WEIGHTED KIRCHHOFF PROBLEM OF N-SCHRÖDINGER TYPE INVOLVING LOGARITHMIC WEIGHT UNDER DOUBLE EXPONENTIAL NONLINEAR GROWTH

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We consider the existence of solutions of the following weighted problem:

$$\begin{cases} L_{(\sigma,\xi)}u &= f(x,u) & \text{in } B \\ u &> 0 & \text{in } B \\ u &= 0 & \text{on } \partial B, \end{cases}$$

where

$$L_{(\sigma,\xi)}u := g\Big(\int_B \big(\sigma(x)|\nabla u|^N + \xi(x)|u|^N\big)dx\Big)\big[-\mathrm{div}\big(\sigma(x)|\nabla u|^{N-2}\nabla u\big) + \xi(x)u^{N-1}\big],$$

B is the unit ball of $\mathbb{R}^N$, $N > 2$, $\sigma(x) = (\log(\frac{e}{|x|}))^{N-1}$ the singular logarithm weight with the limiting exponent $N-1$ in the Trudinger–Moser embedding, $\xi(x)$ is a positive continuous function. The Kirchoff function g is positive and continuous on $(0,+\infty)$. The nonlinearities are critical or subcritical growth in view of Trudinger–Moser inequalities of double exponential type. We prove the existence of positive solution by using Mountain Pass Theorem. In the critical case, the function of Euler–Lagrange does not fulfil the requirements of Palais–Smale conditions at all levels. We dodge this problem by using adapted test functions to identify this level of compactness.

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1. INTRODUCTION

In this paper we study the following weighted problem

$$\begin{cases} L_{(\sigma,\xi)}u &= f(x,u) & \text{in } B \\ u &> 0 & \text{in } B \\ u &= 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where

$$L_{(\sigma,\xi)} := g\Big(\int_B \big(\sigma(x)|\nabla u|^N + \xi(x)|u|^N\big)dx\Big)\big[-\mathrm{div}\big(\sigma(x)|\nabla u|^{N-2}\nabla u\big) + \xi(x)u^{N-1}\big],$$

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B is the unit ball of $\mathbb{R}^N$, $N > 2$, $f(x;t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{e^{\alpha t^{\frac{N}{N-1}}}\}$ as $t \to +\infty$, for some $\alpha > 0$ and $\xi : B \to \mathbb{R}$ is a positive continuous function satisfying some conditions. The weight $\sigma(x)$ is given by

$$\sigma(x) = \Big(\log\Big(\frac{e}{|x|}\Big)\Big)^{N-1}. \tag{2}$$

The Kirchhoff function g is a continuous positive on $(0, +\infty)$, satisfying some mild conditions.

In 1883, Kirchhoff studied the following parabolic problem

$$\sigma \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

The parameters in equation (3) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, σ is the mass density and P_0 is the initial tension. These kinds of problems have physical motivations. Indeed, the Kirchhoff operator $g((\int_B |\nabla u|^2 dx))\Delta u$ also appears in the nonlinear vibration equation namely

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - g(\int_B |\nabla u|^2 dx)\mathrm{div}(\nabla u) & = & f(x,u) & \text{in } B \times (0,T), \\ u & > & 0 & \text{in } B \times (0,T), \\ u & = & 0 & \text{on } \partial B, \\ u(x,0) & = & u_0(x) & \text{in } B, \\ \frac{\partial u}{\partial t}(x,0) & = & u_1(x) & \text{in } B, \end{cases} \tag{4}$$

which have focused the attention of several researchers, mainly as a result of the work of Lions [27]. We mention that non-local problems also arise in other areas, e.g., biological systems where the function u describes a process that depends on the average of itself (for example, population density) see, e.g., [3, 4] and their references.

In the non-weighted case, i.e., when $\sigma(x) = 1$, $\xi(x) = 0$ and when $N = 2$, problem (1) can be seen as a stationary version of the evolution problem (4). For instance, in (1) if we set $\sigma(x) = 1$, $N = 2$ and $g(t) = 1$, then we find the classical Schrödinger equation $-\triangle u + \xi(x)u = f(x,u)$. Also, if we take $\sigma(x) = 1$, $N = 2$, $\xi(x) = 0$ and $g(t) = \bar{a} + \bar{b}t$, with $\bar{a}, \bar{b} > 0$, we find Kirchhoff's classical equation which has been extensively studied. We refer to the work of Chipot [17, 18], Corrêa et al. [24] and their references. We point out that recently, in the case $g(t) = 1$, and $\xi = 0$ or $\xi \neq 0$, Baraket et al. [6], Deng, Hu and Tang, and Calanchi et al. [19, 14] have proved the existence of a nontrivial solution for the following boundary value problem

$$\begin{cases} -\mathrm{div}\big(w(x)|\nabla u(x)|^{N-2}\nabla u(x)\big) + \xi(x)|u|^{N-2}u & = & f(x,u) & \text{in } B, \\ u & = & 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 2$, the radial positive weight $w(x)$ is of logarithmic type, the function $f(x,u)$ is continuous in $B \times \mathbb{R}$ and has critical double exponential growth. In order to motivate our study, we begin by giving a brief survey on Trudinger–Moser inequalities. Since 1970, when Moser gave the famous result on the Trudinger–Moser inequality, many applications have taken place such as in the theory of conformal deformation on collectors, the study of the prescribed Gauss curvature and the mean field equations. After that, a logarithmic Trudinger–Moser inequality was used in a crucial way in [26] to study the Liouville equation of the form

$$\left\{\begin{array}{rcll} -\Delta u & = & \lambda \frac{e^u}{\int_\Omega e^u} & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{5}$$

where Ω is an open domain of $\mathbb{R}^N$, $N \geq 2$ and λ a positive parameter.

The equation (5) has a long history and has been derived in the study of multiple condensate solution in the Chern–Simons–Higgs theory [29, 30] and also, it appeared in the study of Euler Flow [9, 10, 16, 23]. Later, the Trudinger–Moser inequality was improved to a weighted inequalities [1, 11, 12, 15]. The influence of the weight in the Sobolev norm was studied as the compact embedding in [25]. When the weight is of logarithmic type, Calanchi and Ruf [13] extend the Trudinger–Moser inequality and give some applications when $N = 2$ and for prescribed nonlinearities. After that, Calanchi et al. [14] consider a more general nonlinearities and prove the existence of radial solutions. We should also refer to the interesting work of Figueiredo and Severo [22] where they studied the following problem

$$\left\{\begin{array}{rcll} -m(\int_B |\nabla u|^2 dx)\triangle u & = & f(x,u) & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array}\right.$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, the nonlinearity f behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. Also, $m : (0, +\infty) \to (0, +\infty)$ is a continuous function satisfying some conditions. The authors proved that this problem has a positive ground state solution. The existence result was proved by combining minimax techniques and Trudinger–Moser inequality. It should be noted that recently, the following nonhomogeneous Kirchhoff–Schrödinger equation

$$-M\Big(\int_{\mathbb{R}^2} |\nabla u|^2 + V(|x|)u^2 dx\Big)\big(-\triangle u + \xi(|x|)u\big) = Q(x)\tilde{g}(u) + \varepsilon h(x),$$

and

$$u(x) \to 0 \ \text{ as } \ |x| \to +\infty,$$

has been studied in [2], where ε is a positive parameter and $M : \mathbb{R}^+ \to \mathbb{R}^+$, $V, Q : (0. + \infty) \to \mathbb{R}$, are continuous functions satisfying some mild conditions. The nonlinearity $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is continuous and behaves like $\exp(\alpha t^2)$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved the existence of at least two weak solutions for this equation by combining the Mountain Pass Theorem and Ekeland's Variational Principle.

Finally, we note that recently, Xiu, Zhao and Chen [31], studied the following singular nonlocal elliptic problem:

$$-M\Big(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\Big)\mathrm{div}\big(|x|^{-ap}|\nabla u|^{p-2}\nabla u\big) = h(x)|u|^{r-2}u + H(x)|u|^{q-2}u,$$

and

$$u(x) \to 0 \ \text{ as } \ |x| \to +\infty,$$

where $x \in \mathbb{R}^N$, $M(t) = \bar{b}+\bar{a}t$, $\bar{a}, \bar{b} > 0$, $\bar{a} < \frac{N-p}{p}$, $h(x)$ and $H(x)$ are nonnegative function. They proved that this problem has infinitely many solutions by variational methods and the genus theorem.

Inspired by the works cited above, we investigate our problem in adapted weighted Sobolev space setting. We use Trudinger–Moser inequality to study and prove the existence of solutions to (1).

In literature, more attention has been accorded to the subspace of radial functions

$$W^1_{0,\mathrm{rad}}(B,\sigma) = cl\Big\{u \in C^\infty_{0,\mathrm{rad}}(\Omega); \int_\Omega \sigma(x)|\nabla u|^N \, dx < \infty\Big\},$$

endowed with the norm

$$\|\nabla u\|_{N,\sigma} = \Big(\int_\Omega \sigma(x)|\nabla u|^N \, dx\Big)^{\frac{1}{N}}.$$

So, we are motivated by the following double exponential inequality proved in [12], which is an improvement of the Trudinger–Moser inequality in a weighted Sobolev space.

