INEQUALITIES FOR DERIVATIVES AT FIXED POINTS OF CERTAIN ANALYTIC FUNCTIONS ON THE UNIT DISK

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Any analytic self-map f of the open unit disk $\mathbb{D}$ has a distinguished (so-called Denjoy–Wolff) fixed point $\zeta_0 \in \overline{\mathbb{D}}$ at which $|f'(\zeta_0)| \leq 1$. Any other fixed point ζ of f (if exists) is unimodular and $f'(\zeta) > 1$. C. Cowen and Ch. Pommerenke (1982) established sharp estimates for the series $\sum \frac{1}{f'(\zeta)-1}$ (over all "non-distinguished" fixed points) in terms of $f(\zeta_0)$ and $f'(\zeta_0)$ for the elliptic ($|\zeta_0| < 1$) and hyperbolic ($|\zeta_0| = 1$ and $f'(\zeta_0) < 1$) cases. The parabolic case ($|\zeta_0| = 1$ and $f'(\zeta_0) < 1$) was settled by the authors and M. Elin. In this paper, we propose a somewhat different approach that allows to cover all three cases in a unified way. The same approach also applies to a greater class of holomorphic pseudo-contractions and a closely related class of infinitesimal generators, for which similar inequalities for derivatives at fixed points come up with no extra efforts.

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1. INTRODUCTION

We start with the class $\mathcal{S}$ of functions f analytic on the open unit disk $\mathbb{D}$ and such that $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| \leq 1$. This class (the closed unit ball of the Hardy space H^∞) is commonly known as the *Schur class*; by the maximum modulus principle, any $f \in \mathcal{S}$ is either an analytic self-map of $\mathbb{D}$ or a unimodular constant. Adopting the notation

$$f(t) := \lim_{r\to 1^-} f(rt) \quad \text{and} \quad f'(t) := \lim_{r\to 1^-} f'(rt) \quad (|t| = 1) \tag{1.1}$$

for radial boundary limits (whenever they exist), we say that $\zeta \in \overline{\mathbb{D}}$ is *a fixed point* of f if $f(\zeta) = \zeta$. By the Julia–Carathéodory theorem, the boundary limit $f'(\zeta) \geq 0$ exists (finite of infinite) at any boundary fixed point of $f \in \mathcal{S}$. Furthermore, $f'(\zeta) > 0$, unless f is a unimodular constant. Throughout the

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paper, we deal with radial boundary limits, although they can be equivalently replaced by the nontangential ones. By "id" we mean the function $f(z) = z$.

By the Schwarz–Pick theorem, $f \in \mathcal{S}\backslash\{\text{id}\}$ has at most one interior fixed point $z_0 \in \mathbb{D}$ and then necessarily, $|f'(z_0)| \le 1$. In this case, f is called an *elliptic* self-map of $\mathbb{D}$. If f has no fixed point in $\mathbb{D}$, then it has a unique boundary fixed point $t_0 \in \mathbb{T} = \partial\mathbb{D}$ with $f'(t_0) \le 1$, by the Denjoy–Wolff theorem [9, 19]. In this case, f is called *hyperbolic* if $f'(t_0) < 1$ or *parabolic* if $f'(t_0) = 1$.

Thus, any function $f \in \mathcal{S}\backslash\{\text{id}\}$ has a unique fixed point ζ_0 in $\overline{\mathbb{D}}$ such that $|f'(\zeta_0)| \le 1$ (the *Denjoy–Wolff point* of f). Besides, it may have other fixed points $\zeta \in \mathbb{T}$ with $f'(\zeta) > 1$. Although the set of these other fixed points can be uncountable (by the Rudin–Carleson interpolation theorem [17, 6], given any closed subset E of Lebesgue measure zero on the the unit circle $\mathbb{T}$, there is a disk-algebra function f such that $f(z) = z$ for all $z \in \mathbb{T}$ and $|f(z)| < 1$ for all $z \in \mathbb{T}\backslash E$; see also [8, Example 2.2]), there are at most countably many of them with finite boundary derivative, and hence we may consider the positive series

$$\sum_{\zeta\in K_f} \frac{1}{f'(\zeta)-1}, \qquad K_f := \big\{\zeta \in \mathbb{T}:\ f(\zeta) = \zeta \ \text{ and } \ 1 < f'(\zeta) < \infty\big\}. \tag{1.2}$$

This series converges if f is elliptic or hyperbolic. Sharp upper bounds for (1.2) in terms of $f'(\zeta_0)$ (the derivative of f at the Denjoy–Wolff point of f) were established by Cowen and Pommerenke in [8].

