$a(xy+yx) + (sy + \mathfrak{T}x\mathfrak{T}^{-1}t) + (tx + \mathfrak{T}y\mathfrak{T}^{-1}s) = xy + yx \text{ for all } x,\ y,\ s,\ t \in \mathcal{I}.$$

Replace q by $\mathfrak{T}t$ and s by $\mathfrak{T}s$ in above equation, we obtain

$$a(xy+yx) + sy + \mathfrak{T}xt + tx + \mathfrak{T}ys = xy + ys \text{ for all } x,\ y,\ s,\ t \in \mathcal{I}.$$

In particular taking $s = 0$, the above identity reduces to

$$a(xy+yx) + \mathfrak{T}xt + tx = xy + yx \text{ for all } x,\ y,\ t \in \mathcal{I} \tag{2.6}$$

is a polynomial identity satisfied by $\mathcal{Q}$. By the standard arguments, $\mathcal{Q}$ is isomorphic to the dense ring of linear transformations of a vector space $\mathcal{V}$ over

$\mathcal{C}$, containing non-zero linear transformations of finite rank. Assume first that $dim_{\mathcal{C}}\mathcal{V} = \mathcal{K} \geq 3$. Since $\mathfrak{T} \notin \mathcal{C}$, there exists $v \in \mathcal{V}$ such that $v, \mathfrak{T}$ are linearly $\mathcal{C}$-independent, moreover since $dim_{\mathcal{C}}\mathcal{V} = \mathcal{K} \geq 3$, it follows that there exists $w \in \mathcal{V}$ such that $v, \mathfrak{T}v, w$ are linearly $\mathcal{C}$-independent. By the density of $\mathcal{Q}$, there exist x, y, t such that

$$\begin{aligned} xv &= v, \quad yv = 0 \\ tv &= v \end{aligned}$$

Thereby (2.6) we have the following contradiction

$$\Big(a(xy + yx) + \mathfrak{T}xt + tx - xy - yx\Big)v = \mathfrak{T}v \neq 0.$$

On the other hand, if $dim_{\mathcal{C}}\mathcal{V} = 2$, then $\mathcal{Q} = \mathcal{M}_2(\mathcal{C})$ and we write $\mathfrak{T} = \sum \mathfrak{T}_{ij}e_{ij}$, where $\mathfrak{T}_{ij} \in \mathcal{C}$. By putting $x = e_{11},\ y = e_{22},\ t = e_{12}$ in the relation (2.6) it follows that $\mathfrak{T}e_{12} = 0$, which implies that $\mathfrak{T}_{11} = 0$ and $\mathfrak{T}_{21} = 0$, which contradicts the fact that $\mathfrak{T}$ is invertible. Hence the result follows. □

Here an immediate corollary follows from our result :

Corollary 2.1 ([17, Theorem 2.3]). *Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ be a non zero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation $\mathfrak{G}$ associated with a nonzero derivation $\mathfrak{d}$ such that $\mathfrak{G}(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Theorem 2.2. *Let $\mathcal{R}$ be a prime ring ring with characteristic different two, $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. Suppose $\mathfrak{F}$ is a generalized skew derivation with an associated nonzero skew derivation $\mathfrak{D}$ such that $\mathfrak{F}(x) \circ \mathfrak{F}(y) = x \circ y$ for all $x,\ y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Proof. Since every generalized skew derivation $\mathfrak{F}$ with an associated skew derivation $\mathfrak{D}$ can be written of the form $\mathfrak{F}(x) = ax + \mathfrak{D}(x)$, where a is an element of $\mathcal{R}$. So by our hypothesis we an write

$$(ax + \mathfrak{D}(x)) \circ (ay + \mathfrak{D}(y)) = x \circ y \text{ for all } x, y \in \mathcal{I}. \tag{2.7}$$