Theorem 1.1 ([12]). *Let σ given by (3), then*

$$\int_B \exp\{e^{|u|^{\frac{N}{N-1}}}\}dx < +\infty, \qquad \forall\ u \in W^1_{0,\mathrm{rad}}(B,\sigma) \tag{6}$$

and

$$\sup_{\substack{u\in W^1_{0,\mathrm{rad}}(B,\sigma)\\ \|u\|_{N,\sigma}\le 1}} \int_B \exp\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{\frac{N}{N-1}}}\}dx < +\infty \quad \Leftrightarrow \quad \beta \le N, \tag{7}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

The major difficulty in this problem lies in the concurrence between the growths of g and f. To avoid this difficulty, many authors usually assume that g is increasing or bounded (see [3, 4, 22]).

Let us now state our results.

We impose the following conditions for the Kirchoff function g. So, we define the function

$$G(t) = \int_0^t g(s)ds,$$

where the function g is continuous on $\mathbb{R}^+$ and verifies:

(G_1) There exists $g_0 > 0$ such that $g(t) \geq g_0$ for all $t \geq 0$ and

$$G(t+s) \geq G(t) + G(s) \quad \forall\, s,t \geq 0;$$

(G_2) $\frac{g(t)}{t}$ is nonincreasing for $t > 0$.

The assumption (G_2) implies that $\frac{g(t)}{t} \leq g(1)$ for all $t \geq 1$. Then, one has $g(t) \leq g(1)t$ for $t \geq 1$. As a consequence of (G_2), a simple calculation shows that

$$\frac{1}{N}G(t) - \frac{1}{2N}g(t)t \ \text{ is nondecreasing for } \ t \geq 0.$$

Consequently, one has

$$\frac{1}{N}G(t) - \frac{1}{2N}g(t)t \geq 0, \quad \forall\, t \geq 0. \tag{8}$$

A typical example of a function g fulfilling the conditions (G_1), (G_2) and (G_3) is given by

$$g(t) = g_0 + at, g_0, a > 0.$$

Another example is given by $g(t) = 1 + \ln(1+t)$.

For this paper, we hypothesize that the nonlinearity $f(x,t)$ verifies the following assumptions:

(A_1) The nonlinearity $f : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is positive, continuous, radial in x, and $f(x,t) = 0$ for $t \leq 0$.

(A_2) There exist $t_0 > 0$ and $M_0 > 0$ such that for all $t > t_0$ and for all $x \in B$ we have

$$0 < F(x,t) \leq M_0 f(x,t),$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

(A_3) For each $x \in B$, $\frac{f(x,t)}{t^{2N-1}}$ is increasing for $t > 0$.

(A_4) $\lim_{t\to\infty} \frac{f(x,t)t}{\exp(Ne^{\alpha_0 t^{N'}})} \geq \gamma_0$ uniformly in x, with

$$\gamma_0 > \frac{Ng(\frac{\omega_{N-1}}{\alpha_0^{N-1}})}{\alpha_0^{N-1}e^N(N-1+e^{-\frac{\mathfrak{C}(m,N)}{N-1}})},$$

and

$$\mathfrak{C}(m,N) = m\Big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1\Big),$$

where

$$\mathfrak{S}(N) = \frac{2}{N} + \frac{N-1}{N^2} + \frac{(N-1)(N-2)}{N^3} + \cdots + \frac{(N-1)(N-2)\cdots 3}{N^{N-2}}.$$

The condition (A_2) implies that for any $\varepsilon > 0$, there exists a real $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \quad \forall |t| > t_\varepsilon, \quad \text{uniformly in} \ \ x \in B. \tag{9}$$

Also, we have that the condition (A_3) leads to

$$\lim_{t\to 0} \frac{f(x,t)}{t^\theta} = 0 \ \ \text{for all} \ \ 0 \leq \theta < 2N-1. \tag{10}$$

The asymptotic condition (A_4) would be crucial to identify the minimax level of the energy associated to the problem (1).

An example of such nonlinearity, is given by $f(t) = F'(t)$, with

$$F(t) = \frac{t^{2N+2}}{2N+2} + t^\tau \exp(Ne^{\alpha_0 t^{N'}}),$$

where $\tau > 2N$ and N' is the conjugate of N. A simple calculation shows that f verifies the conditions (A_1), (A_2), (A_3) and (A_4).

The potential ξ is continuous on $\overline{B}$ and verifies

(ξ_1) $\xi(x) \geq \xi_0 > 0$ in B for some $\xi_0 > 0$. So the function $\frac{1}{\xi}$ belongs to $L^{\frac{1}{N-1}}(B)$.

In view of (6) and (7), we say that f has subcritical growth at $+\infty$ if

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = 0, \qquad \text{for all} \ \ \alpha > 0 \tag{11}$$

and f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$ such that

$$\begin{aligned} &\lim_{s\to+\infty} \frac{|f(x,s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = 0, \ \ \forall\, \alpha > \alpha_0 \quad \text{and} \\ &\lim_{s\to+\infty} \frac{|f(x,s)|}{\exp\{Ne^{\alpha s^{N'}}\}} = +\infty, \ \ \forall\, \alpha < \alpha_0. \end{aligned} \tag{12}$$

To study the solvability of the problem (1), consider the space

$$\mathfrak{W} = \Big\{ u \in W^1_{0,\mathrm{rad}} \mid \int_B \xi(x)|u|^N dx < +\infty \Big\},$$

endowed with the norm

$$\|u\| = \Big(\int_B \sigma(x)|\nabla u|^N dx + \int_B \xi(x)|u|^N dx \Big)^{\frac{1}{N}}.$$

We say that u is a solution to the problem (1), if u is a weak solution in the following sense.

Definition 1.2. A function u is called a solution to (1) if $u \in \mathfrak{W}$ and

$$g(\|u_n\|^N)\Big[\int_B \big(\sigma(x)|\nabla u|^{N-2}\nabla u \nabla \varphi + \xi |u|^{N-2} u\varphi\big)dx\Big] = \int_B f(x,u)\varphi dx, \ \forall\, \varphi \in \mathfrak{W}.$$

It is clear that finding weak solutions of the problem (1) is equivalent to finding nonzero critical points of the following functional on $\mathfrak{W}$:

$$\mathcal{E}(u) = \frac{1}{N} G(\|u\|^N) - \int_B F(x,u)dx, \tag{13}$$

where $F(x,u) = \int_0^u f(x,t)dt$.

In order to find critical points of the functional $\mathcal{E}$ associated with (1), one generally applies the mountain pass given by Ambrosetti and Rabinowitz, see [5].

We start by the first result. In the subcritical double exponential growth, we have the following result.

Theorem 1.3. *Let $f(x,t)$ be a function that has a subcritical growth at $+\infty$ and satisfies* (A_1), (A_2) *and* (A_3). *In addition, suppose that* (G_1) *and* (G_2) *hold, then problem* (1) *has a nontrivial radial solution.*

In the context of the critical double exponential growth, the study of the problem (1) becomes more difficult than in the subcritical case. Our Euler–Lagrange function is losing compactness at a certain level. To overcome this lack of compactness, we choose test functions, which are extremal for the Trudinger–Moser inequality (7). Our result is as follows:

Theorem 1.4. *Assume that $f(x,t)$ has a critical growth at $+\infty$ and satisfies the conditions* $(A_1), (A_2), (A_3)$ *and* (A_4). *If, in addition,* (G_1) *and* (G_2) *are satisfied, then the problem* (1) *has a nontrivial solution.*

To the best of our knowledge, the present paper's results have not been covered yet in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In Section 3, we give some useful lemmas for the compactness analysis. In Section 4, we prove that the energy $\mathcal{E}$ satisfies the two geometric properties. Section 5 is devoted to estimate the minimax level of the energy. Finally, we conclude with the proofs of the main results in Section 6. Through this paper, the constant C may change from one line to another and we sometimes index the constants in order to show how they change.

2. WEIGHTED LEBESGUE AND SOBOLEV SPACES SETTING

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $\sigma \in L^1(\Omega)$ be a nonnegative function. Following Drabek et al., and Kufner in [20, 25], the weighted Lebesgue space $L^p(\Omega, \sigma)$ is defined as follows:

$$L^p(\Omega, \sigma) = \Big\{ u : \Omega \to \mathbb{R} \text{ measurable}; \quad \int_\Omega \sigma(x)|u|^p \, dx < \infty \Big\},$$

for any real number $1 \leq p < \infty$. This is a normed vector space equipped with the norm

$$\|u\|_{p,\sigma} = \Big(\int_\Omega \sigma(x)|u|^p \, dx \Big)^{\frac{1}{p}}$$

and for $\sigma(x) = 1$, we find the standard Lebesgue space $L^p(\Omega)$ and its norm

$$\|u\|_p = \Big(\int_\Omega |u|^p \, dx \Big)^{\frac{1}{p}}.$$

In [20], the corresponding weighted Sobolev space was defined as

$$W^{1,p}(\Omega, \sigma) = \big\{ u \in L^p(\Omega); \quad \nabla u \in L^p(\Omega, \sigma) \big\}$$

and equipped with the norm defined on $W^{1,p}(\Omega)$ by

$$\|u\|_{W^{1,p}(\Omega,\sigma)} = \big(\|u\|_p^p + \|\nabla u\|_{p,\sigma}^p \big)^{\frac{1}{p}}. \tag{14}$$

$L^p(\Omega, \sigma)$ and $W^{1,p}(\Omega, \sigma)$ are separable, reflexive Banach spaces provided that $\sigma(x)^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega)$.

Furthemore, if $\sigma(x) \in L^1_{\text{loc}}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, \sigma)$ and we can introduce the space $W_0^{1,p}(\Omega, \sigma)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \sigma)$. The space $W_0^{1,p}(\Omega, \sigma)$ is equipped with the following norm:

$$\|u\|_{W_0^{1,p}(\sigma,\Omega)} = \Big(\int_\Omega \sigma(x)|\nabla u|^p \, dx \Big)^{\frac{1}{p}}, \tag{15}$$

which is equivalent to the one given by (14).

Also, we use the space $W_0^{1,N}(\Omega,\sigma)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,N}(\Omega,\sigma)$, equipped with the norm

$$\|u\|_{W_0^1(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)|\nabla u|^N \ dx\Big)^{\frac{1}{N}}.$$

Let s the real such that

$$s \in (1,+\infty) \ \text{ and } \ \sigma^{-s} \in L^1(\Omega). \tag{16}$$

The last condition gives important embedding of the space $W^{1,N}(\Omega,\sigma)$ into usual Lebesgues spaces without weight. More precisely, following [20] we have

$$W^{1,N}(\Omega,\sigma) \hookrightarrow L^N(\Omega) \ \text{ with compact injection} \tag{17}$$

and

$$W^{1,N}(\Omega,\sigma) \hookrightarrow L^{N+\eta}(\Omega) \ \text{ with compact injection } \quad \text{for } \ 0 \leq \eta < N(s-1),$$

provided

$$\sigma^{-s} \in L^1(\Omega) \ \text{ with } \ s \in (1,+\infty).$$

Let the subspace

$$W^1_{0,\mathrm{rad}}(B,\sigma) = cl\Big\{u \in C^\infty_{0,\mathrm{rad}}(\Omega);\int_\Omega \sigma(x)|\nabla u|^N \ dx < \infty\Big\},$$

with

$$\sigma(x) = \Big(\log\Big(\frac{e}{|x|}\Big)\Big)^{N-1}.$$

Then the space

$$\mathfrak{W} = \Big\{u \in W^1_{0,\mathrm{rad}}(B,\sigma) \mid \int_B \xi(x)|u|^N dx < +\infty\Big\},$$

is a Banach and reflexive space provided (ξ_1) is satisfied. $\mathfrak{W}$ is endowed with the norm

$$\|u\| = \Big(\int_B \sigma(x)|\nabla u|^N dx + \int_B \xi(x)|u|^N dx\Big)^{\frac{1}{N}},$$

which is equivalent to the following norm

$$\|u\|_{W^1_{0,\mathrm{rad}}(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)|\nabla u|^N \ dx\Big)^{\frac{1}{N}}.$$

3. PRELIMINARIES FOR THE COMPACTNESS ANALYSIS

In this section, we present a number of technical lemmas for our future use. We begin by the radial lemma.

LEMMA 3.1 ([6]). *Assume that ξ is continuous and verifies (ξ_1).*

(i) *Let u be a radially symmetric C_0^1 function on the unit ball B [11]. Then, we have*

$$|u(x)| \leq \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \log^{\frac{1}{N'}}\Big(\log\Big(\frac{e}{|x|}\Big)\Big)\|u\|,$$

where ω_{N-1} is the area of the unit sphere S_{N-1} in $\mathbb{R}^{\mathbb{N}}$.

(ii) *There exists a positive constant C such that for all $u \in \mathfrak{W}$,*

$$\int_B \xi|u|^N dx \leq C\|u\|^N$$

and then the norms

$$\|.\| \quad and \quad \|.\|_{W^1_{0,rad}(\Omega,\sigma)} = \Big(\int_\Omega \sigma(x)|\nabla .|^N\ dx\Big)^{\frac{1}{N}}$$

are equivalents.

(iii) *The following embedding is continuous*

$$\mathfrak{W} \hookrightarrow L^q(B) \quad for\ all \quad q \geq 1.$$

(iv) *$\mathfrak{W}$ is compactly embedded in $L^q(B)$ for all $q \geq 1$.*

Next, we give an important lemma.

Lemma 3.2 ([21]). *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n)u_n|dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \to (x, u) \quad in \quad L^1(\Omega).$$

In an attempt to prove a compactness condition for the energy $\mathcal{E}$, we need a Lions type result [28] about an improved Trudinger–Moser inequality when we deal with weakly convergent sequences and double exponential case.

Lemma 3.3 ([6]). *Let $\{u_k\}_k$ be a sequence in $\mathfrak{W}$. Suppose that $\|u_k\| = 1$, $u_n \rightharpoonup u$ weakly in $\mathfrak{W}$, $u_n(x) \to u(x)$ and $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in B. Then*

$$\sup_k \int_B \exp\big(Ne^{p\omega_{N-1}^{\frac{1}{N-1}}|u_k|^{N'}}\big)dx < +\infty,$$

for all $1 < p < \mathcal{P}$, where

$$\mathcal{P} := \begin{cases} (1 - \|u\|^N)^{\frac{-1}{N-1}} & if\ \|u\| < 1, \\ +\infty & if\ \|u\| = 1. \end{cases}$$

Proof. For the proof, see [6]. $\square$

4. THE MOUNTAIN PASS GEOMETRY OF THE ENERGY

Since the nonlinearity f is critical or subcritical at $+\infty$, there exist positive constants $a, C > 0$, and there exists $t_2 > 1$ such that

$$|f(x,t)| \leq C \exp(e^{a\, t^{N'}}), \qquad \forall |t| > t_2. \tag{18}$$

So the functional $\mathcal{E}$ given by (13) is well defined and of class C^1.

In order to prove the existence of nontrivial solution to the problem (1), we prove the existence of nonzero critical point of the functional $\mathcal{E}$ by using the result introduced by Ambrosetti and Rabinowitz in [5] (Mountain Pass Theorem).

Definition 4.1. Let (u_n) be a sequence in a Banach space E, $J \in C^1(E, \mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence (u_n) is a Palais–Smale sequence at level c (or $(PS)_c$ sequence) for the functional J if

$$J(u_n) \to c \ \text{ in } \ \mathbb{R}, \quad \text{as } \ n \to +\infty$$

and

$$J'(u_n) \to 0 \ \text{ in } \ E', \quad \text{as } \ n \to +\infty.$$

We say that the functional J satisfies the Palais–Smale condition $(PS)_c$ at the level c if every $(PS)_c$ sequence (u_n) is relatively compact in E.

THEOREM 4.2 ([5]). *Let E be a Banach space and $J : E \to \mathbb{R}$ a C^1 functional satisfying $J(0) = 0$. Suppose that*

(i) *There exist $\rho, \beta > 0$ such that for all $u \in \partial B(0, \rho)$, $J(u) \geq \beta$;*

(ii) *There exists $x_1 \in E$ such that $\|x_1\| > \rho$ and $J(x_1) < 0$;*

(iii) *J satisfies the Palais–Smale condition (PS), that is any Palais–Smale sequence (u_n) in E is relatively compact.*

Then, J has a critical point u and the critical value $c = J(u)$ verifies

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J\big(\gamma(t)\big)$$

where $\Gamma := \{\gamma \in C([0,1], E)$ such that $\gamma(0) = 0$ and $\gamma(1) = x_1\}$ and $c \geq \beta$.

Before starting the proof of the geometric properties for the functional $\mathcal{E}$, it follows from the continuous embedding $\mathfrak{W} \hookrightarrow L^q(B)$ for all $q \geq 1$, that there exists a constant $C > 0$ such that $\|u\|_{N'q} \leq c\|u\|$, for all $u \in \mathfrak{W}$.

In the next lemmas, we prove that the functional $\mathcal{E}$ has the mountain pass geometry of Theorem 4.2.

Lemma 4.3. *Suppose that f has critical growth at $+\infty$. In addition, if (A_1), (A_3) and (G_1) hold, then, there exist $\rho, \beta > 0$ such that $\mathcal{J}(u) \geq \beta$ for all $u \in \mathfrak{W}$ with $\|u\| = \rho$.*

Proof. It follows from (10) that there exists $\delta_0 > 0$ such that

$$F(x,t) \leq \epsilon |t|^N, \qquad \text{for } \ |t| < \delta_0. \tag{19}$$

From (A_3) and (18), for all $q > N$, there exists a positive constant $C > 0$ such that

$$F(x,t) \leq C|t|^q \exp(e^{a\ t^{N'}}), \quad \forall\ |t| > \delta_1. \tag{20}$$

Using (19), (20) and the continuity of F, we get for all $q > N$,

$$F(x,t) \leq \epsilon |t|^N + C|t|^q \exp(e^{a\ t^{N'}}), \qquad \text{for all } \ t \in \mathbb{R}. \tag{21}$$

Since

$$\mathcal{E}(u) = \frac{1}{N} G(\|u\|^N) - \int_B F(x,u)dx,$$

we get from (G_1) and (21)

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N dx - C\int_B |u|^q \exp(e^{a\ u^{N'}})\ dx.$$

From the Hölder inequality, we obtain

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N dx - C\Big(\int_B \exp(N e^{a\ |u|^{N'}})dx\Big)^{\frac{1}{N}} \|u\|^q_{N'q}.$$

From the Theorem 1.1, if we choose $u \in \mathfrak{W}$ such that

$$a\|u\|^{N'} \leq \omega_{N-1}^{\frac{1}{N-1}}, \tag{22}$$

we get

$$\int_B \exp\big(N e^{a\ |u|^{N'}}\big)dx = \int_B \exp\big(N e^{a\ \|u\|^{N'}(\frac{|u|}{\|u\|})^{N'}}\big)dx < +\infty.$$

On the other hand, $\|u\|_{N'q} \leq C_1\|u\|$, so

$$\mathcal{E}(u) \geq \frac{g_0}{N}\|u\|^N - \epsilon C_1\|u\|^N - C\|u\|^q = \|u\|^N\Big(\frac{g_0}{N} - \epsilon C_1 - C\|u\|^{q-N}\Big),$$

for all $u \in \mathfrak{W}$ satisfying (22). Since $N < q$, we can choose $\rho = \|u\| \leq \frac{\omega_{N-1}^{\frac{1}{N}}}{a^{N'}}$ and for fixed ϵ such that $\frac{g_0}{N} - \epsilon C_1 > 0$, there exists

$$\beta = \rho^N\Big(\frac{g_0}{N} - \epsilon C_1 - C\rho^{q-N}\Big) > 0 \quad \text{with } \ \mathcal{E}(u) \geq \beta > 0. \qquad \square$$

By the following lemma, we prove the second geometric property for the functional $\mathcal{E}$.