Case 1. Now if $\mathfrak{D}$ is an inner skew derivation with an associated automorphism α then $\mathfrak{D}(x) = \alpha(x)q - qx$ for all $x \in \mathcal{R}$ and $q \in \mathcal{Q}$. Hence by [7, Theorem 2], relation (2.7) implies

$$(ax + \alpha(x)q - qx) \circ (ay + \alpha(y)q - qy) = x \circ y \text{ for all } x,\ y \in \mathcal{Q}. \tag{2.8}$$

If α is an outer automorphism, then by using the Fact 2.7, we have $\mathcal{Q}$ satisfies the following polynomial identity

$$(ax + uq - qx) \circ (ay + vq - qy) = x \circ y \text{ for all } x,\ y,\ u,\ v \in \mathcal{Q}.$$

In particular take $x = y = 0$, then $\mathcal{Q}$ satisfies the following generalized polynomial identity

$$uq \circ vq = 0.$$

Proceeding in the same way as in the proof of Theorem 2.1 with necessarily variations, we arrive at a similar contradiction.

Next we consider that α is $\mathcal{Q}$-inner automorphism, then there exists an invertible element $\mathfrak{T} \in \mathcal{Q}$, such that $\alpha(x) = \mathfrak{T}x\mathfrak{T}^{-1}$ for all $x \in \mathcal{R}$. Thus from (2.8), we get

$$(ax + \mathfrak{T}x\mathfrak{T}^{-1}q - qx) \circ (ay + \mathfrak{T}y\mathfrak{T}^{-1}q - qy) = xy \text{ for all } x,\ y \in \mathcal{Q}.$$

Now if $\mathfrak{T}^{-1}q \in \mathcal{C}$, then

$$\mathfrak{D}(x) = \alpha(x) - qx = \mathfrak{T}x\mathfrak{T}^{-1}q - qx = \mathfrak{T}(x\mathfrak{T}^{-1}q - \mathfrak{T}^{-1}qx) = \mathfrak{T}[x, \mathfrak{T}^{-1}q] = 0,$$

which is a contradiction. Hence $\mathfrak{T}^{-1}q \notin \mathcal{C}$, then we have the following non trivial generalized polynomial identity.

$$(ax + \mathfrak{T}x\mathfrak{T}^{-1}q - qx) \circ (ay + \mathfrak{T}x\mathfrak{T}^{-1}q - qy) = x \circ y \text{ for all } x,\ y \in \mathcal{R}$$

where $\mathcal{R}$ is a primitive ring with $\mathcal{D}$ as an associated division ring. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_K(\mathcal{D})$, where $K = dim(\mathcal{V}_\mathcal{D})$. We first assume that $dim(\mathcal{V}_\mathcal{D}) \geq 2$. Here we show $\{v, \mathfrak{T}^{-1}qv\}$ is linearly $\mathcal{D}$-dependent set for every $v \in \mathcal{V}$. If $v = 0$, then v and $\mathfrak{T}^{-1}qv$ are linearly $\mathcal{D}$-dependent. Suppose on the contrary that v_0 and $\mathfrak{T}^{-1}qv_0$ are linearly $\mathcal{D}$-independent for some $v_0 \in \mathcal{D}$. By Invoking the density theorem, there exists $x, y \in \mathcal{R}$, such that

$$\begin{aligned} xv_0 &= 0, \quad x\mathfrak{T}^{-1}qv_0 = -\mathfrak{T}^{-1}v_0 \\ yv_0 &= 0, \quad y\mathfrak{T}^{-1}qv_0 = \mathfrak{T}^{-1}v_0 \end{aligned}$$

and thus we have

$$0 = \Big((ax + \mathfrak{T}x\mathfrak{T}^{-1}q - qx) \circ (ay + \mathfrak{T}y\mathfrak{T}^{-1}q - qy) - (x \circ y)\Big)v_0 = -2v_0,$$