Lemma 4.4. *Suppose that* (A_1), (A_2), *and* (G_2) *hold. Then there exists* $e \in \mathfrak{W}$ *with* $\mathcal{E}(e) < 0$ *and* $\|e\| > \rho$.

Proof. From the condition (G_2), for all $t \geq 1$, we have that

$$G(t) \leq \frac{g(1)}{2} t^2. \tag{23}$$

It follows from the condition (A_2), for all $t \geq t_0$

$$f(x,t) = \frac{\partial}{\partial t} F(x,t) \geq \frac{1}{M_0} F(x,t)\cdot$$

So

$$F(x,t) \geq C\ e^{\frac{t}{M_0}}, \quad \forall\ t \geq t_0.$$

In particular, for $p > 2N$ there exists C_1 and C_2 such that

$$F(x,t) \geq C_1 |t|^p - C_2, \quad \forall t \in \mathbb{R}, \quad x \in B. \tag{24}$$

Next, one arbitrarily picks $\bar{u} \in \mathfrak{W}$ such that $\|\bar{u}\| = 1$. Thus, from (23) and (24) for all $t \geq 1$

$$\mathcal{J}(t\bar{u}) \leq \frac{g(1)}{2N} t^{2N} - C_1 \|\bar{u}\|_p^p \ \ t^p - \frac{\omega_{N-1}}{N} C_2.$$

Therefore,

$$\lim_{t \to +\infty} \mathcal{J}(t\bar{u}) = -\infty.$$

We take $e = \bar{t}\bar{u}$, for some $\bar{t} > 0$ large enough. So, Lemma 4.4 follows. $\square$

5. THE MINIMAX ESTIMATE OF THE ENERGY

According to Lemmas 4.3 and 4.4, let

$$d_* := \inf_{\gamma \in \Lambda} \max_{t \in [0,1]} \mathcal{E}\big(\gamma(t)\big) > 0$$

and

$$\Lambda := \Big\{ \gamma \in C\big([0,1], \mathfrak{W}\big) \ \text{ such that } \ \gamma(0) = 0 \quad \text{and} \quad \mathcal{E}\big(\gamma(1)\big) < 0 \Big\}.$$

We are going to estimate the minimax value of the functional $\mathcal{E}$. The idea is to construct a sequence of functions $(v_n) \in \mathfrak{W}$ and estimate $\max\{\mathcal{E}(tv_n) : t \geq 0\}$. For this goal, let us consider the following Moser function

$$w_n(x) = \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \begin{cases} \dfrac{\log(\log(\frac{e}{|x|}))}{\log^{\frac{1}{N}}(1+n)} & \text{if } e^{-n} \leq |x| \leq 1, \\ \log^{\frac{N-1}{N}}(1+n) & \text{if } 0 \leq |x| \leq e^{-n}. \end{cases}$$

Let $v_n(x) = \frac{w_n(x)}{\|w_n\|}$. Then $v_n \in \mathfrak{W}$ and $\|v_n\| = 1$.

5.1. **Helpful lemmas**

We need two technical lemmas that help us reach our aims and objectives.

Lemma 5.1 ([6]). *Assume $\xi(x)$ is continuous and (ξ_1) is satisfied. Then there holds*

(i)

$$\|w_n\|^N \leq 1 + \frac{m\Big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1)\Big)}{\log(1+n)} + o_n(1),$$

where $m = \max_{x\in\overline{B}} \xi(x)$, $o_n(1) \to 0$ as $n \to +\infty$ and

$$\mathfrak{S}(N) = \frac{2}{N} + \frac{N-1}{N^2} + \frac{(N-1)(N-2)}{N^3} + \cdots + \frac{(N-1)(N-2)\cdots 3}{N^{N-2}}.$$

(ii)

$$E(m,N,n) \leq \frac{1}{\|w_n\|^{N'}} \leq D(\xi_0,N,n),$$

where

$$E(m,N,n) = 1 - \frac{m\Big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1)\Big)}{(N-1)\log(1+n)} + o_n(1)$$

and

$$D(\xi_0,N,n) = 1 - \frac{\xi_0\Big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1)\Big)}{(N-1)\log(1+n)} + o_n(1).$$

Proof. For the proof, see [6]. $\square$

Now, we have the following estimation:

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{e^{-n}\leq|x|\leq 1} \exp\big\{N e^{\omega_{N-1}^{\frac{1}{N-1}} v_n^{N'}}\big\}dx \\
&\qquad = \lim_{n\to+\infty} \omega_{N-1} \int_0^n \exp\Big\{N e^{\frac{\log^{N'}(1+t)}{\log^{\frac{1}{N-1}}(1+n)\|w_n\|^{N'}}} - Nt\Big\}dt \\
&\qquad \geq \omega_{N-1} e^N.
\end{aligned}$$

5.2. **The minimax value of the energy $\mathcal{E}$**

Finally, we give the desired estimate.

Lemma 5.2. *Assume that (G_1), (G_2) and (A_4) holds, then*

$$d_* < \frac{1}{N} G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big).$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. So, from Lemma 4.4, $\mathcal{E}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$d_* \leq \max_{t\geq 0} \mathcal{E}(tv_n)\cdot$$

We argue by contradiction and suppose that for all $n \geq 1$,

$$\max_{t\geq 0} \mathcal{E}(tv_n) \geq \frac{1}{N} G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big).$$

Since $\mathcal{E}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0} \mathcal{E}(tv_n) = \mathcal{E}(t_n v_n) \geq \frac{1}{N} G\Big(\frac{\omega_{N-1}}{\alpha_0}\Big).$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$, we get

$$G(t_n^N) \geq G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big).$$

On one hand, the condition (G_1) implies that $G : [0,+\infty) \to [0,+\infty)$ is an increasing bijection. So

$$t_n^N \geq \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{25}$$

On the other hand,

$$\frac{d}{dt} J(tv_n)\big|_{t=t_n} = g(t_n^N) t_n^{N-1} - \int_B f(x, t_n v_n) v_n dx = 0,$$

that is

$$g(t_n^N) t_n^N = \int_B f(x, t_n v_n) t_n v_n dx. \tag{26}$$

Now, we claim that the sequence (t_n) is bounded in $(0,+\infty)$. Indeed, it follows from (A_4) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon) \exp\big(N e^{\alpha_0 t^{N'}}\big) \quad \forall |t| \geq t_\varepsilon, \quad \text{uniformly in } \; x \in B. \tag{27}$$

By Lemma 5.1, if $|x| \leq e^{-n}$ we have

$$\begin{aligned} v_n^{N'} &\geq \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \frac{\log(1+n)}{1 + \frac{m\big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1 + o_n(1)\big)}{\log(1+n)} + o_n(1)} \\ &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{m\big(\mathfrak{S}(N) + \frac{(N+1)(N-1)!}{N^N} + 1\big)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o_n(1) \\ &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \log(1+n) - \frac{\mathfrak{C}(m,N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}} + o_n(1), \end{aligned} \tag{28}$$

where
$$\mathfrak{C}(m,N)=m\Big(\mathfrak{S}(N)+\frac{(N+1)(N-1)!}{N^N}+1\Big).$$
We have
(29)
$$\begin{aligned}
g(t_n^N)t_n^N &\geq (\gamma_0-\varepsilon)\int_{0\leq|x|\leq e^{-n}}\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx\\
&\geq (\gamma_0-\varepsilon)\int_{0\leq|x|\leq e^{-n}}\exp\Big\{Ne^{\alpha_0 t_n^{N'}\Big(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\log(1+n)-\frac{\mathfrak{C}(m,N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}}+o_n(1)\Big)}\Big\}dx\\
&=\omega_{N-1}(\gamma_0-\varepsilon)\exp\Big\{Ne^{\alpha_0 t_n^{N'}\Big(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\log(1+n)-\frac{\mathfrak{C}(m,N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}}+o(1)\Big)}-Nn\Big\}.
\end{aligned}$$

Using the condition (G_2), we obtain

$$g(1)t_n^{2N}\geq\frac{\omega_{N-1}}{N}(\gamma_0-\varepsilon)\exp\Big(Ne^{\frac{\alpha_0}{\omega_{N-1}^{\frac{1}{N-1}}}t_n^{N'}\log(1+n)}-Nn\Big). \tag{30}$$

From (30), we obtain for n large enough

$$1\geq\frac{\omega_{N-1}}{N}(\gamma_0-\varepsilon)\exp\Big(Ne^{\alpha_0 t_n^{N'}\Big(\frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\log(1+n)-\frac{\mathfrak{C}(m,N)}{(N-1)\omega_{N-1}^{\frac{1}{N-1}}}+o(1)\Big)}-\log\big(g(1)t_n^{2N}\big)\Big).$$

Therefore, (t_n) is bounded in $\mathbb{R}$. Also, we have
$$t_n^N\geq\frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$

Now, suppose that
$$\lim_{n\to+\infty}t_n^N>\frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$
For n large enough, $t_n^N>\frac{\omega_{N-1}}{\alpha_0^{N-1}}$ and in this case, the right-hand side of the inequality (29) gives the unboundedness of the sequence (t_n). Since (t_n) is bounded, we get

$$\lim_{n\to+\infty}t_n^N=\frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{31}$$

We claim that (31) leads to a contradiction with (A_4). For this purpose, the following sets should be used
$$\mathcal{A}_n=\{x\in B|t_nv_n\geq t_\varepsilon\}\ \text{ and }\ \mathcal{C}_n=B\setminus\mathcal{A}_n,$$
where t_ε is given in (27). We have
$$g(t_n^N)t_n^N=\int_B f(x,t_nv_n)t_nv_n dx=\int_{\mathcal{A}_n}f(x,t_nv_n)t_nv_n dx+\int_{\mathcal{C}_n}f(x,t_nv_n)t_nv_n$$

$$
\begin{aligned}
&\geq (\gamma_0-\varepsilon)\int_{\mathcal{A}_n}\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx+\int_{\mathcal{C}_n}f(x,t_nv_n)t_nv_ndx\\
&=(\gamma_0-\varepsilon)\int_{B}\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx-(\gamma_0-\varepsilon)\int_{\mathcal{C}_n}\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx\\
&\qquad+\int_{\mathcal{C}_n}f(x,t_nv_n)t_nv_ndx.
\end{aligned}
$$

Since $v_n\to 0$ a.e in B, $\chi_{\mathcal{C}_n}\to 1$ a.e in B, therefore, using the dominated convergence theorem, we get

$$
\begin{aligned}
\lim_{n\to+\infty}g(t_n^N)t_n^N&=g\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big)\frac{\omega_{N-1}}{\alpha_0^{N-1}}\\
&\geq(\gamma_0-\varepsilon)\lim_{n\to+\infty}\int_B\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx-(\gamma_0-\varepsilon)\frac{\omega_{N-1}}{N}e^N.
\end{aligned}
$$

By using the fact that

$$t_n^N\geq\frac{\omega_{N-1}}{\alpha_0^{N-1}},$$

we get

$$
\begin{aligned}
\int_B\exp\{Ne^{\alpha_0 t_n^{N'}v_n^{N'}}\}dx\geq&\int_{0\leq|x|\leq e^{-n}}\exp\Big\{Ne^{N\omega_{N-1}^{\frac{1}{N-1}}v_n^{N'}}\Big\}dx\\
&+\int_{e^{-k}\leq|x|\leq 1}\exp\Big\{Ne^{\omega_{N-1}^{\frac{1}{N-1}}v_n^{N'}}\Big\}dx.
\end{aligned}
$$

On one hand, we have by (29)

$$
\begin{aligned}
&\int_{0\leq|x|\leq e^{-n}}\exp\Big\{Ne^{\omega_{N-1}^{\frac{1}{N-1}}v_n^{N'}}\Big\}dx\\
&\qquad\geq\int_{0\leq|x|\leq e^{-n}}\exp\Big\{Ne^{\log\big(1+n)-\frac{\mathfrak{C}(m,N)}{N-1}+o_n(1)\big)}\Big\}dx\\
&\qquad=\frac{\omega_{N-1}}{N}\exp\Big\{N+Nn-\frac{\mathfrak{C}(m,N)}{N-1}+o_n(1)\Big\}e^{-Nn}\\
&\qquad=e\frac{\omega_{N-1}}{N}\exp\Big\{N-\frac{\mathfrak{C}(m,N)}{N-1}+o_n(1)\Big\}\to\frac{\omega_{N-1}}{N}e^{N-\frac{\mathfrak{C}(m,N)}{N-1}}.
\end{aligned}
$$

On other side, we have by the definition of v_n and previous estimates,

$$
\begin{aligned}
\int_{e^{-n}\leq|x|\leq 1}\exp\Big\{Ne^{\omega_{N-1}^{\frac{1}{N-1}}v_n^{N'}}\Big\}dx&=\int_{e^{-n}\leq|x|\leq 1}\exp\Big\{Ne^{\frac{\log^{N'}(\log(\frac{e}{|x|}))}{\|w_n\|^{N'}\log^{\frac{1}{N-1}}(1+n)}}\Big\}dx\\
&\geq\omega_{N-1}e^N.
\end{aligned}
$$

Hence

$$\lim_{n\to+\infty}g(t_n^N)t_n^N=g\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big)\frac{\omega_{N-1}}{\alpha_0^{N-1}}\geq(\gamma_0-\varepsilon)\omega_{N-1}\frac{e^N}{N}\big(N-1+e^{-\frac{\mathfrak{C}(m,N)}{N-1}}\big).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\frac{Ng\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right)}{\alpha_0^{N-1}e^N\left(N-1+e^{-\frac{\mathfrak{C}(m,N)}{N-1}}\right)} \geq \gamma_0.$$

This contradicts (A_4). The lemma is proved. $\square$

6. PROOF OF MAIN RESULTS

First, we begin by some crucial lemmas. We consider the Nehari manifold associated to the functional $\mathcal{E}$, namely,

$$\mathcal{N} = \left\{u \in \mathfrak{W} : \mathcal{E}'(u)u = 0, u \neq 0\right\},$$

and the number $c = \inf\limits_{u\in\mathcal{N}} \mathcal{E}(u)$. We have the following lemmas.

Lemma 6.1. *Assume that the condition (A_3) holds, then for each $x \in B$,*

$$tf(x,t) - 2NF(x,t) \quad \textit{is increasing for} \quad t > 0.$$

In particular, $tf(x,t) - 2NF(x,t) \geq 0$ for all $(x,t) \in B \times [0,+\infty)$.

Proof. Assume that $0 < t < s$. For each $x \in B$, we have

$$\begin{aligned} tf(x,t) - 2NF(x,t) &= \frac{f(x,t)}{t^{2N-1}}t^{2N} - 2NF(x,s) + 2N\int_t^s f(x,\nu)d\nu \\ &< \frac{f(x,t)}{s^{2N-1}}t^{2N} - 2NF(x,s) + \frac{f(x,s)}{s^{2N-1}}(s^{2N} - t^{2N}) \\ &= sf(x,s) - 2NF(x,s). \end{aligned}$$

$\square$

Lemma 6.2. *If (G_2) and (A_3) are satisfied, then $d_* \leq c$.*

Proof. Let $\bar{u} \in \mathcal{N}$, $\bar{u} > 0$ and consider the function $\psi : (0,+\infty) \to \mathbb{R}$ defined by $\psi(t) = \mathcal{E}(t\bar{u})$. ψ is differentiable and we have

$$\psi'(t) = \mathcal{E}'(t\bar{u})\bar{u} = g(t^N\|\bar{u}\|^N)t^{N-1}\|\bar{u}\|^N - \int_B f(x,t\bar{u})\bar{u}dx, \quad \text{for all } t > 0.$$

Since $\bar{u} \in \mathcal{N}$, we have $\mathcal{E}'(\bar{u})\bar{u} = 0$ and therefore $g(\|\bar{u}\|^N)\|\bar{u}\|^N = \int_B f(x,\bar{u})\bar{u}dx$. Hence,

$$\begin{aligned} \psi'(t) = t^{2N-1}\|\bar{u}\|^{2N}\Big(&\frac{g(t^N\|\bar{u}\|^N)\|}{t^N\|\bar{u}\|^N} - \frac{g(\|\bar{u}\|^N)}{\|\bar{u}\|^N}\Big) \\ &+ t^{2N-1}\int_B\Big(\frac{f(x,\bar{u})}{\bar{u}^{2N-1}} - \frac{f(x,t\bar{u})}{(t\bar{u})^{2N-1}}\Big)\bar{u}^{2N}dx. \end{aligned}$$

We have that $\psi'(1) = 0$. We have also by the conditions (G_2) and (A_3) that $\psi'(t) > 0$ for all $0 < t < 1$, $\psi'(t) \leq 0$ for all $t > 1$. It follows that

$$\mathcal{E}(\bar{u}) = \max_{t\geq 0} \mathcal{E}(t\bar{u}).$$

We define the function $\lambda : [0,1] \to \mathcal{E}$ such that $\lambda(t) = t\bar{t}\bar{u}$, with $\mathcal{E}(\bar{t}\bar{u}) < 0$. We have $\lambda \in \Lambda$, and hence

$$d_* \leq \max_{t\in[0,1]} \mathcal{E}\big(\lambda(t)\big) \leq \max_{t\geq 0} \mathcal{E}(t\bar{u}) = \mathcal{E}(\bar{u}).$$

Since $\bar{u} \in \mathcal{N}$ is arbitrary then $d_* \leq c$.