which is a contradiction. Hence we reached to the conclusion that v_0 and $\mathfrak{T}^{-1}qv_0$ are linearly $\mathcal{D}$-dependent for all $v_0 \in \mathcal{V}$. Thus by the Fact 2.1, there exists $\beta \in \mathcal{D}$ such that $\mathfrak{T}^{-1}qv = \beta v$ for all $v \in \mathcal{V}$. Thus for $z \in \mathcal{R}$ and $w \in \mathcal{V}$, we can write $\mathfrak{T}^{-1}qw = \lambda w$, $z(\mathfrak{T}^{-1}qw) = z(w\lambda)$ and also $\mathfrak{T}^{-1}q(zw) = (zw)\lambda$. Thus $0 = [\mathfrak{T}^{-1}q,\ z]w$ for any $w \in \mathcal{V}$, i.e., $[\mathfrak{T}^{-1}q,\ z]\mathcal{V} = 0$. Since $\mathcal{V}$ is left faithful irreducible $\mathcal{R}$ module, we have $[\mathfrak{T}^{-1}q, z] = 0$ for every $z \in \mathcal{R}$, i.e., $\mathfrak{T}^{-1}q \in \mathcal{C}$, a contradiction. Hence $dim(\mathcal{V}_\mathcal{D}) = 1$, which implies that $\mathcal{R}$ is commutative.

Case 2. In this case we consider when $\mathfrak{D}$ is an $\mathcal{Q}$-outer skew derivation. Therefore $\mathcal{I}$ satisfies the identity

$$(ax + \mathfrak{D}(x)) \circ (ay + \mathfrak{D}(y)) = x \circ y. \tag{2.9}$$

Also we assume that α is not $\mathcal{Q}-inner$. Thus from the relation ((2.9)) and using the Fact 2.7, $\mathcal{I}$ satisfies the following identity

$$(ax+s)\circ(ay+t)=x\circ y.$$

In particular taking $x=y=0$, we have

$$s\circ t=0 \quad \text{for} \quad \text{all} \quad s,t\in\mathcal{I}. \tag{2.10}$$

Thus by [6] there exists a suitable field $\mathbb{F}$ such that $\mathcal{R}\subseteq\mathcal{M}_K(\mathbb{F})$, the ring of $K\times K$ matrices over a field $\mathbb{F}$, where K is positive integer greater than 1. Moreover $\mathcal{R}$ and $\mathcal{M}_K(\mathbb{F})$ satisfy the same polynomial identity . Let e_{ij} denote the matrix unit with 1 in the (i,j)-th entry and zero elsewhere. By taking $s=e_{11},\ t=e_{12}$ in (2.10) we get $e_{12}=0$, a contradiction. By this, we complete the proof of the theorem. □

Finally, we furnish an example that shows that the primeness condition on $\mathcal{R}$ is not superfluous in our hypothesis.

Example 2.1. Let $\mathfrak{S}$ be any ring and let

$$\mathcal{R}=\left\{\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix} \mid a,b\in\mathfrak{S}\right\} \text{ and } \mathcal{I}=\left\{\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix} \mid b\in\mathfrak{S}\right\}.$$

Define a map $\mathfrak{F}:\mathcal{R}\to\mathcal{R}$ by $\mathfrak{F}(x)=2e_{11}x-xe_{11}$ for all $x\in\mathcal{R}$. Clearly $\mathfrak{F}$ is a generalized skew derivation on $\mathcal{R}$ induced by an associated skew derivation $\mathfrak{D}$ defined by $\mathfrak{D}(x)=e_{11}x-xe_{11}$ and an associated automorphism $\alpha:\mathcal{R}\to\mathcal{R}$ given by $\alpha\left(\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}a & -b\\ 0 & 0\end{bmatrix}$. It is easy to see that $\mathfrak{F}$ satisfies the properties $\mathfrak{F}(x\circ y)=x\circ y$ and $\mathfrak{F}(x)\circ\mathfrak{F}(y)=x\circ y$ for all x,y in the ideal $\mathcal{I}$ of $\mathcal{R}$. However, $\mathcal{R}$ is not commutative.

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