The main difficulty in the approach to the critical problem of growth is the loss of compactness. Precisely, the overall conditions of Palais–Smale are not verified for a certain level of energy. In the following proposition, we identify the first level of non-compactness below which the $(PS)_d$ condition holds. □

Proposition 6.3. *Let $\mathcal{E}$ be the energy associated to the problem* (1) *defined by* (13). *Assume that the conditions* (G_1), (G_2), (A_1), (A_2), (A_3) *and* (A_4) *are satisfied, then*

(i) *In the subcritical case, the functional $\mathcal{E}$ satisfies the Palais–Smale condition $(PS)_d$ at all level $d \in \mathbb{R}$.*

(ii) *In the critical case, the functional $\mathcal{E}$ satisfies the Palais–Smale condition $(PS)_d$ only for level d such that $d < \frac{1}{N} G(\frac{\omega_{N-1}}{\alpha_0^{N-1}})$.*

Proof. We start with the second item. Consider a $(PS)_d$ sequence in $\mathfrak{W}$, for some $d \in \mathbb{R}$, that is

$$\mathcal{E}(u_n) = \frac{1}{N} G(\|u_n\|^N) - \int_B F(x,u_n)dx \to d, \quad n \to +\infty \tag{32}$$

and

$$\begin{aligned} |\mathcal{E}'(u_n)\varphi| =& \Big| g(\|u_n\|^N)\Big[\int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla\varphi + \xi(x)|u_n|^{N-2}u_n\varphi dx\Big] \\ & - \int_B f(x,u_n)\varphi dx\Big| \leq \epsilon_n\|\varphi\|, \end{aligned} \tag{33}$$

for all $\varphi \in \mathfrak{W}$, where $\epsilon_n \to 0$, when $n \to +\infty$.

By (32), for all $\epsilon > 0$ there exists a constant $C > 0$

$$\frac{1}{N} G(\|u_n\|^N) \leq C + \int_B F(x,u_n)dx.$$

From (9), for all $\epsilon > 0$, there exists $t_\epsilon > 0$ such that

$$F(x,t) \leq \epsilon t f(x,t), \quad \text{for all } |t| > t_\epsilon \text{ and uniformly in } x \in B.$$

It follows that,

$$\frac{1}{N}G(\|u_n\|^N) \leq C + \int_{|u_n|\leq t_\epsilon} F(x,u_n)dx + \epsilon \int_B f(x,u_n)u_n dx.$$

From (33), we get

$$\frac{1}{2N}g(\|u_n\|^N)\|u_n\|^N \leq \frac{1}{N}G(\|u_n\|^N) \leq C_1 + \epsilon\epsilon_n\|u_n\| + \epsilon g(\|u_n\|^N)\|u_n\|^N,$$

for some constant $C_1 > 0$. Using (8) and the condition (G_1), for all ϵ such that $0 < \epsilon < \frac{1}{2N}$, we get

$$g_0\Big(\frac{1}{2N} - \epsilon\Big)\|u_n\|^N \leq C_1 + \epsilon\epsilon_n\|u_n\|.$$

We deduce that the sequence (u_n) is bounded in $\mathfrak{W}$. As a consequence, there exists $u \in \mathfrak{W}$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathfrak{W}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \geq 1$.

Furthermore, we have from (33) and (34), that

$$0 < \int_B f(x,u_n)u_n \leq C,$$

and

$$0 < \int_B F(x,u_n) \leq C.$$

Since by Lemma 3.2 we have

$$f(x,u_n) \to f(x,u) \quad \text{in} \quad L^1(B) \quad \text{as} \quad n \to +\infty, \tag{34}$$

then, it follows from (A_2) and the generalized Lebesgue dominated convergence theorem that

$$F(x,u_n) \to F(x,u) \quad \text{in} \quad L^1(B) \quad \text{as} \quad n \to +\infty. \tag{35}$$

So,

$$\lim_{n\to+\infty} G(\|u_n\|^N) = N\big(d + \int_B F(x,u)dx\big). \tag{36}$$

Next, we are going to make some claims.

Claim 1. $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in B$. Indeed, for any $\eta > 0$, let

$$\mathcal{A}_\eta = \big\{x \in B, |u_n - u| \geq \eta\big\}.$$

For all $t \in \mathbb{R}$, for all positive $c > 0$, we have $ct \leq Ne^t + \frac{c^2}{N}$. It follows that for

$$t = \omega_{N-1}^{\frac{1}{N-1}}\Big(\frac{|u_n - u|}{\|u_n - u\|}\Big)^{N'}, \quad c = \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}}\|u_n - u\|^{N'},$$

we get

$$
\begin{aligned}
|u_n-u|^{N'} &\le Ne^{\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_n-u|}{\|u_n-u\|})^{N'}} + \frac{\frac{1}{\omega_{N-1}^{\frac{2}{N-1}}}\|u_n-u\|^{2N'}}{N} \\
&\le Ne^{\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_n-u|}{\|u_n-u\|})^{N'}} + C_1(N),
\end{aligned}
$$

where $C_1(N)$ is a constant depending only on N and the upper bound of $\|u_n\|$. So, if we denote by $\mathcal{L}(\mathcal{A}_\eta)$ the Lebesgue measure of the set $\mathcal{A}_\eta$, we obtain

$$
\begin{aligned}
\mathcal{L}(\mathcal{A}_\eta)=\int_{\mathcal{A}_\eta} e^{|u_n-u|^{N'}}e^{-|u_n-u|^{N'}}dx &\le e^{-\eta^{N'}}\int_{\mathcal{A}_\eta}\exp\Big(Ne^{\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_n-u|}{\|u_n-u\|})^{N'}}+C_1(N)\Big)dx \\
&\le e^{-\eta^{N'}}e^{C_1(N)}\int_B \exp\Big(Ne^{\omega_{N-1}^{\frac{1}{N-1}}(\frac{|u_n-u|}{\|u_n-u\|})^{N'}}\Big)dx \\
&\le e^{-\eta^{N'}}C_2(N)\to 0 \quad\text{as}\quad \eta\to+\infty,
\end{aligned}
$$

where $C_2(N)$ is a positive constant depending only on N and the upper bound of $\|u_n\|$. It follows that

$$
(37)\quad \int_{\mathcal{A}_\eta}|\nabla u_n-\nabla u|dx \le Ce^{-\frac{1}{2}\eta^{N'}}\Big(\int_B|\nabla u_n-\nabla u|^2\sigma(x)dx\Big)^{\frac{1}{2}}\to 0 \text{ as } \eta\to+\infty.
$$

We define for $\eta>0$, the truncation function used in [7]

$$
T_\eta(s):=\begin{cases} s & \text{if } |s|<\eta, \\ \eta\frac{s}{|s|} & \text{if } |s|\ge\eta. \end{cases}
$$

We take $\varphi=T_\eta(u_n-u)\in\mathfrak{W}$ in (33) and since $\nabla\varphi=\chi_{\mathcal{A}_\eta}\nabla(u_n-u)$, we obtain

$$
\begin{aligned}
\Big|g(\|u_n\|^N)&\int_{B\backslash\mathcal{A}_\eta}\sigma(x)(|\nabla u_n|^{N-2}\nabla u_n-|\nabla u|^{N-2}\nabla u).(\nabla u_n-\nabla u)dx \\
&\quad+\int_{B\backslash\mathcal{A}_\eta} g(\|u_n\|^N)\xi(x)(|u_n|^{N-2}u_n-|u|^{N-2}u)(u_n-u)\Big| \\
&\le \Big|\int_{B\backslash\mathcal{A}_\eta} g(\|u_n\|^N)|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u) \\
&\quad+g(\|u_n\|^N)\xi(x)(|u|^{N-2}u)(u_n-u)dx\Big| \\
&\quad+\int_B f(x,u_n)T_\eta(u_n-u)dx+\varepsilon_n\|u_n-u\| \\
&\le g(\|u_n\|^N)\Big|\int_B|\nabla u|^{N-2}\nabla u.(\nabla u_n-\nabla u+\xi(x)\big(|u|^{N-2}u)(u_n-u)\big)dx\Big| \\
&\quad+\int_B f(x,u_n)T_\eta(u_n-u)dx+\varepsilon_n\|u_n-u\|.
\end{aligned}
$$

Since $u_n \rightharpoonup u$ weakly, then

$$g(\|u_n\|^N)\int_B \big(\sigma(x)|\nabla u|^{N-2}\nabla u.(\nabla u_n - \nabla u) + \xi(x)(|u|^{N-2}u)(u_n - u)\big)dx \to 0.$$

Moreover, using (34) and the Lebesgue dominated convergence theorem, we get

$$\int_B f(x,u_n)T_\eta(u_n-u)dx \to 0 \quad \text{as} \quad n\to+\infty.$$

Using the well-known inequality,

$$\text{(38)}\qquad \big\langle |x|^{N-2}x - |y|^{N-2}y, x-y\big\rangle \geq 2^{2-N}|x-y|^N \;\forall\; x,y\in\mathbb{R}^N,\quad N\geq 2,$$

$\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^{\mathbb{N}}$ and the fact that $0 < g_0 \leq g(\|u_n\|^N)$, one has

$$\int_{B\setminus\mathcal{A}_\eta}\big(\sigma(x)|\nabla u_n - \nabla u|^N + \xi(x)|u_n-u|^N\big)dx \to 0.$$

Since $\xi(x) > 0$, then

$$\int_{B\setminus\mathcal{A}_\eta}\sigma(x)|\nabla u_n - \nabla u|^N dx \to 0.$$

Therefore,

$$\text{(39)}\qquad \begin{aligned}&\int_{B\setminus\mathcal{A}_\eta}|\nabla u_n - \nabla u|dx\\ &\leq \Big(\int_{B\setminus\mathcal{A}_\eta}\sigma(x)|\nabla u_n - \nabla u|^N dx\Big)^{\frac{1}{N}}\mathcal{L}^{\frac{1}{N'}}(B\setminus\mathcal{A}_\eta)\to 0 \text{ as } n\to+\infty.\end{aligned}$$

From (37) and (39), we deduce that

$$\int_B|\nabla u_n - \nabla u|dx \to 0 \quad \text{as} \quad n\to+\infty.$$

Therefore, $\nabla u_n(x)\to\nabla u(x)$ a.e. $x\in B$ and Claim 1 is proved.

Claim 2. $g(\|u\|^N)\|u\|^N \geq \int_B f(x,u)udx$. This inequality holds in the case $\|u\| = 0$. So we suppose that $\|u\|\neq 0$ and we argue by contradiction. Suppose that

$$g(\|u\|^N)\|u\|^N < \int_B f(x,u)udx.$$

Hence, $\mathcal{E}'(u)u < 0$. The function $\psi: t\to\psi(t) = \mathcal{E}'(tu)u$ is positive for t small enough. Indeed, from (9) and the critical (respectively, subcritical) growth of the nonlinearity f, for every $\epsilon > 0$, for every $q > N$, there exist positive constants C and c such that

$$|f(x,t)| \leq \epsilon|t|^{N-1} + C|t|^q\exp(e^{c\,t^{N'}}),\qquad \forall,(t,x)\in\mathbb{R}\times B.$$

Then using the condition (G_1), the last inequality and the Hölder inequality, we obtain

$$
\begin{aligned}
\psi(t) &= g(t^N\|u\|^N)t^{N-1}\|u\|^N - \int_B f(x,tu)u dx \\
&\geq g_0 t^{N-1}\|u\|^N - \epsilon t^{N-1}\int_B u^N dx \\
&\quad - C\Big(\int_B \exp(Ne^{c\, t^{N'}u^{N'}})dx\Big)^{\frac{1}{N}}\Big(\int_B u^{N'q}dx\Big)^{\frac{1}{N'}}.
\end{aligned}
$$

In view of (7) the integral

$$
\int_B \exp\big(Ne^{c\, t^{N'}u^{N'}}\big)dx \leq \int_B \exp\big(Ne^{c\, t^{N'}\frac{u^{N'}}{\|u\|^{N'}}\|u\|^{N'}}\big)dx \leq C,
$$

provided $t \leq \frac{\omega_{N-1}^{\frac{1}{N}}}{c^{N'}\|u\|}$. Using the radial Lemma 3.1, we get $\|u\|^q_{N'q} \leq C'\|u\|^q$. Then,

$$
\begin{aligned}
\psi(t) &\geq g_0 t^{N-1}\|u\|^N - C_1\epsilon t^{N-1}\|u\|^N - C_2\|u\|^q \\
&= \|u\|^N t^{N-1}\big[(g_0 - C_1\epsilon) - C_2 t^{q-(N-1)}\|u\|^{q-N}\big].
\end{aligned}
$$

We chose $\epsilon > 0$, such that $g_0 - C_1\epsilon > 0$ and since $q > N$, for small t, we get $\psi : t \to \psi(t) = \mathcal{E}'(tu)u > 0$. So there exists $\eta \in (0,1)$ such that $\psi(\eta u) = 0$. Therefore, $\eta u \in \mathcal{N}$. Using (10), the semicontinuity of norm and Fatou's Lemma, we get

$$
\begin{aligned}
d \leq d_* \leq c \leq \mathcal{E}(\eta u) &= \mathcal{E}(\eta u) - \frac{1}{2N}\mathcal{E}'(\eta u)\eta u \\
&= \frac{1}{N}G(\|\eta u\|^N) - \frac{1}{2N}g(\|\eta u\|^N)\|\eta u\|^N \\
&\qquad + \frac{1}{2N}\int_B \big(f(x,\eta u)\eta u - 2NF(x,\eta u)\big)dx \\
&< \frac{1}{N}G(\|u\|^N) - \frac{1}{2N}g(\|u\|^N)\|u\|^N \\
&\qquad + \frac{1}{2N}\int_B \big(f(x,u)u - 2NF(x,u)\big)dx \\
&\leq \liminf_{n\to+\infty}\Big[\frac{1}{N}G(\|u_n\|^N) - \frac{1}{2N}g(\|u_n\|^N)\|u_n\|^N \\
&\qquad + \frac{1}{2N}\int_B \big(f(x,u_n)u_n - 2NF(x,u_n)\big)dx\Big] \\
&\leq \lim_{n\to+\infty}\big[\mathcal{E}(u_n) - \frac{1}{2N}\mathcal{E}'(u_n)u_n\big] = d
\end{aligned}
$$

which is absurd and the claim is well established.

On the other hand, by Claim 2, (8) and Lemma 6.1 we obtain

(40)
$$\begin{aligned}\mathcal{E}(u)\geq &\frac{1}{N}G(\|u\|^N)-\frac{1}{2N}g(\|u\|^N)\|u\|^N+\frac{1}{2N}\int_B\big[f(x,u)u-2NF(x,u)\big]dx\\ \geq &\ 0.\end{aligned}$$

Now, using the semicontinuity of the norm and (35) we get,

$$\mathcal{E}(u)\leq \frac{1}{N}\liminf_{n\to\to\infty}G(\|u_n\|^N)-\int_B F(x,u)dx=d.$$

So from (40), $d\geq 0$. We also have $u\geq 0$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\|\to\rho>0$. In addition, $\mathcal{E}'(u_n)\to 0$ leads to

$$g(\rho^N)\Big[\int_B\sigma(x)|\nabla u|^{N-2}\nabla u.\nabla\varphi+\xi(x)|u|^{N-2}u\varphi dx\Big]=\int_B f(x,u)\varphi dx,\quad\forall\varphi\in\mathfrak{W}.$$

By taking $\varphi=u^-$, with $w^{\pm}=\max(\pm w,0)$, we obtain $\|u^-\|^N=0$ and so $u=u^+\geq 0$.

We finish the proof by considering three cases.

Case 1. $d=0$. In this case,

$$0\leq\mathcal{E}(u)\leq\liminf_{n\to+\infty}\mathcal{E}(u_n)=0.$$

So,

$$\mathcal{E}(u)=0$$

and then

$$\lim_{n\to+\infty}\frac{1}{N}G(\|u_n\|^N)=\int_B F(x,u)dx=\frac{1}{N}G(\|u\|^N).$$

Consequently,

$$\|u_n\|\to\|u\|.$$

Therefore, by Brezis–Lieb's Lemma [8] $u_n\to u$ in $\mathfrak{W}$.

Case 2. $d>0$ and $u=0$. This case is impossible. Indeed, suppose that $u=0$. Then, $\int_B F(x,u_n)dx\to 0$ and consequently, we get

$$\frac{1}{N}G(\|u_n\|^N)\to d<\frac{1}{N}G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big). \tag{41}$$

By (33), we have

$$\Big|g(\|u_n\|^N)\|u_n\|^N-\int_B f(x,u_n)u_n dx\Big|\leq C\epsilon_n.$$

First, we claim that there exists $q>1$ such that

$$\int_B|f(x,u_n)|^q dx\leq C. \tag{42}$$

So

$$g(\|u_n\|^N)\|u_n\|^N\leq C\epsilon_n+\Big(\int_B|f(x,u_n)|^q\Big)^{\frac{1}{q}}dx\Big(\int_B|u_n|^{q'}\Big)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Since (u_n) converge to $u=0$ in $L^{q'}(B)$

$$\lim_{n\to+\infty} g(\|u_n\|^N)\|u_n\|^N = 0.$$

From the condition (G_1), we obtain

$$\lim_{n\to+\infty} \|u_n\|^N = 0.$$

So, $u_n \to 0$ in $\mathfrak{W}$. Therefore, $\mathcal{E}(u_n)\to 0$ which is in contradiction with $d>0$. For the proof of the claim (42), since f has critical growth, for every $\epsilon>0$ and $q>1$ there exists $t_\epsilon>0$ and $C>0$ such that for all $|t|\geq t_\epsilon$, we have

$$|f(x,t)|^q \leq C\exp\big(Ne^{\alpha_0(\epsilon+1)t^{N'}}\big).$$

Consequently,

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\leq t_\epsilon} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q dx\\ &\leq \omega_{N-1}\max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B \exp\big(Ne^{\alpha_0(\epsilon+1)|u_n|^{N'}}\big)dx.\end{aligned}$$

Since

$$\big(G^{-1}(Nd)\big)^{\frac{1}{N-1}} < \frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0},$$

there exists $\eta\in(0,\frac{1}{2})$ such that

$$\big(G^{-1}(Nd)\big)^{\frac{1}{N-1}} = (1-2\eta)\frac{\omega_{N-1}^{\frac{1}{N-1}}}{\alpha_0}.$$

From (41),

$$\|u_n\|^{N'} \to \big(G^{-1}(Nd)\big)^{\frac{1}{N-1}},$$

so there exist $n_\eta\in\mathbb{N}$ such that

$$\alpha_0\|u_n\|^{N'} \leq (1-\eta)\omega_{N-1}^{\frac{1}{N-1}},$$

for all $n\geq n_\eta$. Therefore,

$$\alpha_0(1+\epsilon)\Big(\frac{|u_n|}{\|u_n\|}\Big)^{N'}\|u_n\|^{N'} \leq (1+\epsilon)(1-\eta)\Big(\frac{|u_n|}{\|u_n\|}\Big)^{N'}\omega_{N-1}^{\frac{1}{N-1}}.$$

We choose $\epsilon>0$ small enough to get

$$(1+\epsilon)(1-\eta)<1,$$

hence, the second integral is uniformly bounded in view of (7).

Case 3. $d>0$, $u>0$. In this case, we claim that $\mathcal{E}(u)=d$. Now, using the semicontinuity of the norm and (32) we get,

$$\mathcal{E}(u) \leq \frac{1}{N}\liminf_{n\to\to\infty} G(\|u_n\|^N) - \int_B F(x,u)dx = d.$$

We argue by contradiction and we suppose that

$$\mathcal{E}(u) < d. \tag{43}$$

Then

$$\|u\|^N < \rho^N. \tag{44}$$

In addition,

$$\frac{1}{N}G(\rho^N) = \frac{1}{N}\lim_{n\to+\infty} G(\|u_n\|^N) = \Big(d + \int_B F(x,u)dx\Big), \tag{45}$$

which means that

$$\rho^N = G^{-1}\Big(\big(N(d + \int_B F(x,u)dx)\Big).$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\rho}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathfrak{W}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 3.3, we get

$$\sup_n \int_B \exp\big(Ne^{p \quad \omega_{N-1}^{\frac{1}{N-1}} |v_n|^{N'}}\big)dx < \infty,$$

for

$$1 < p < (1 - \|v\|^N)^{-\frac{1}{N-1}}.$$

From (40), (45) and the following equality

$$Nd - N\mathcal{E}(u) = G(\rho^N) - G(\|u\|^N),$$

we get

$$G(\rho^N) \leq Nd + G(\|u\|^N) < G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big) + G(\|u\|^N).$$

Now, using the condition (G_1) one has

$$\rho^N < G^{-1}\Big(G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big) + G(\|u\|^N)\Big) \leq \frac{\omega_{N-1}}{\alpha_0} + \|u\|^N. \tag{46}$$

Since

$$\rho^{N'} = \Big(\frac{\rho^N - \|u\|^N}{1 - \|v\|^N}\Big)^{\frac{1}{N-1}},$$

we deduce from (46) that

$$\rho^{N'} < \Big(\frac{\frac{\omega_{N-1}}{\alpha_0^{N-1}}}{1 - \|v\|^N}\Big)^{\frac{1}{N-1}}. \tag{47}$$

On one hand, we have this estimate $\int_B |f(x,u_n)|^q dx < C$. Indeed, for $\epsilon > 0$, we get

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\le t_\epsilon} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q dx\\ &\le \omega_{N-1} \max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B \exp\big(Ne^{\alpha_0(1+\epsilon)|u_n|^{N'}}\big)dx\\ &\le C_\epsilon + C\int_B \exp\big(Ne^{\alpha_0(1+\epsilon)\|u_n\|^{N'}|v_n|^{N'}}\big)dx \le C\end{aligned}$$

if we have $\alpha_0(1+\epsilon)\|u_n\|^{N'} \le p\omega_{N-1}^{\frac{1}{N-1}}$, with $1<p<(1-\|v\|^N)^{\frac{-1}{N-1}}$. From (47), there exists $\delta \in (0,\frac{1}{2})$ such that

$$\rho^{N'} = (1-2\delta)\Big(\frac{\rho^N - \|u\|^N}{1-\|v\|^N}\Big)^{\frac{1}{N-1}}.$$

Since $\lim_{n\to+\infty}\|u_n\|^{N'} = \rho^{N'}$, then, for n large enough

$$\alpha_0(1+\epsilon)\|u_n\|^{N'} \le (1+\epsilon)(1-\delta)\ \omega_{N-1}^{\frac{1}{N-1}}\Big(\frac{1}{1-\|v\|^N}\Big)^{\frac{1}{N-1}}.$$

We choose $\epsilon>0$ small enough such that $(1+\epsilon)(1-\delta)<1$ which means

$$\alpha_0(1+\epsilon)\|u_n\|^{N'} < \omega_{N-1}^{\frac{1}{N-1}}\Big(\frac{1}{1-\|v\|^N}\Big)^{\frac{1}{N-1}}$$

and so, the sequence $(f(x,u_n))$ is bounded in L^q, $q>1$. Using the Hölder inequality, we deduce that

$$\begin{aligned}\Big|\int_B f(x,u_n)(u_n-u)dx\Big| &\le \Big(\int_B |f(x,u_n)|^q dx\Big)^{\frac{1}{q}}\Big(\int_B |u_n-u|^{q'}\Big)^{\frac{1}{q'}}dx\\ &\le C\Big(\int_B |u_n-u|^{q'}\Big)^{\frac{1}{q'}}dx \to 0 \ \text{ as } \ n\to+\infty\end{aligned} \tag{48}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Since $\mathcal{E}'(u_n)(u_n-u)=o_n(1)$, it follows that

$$g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n-\nabla u)+\xi(x)|u_n|^{N-2}u_n(u_n-u)dx) \to 0.$$

On the other side,

$$\begin{aligned}&g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.(\nabla u_n-\nabla u)+\xi(x)|u_n|^{N-2}u_n(u_n-u)dx)\\ &= g(\|u_n\|^N)\|u_n\|^N - g(\|u_n\|^N)\int_B (\sigma(x)|\nabla u_n|^{N-2}\nabla u_n.\nabla u+\xi(x)|u_n|^{N-2}u_n u)dx.\end{aligned}$$

Passing to the limit in the last equality, using the result of Claim 1 and (48), we get

$$g(\rho^N)\rho^N - g(\rho^N)\|u\|^N = 0.$$

Therefore, $\|u\| = \rho$ and $\|u_n\| \to \|u\|$. This is in contradiction with (44). It follows that $G(\rho^N) = G(\|u\|^N)$ and consequently, $\mathcal{E}(u) = d$. Therefore, again by Brezis–Lieb's Lemma $u_n \to u$ in $\mathfrak{W}$. Also,

$$g(\|u\|^N)\int_B (\sigma(x)|\nabla u|^{N-2}\nabla u\nabla\varphi+\xi(x)|u|^{N-2}u\varphi)dx = \int_B f(x,u)\varphi dx,\ \ \forall\varphi\in\mathfrak{W}.$$

So u is a solution of the problem (1).

(i) In the subcritical case, the Palais–Smale condition is satisfied for all level $d\in\mathbb{R}$. Indeed, up to subsequences, we can assume that

$$\begin{cases} \|u_n\| \le M & \text{in } \mathfrak{W} \\ u_n \rightharpoonup u & \text{weakly in } \mathfrak{W} \\ u_n \to u & \text{strongly in} L^q(B)\ \ \forall q\ge 1 \\ u_n(x)\to u(x) & \text{almost everywhere in } B. \end{cases}$$

Since f is subcritical at $+\infty$, there exists a constant $C_M>0$ such that

$$f(x,s) \le C_M \exp\{e^{\frac{w_{N-1}^{\frac{1}{N-1}}}{M^{\frac{N}{N-1}}}s^{\frac{N}{N-1}}}\},\ \ \forall (x,s)\in B\times(0,+\infty).$$

Using the Hölder inequality

$$\begin{aligned} \Big|\int_B f(x,u_n)(u_n-u)dx\Big| &\le \int_B |f(x,u_n)(u_n-u)|dx \\ &\le \Big(\int_B |f(x,u_n)|^2dx\Big)^{\frac12}\Big(\int_B |u_n-u|^2dx\Big)^{\frac12} \\ &\le C\Big(\int_B \exp\{2e^{\frac{w_{N-1}^{\frac{1}{N-1}}}{M^{\frac{N}{N-1}}}u_n^{\frac{N}{N-1}}}\}dx\Big)^{\frac12}\|u_n-u\|_2 \\ &\le C\Big(\int_B \exp\{2e^{\frac{w_{N-1}^{\frac{1}{N-1}}}{M^{\frac{N}{N-1}}}\|u_n\|^{\frac{N}{N-1}}\frac{|u_n|^{\frac{N}{N-1}}}{\|u_n\|^{\frac{N}{N-1}}}}\}dx\Big)^{\frac12}\|u_n-u\|_2 \\ &\le C\|u_n-u\|_2 \to 0 \ \text{ as }\ n\to+\infty. \end{aligned}$$

It is easy to check, $\mathcal{J}(u)=d$. As a consequence, $\|u_n\|\to\|u\|$ and $\mathcal{E}(u)=d$. Also, u is a solution of (1). This completes the proof of the Proposition 6.3. □

Proof of Theorem 1.3. Since $f(x,t)$ satisfies the condition (11) for all $\alpha_0>0$, then by Proposition 6.3, the functional $\mathcal{E}$ satisfies the (PS) condition (at each possible level d). So, by Lemma 4.3 and Lemma 4.4, we deduce that the functional $\mathcal{E}$ has a nonzero critical point u in $\mathfrak{W}$. □

Proof of Theorem 1.4. In the critical case, again using Proposition 6.3, Lemma 4.3 and Lemma 4.4, the energy $\mathcal{E}$ satisfies the $(PS)_d$ condition for all

$$d < \frac{1}{N} G\Big(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\Big).$$

Therefore, $\mathcal{E}$ has a nonzero critical point u in $\mathfrak{W}$. $\square$

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