THEOREM 1.1 ([8]). *Let ζ_0 be the Denjoy–Wolff point of a function $f \in \mathcal{S}$ with $f'(\zeta_0) \ne 1$ and let $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ be other fixed points of f. Then*

$$\sum_{i=1}^{n} \frac{1}{f'(\zeta_i)-1} \le \begin{cases} \frac{1-|f'(\zeta_0)|^2}{|1-f'(\zeta_0)|^2} & \text{if } \zeta_0 \in \mathbb{D}, \\ \frac{f'(\zeta_0)}{1-f'(\zeta_0)} & \text{if } \zeta_0 \in \mathbb{T}. \end{cases} \tag{1.3}$$

Moreover, equality holds if and only if f is a finite Blaschke product: of degree $n+1$ if $\zeta_0 \in \mathbb{D}$ or of degree n if $\zeta_0 \in \mathbb{T}$.

The elliptic case $f'(\zeta_0) = 1$ was handled in [2]: it was shown that under the assumption that the boundary derivative $f''(\zeta_0) \ne 0$ exists and $\Re(\zeta_0 f''(\zeta_0)) = 0$, then $f'''(\zeta_0)$ exists (finitely or infinitely), and the series in equation (1.3) is dominated by $\frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2} - 1$ with equality if and only if f is a finite Blaschke product of degree $n+1$. In this paper, we recapture these results in a more unified way, which admits an almost immediate extension to the class of holomorphic pseudo-contractions and a closely related class of infinitesimal generators. We explain why the sufficient conditions imposed on f in the elliptic case are essentially necessary and we consider the case where $n = \infty$ in some detail.

2. BACKGROUND: THE JULIA–CARATHÉODORY THEOREM

A closely related (via Cayley's transform) to the Schur class $\mathcal{S}$ is the Carathéodory class $\mathcal{C}$ of functions analytic and with positive real part in $\mathbb{D}$. Any function $p \in \mathcal{C}$ admits the Riesz–Herglotz representation [16, 13]

$$p(z) = i\Im\big(p(0)\big) + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) = -\overline{p(0)} + \int_{\mathbb{T}} \frac{2\zeta d\mu(\zeta)}{\zeta - z} \tag{2.1}$$

with a positive Borel measure μ on $\mathbb{T}$, which in turn is recovered from p by the Stieltjes inversion formula. In particular,

$$2\mu\big(\{t_0\}\big) = \lim_{r\to 1}(1-r)p(rt_0) \quad \text{for any } t_0 \in \mathbb{T}. \tag{2.2}$$

We recall the classical Julia–Carathéodory theorem which establishes connections between the boundary behavior of p, p' and of the quantity

$$\Delta_p(z) = \frac{p(z) + \overline{p(z)}}{2(1-|z|^2)} = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2}. \tag{2.3}$$

THEOREM 2.1. *For $p \in \mathcal{C}$ of the form* (2.1) *and $t_0 \in \mathbb{T}$, the following are equivalent:*

(1) $\Delta = \liminf_{z\to t_0} \Delta_p(z) < \infty$ *($z \to t_0$ unrestrictedly in $\mathbb{D}$).*

(2) $\Delta_p(t_0) := \lim_{r\to 1} \Delta_p(rt_0) < \infty$.

(2.4) (3) $\displaystyle\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - t_0|^2} < \infty.$

(2.5) (4) *The limits $p(t_0)$, $p'(t_0)$ exist and $\Re(p(t_0)) = 0$.*

Moreover, if these conditions hold, then

$$\Delta = \Delta_p(t_0) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - t_0|^2} = -\frac{t_0 p'(t_0)}{2}. \tag{2.6}$$

Finally, if $p'(t_0) = 0$ then p is an imaginary constant.

Proof. The implication (1) $\Rightarrow$ (3) follows by Fatou's lemma:

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - t_0|^2} \le \liminf_{z\to t_0} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2} = \liminf_{z\to t_0} \Delta_p(z) < \infty.$$

The implication (3) $\Rightarrow$ (2) follows by Lebesgue's dominated convergence theorem:

$$\lim_{r\to 1} \Delta_p(rt_0) = \lim_{r\to 1} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - rt_0|^2} = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - t_0|^2} = \Delta_p(t_0),$$

which applies since $|\zeta - t_0| \le 2|\zeta - rt_0|$. Since the implication (2) $\Rightarrow$ (1) is trivial, we already have the equivalence of (1), (2), (3) and the equality $\Delta = \Delta_p(t_0)$.

To verify (3) ⇒ (4), let us assume that condition (2.4) is in force. Then

$$p(t_0) := \lim_{r\to 1} p(rt_0) = -\overline{p(0)} + \lim_{r\to 1}\int_{\mathbb{T}} \frac{2\zeta d\mu(\zeta)}{\zeta - rt_0} = -\overline{p(0)} + \int_{\mathbb{T}} \frac{2\zeta d\mu(\zeta)}{\zeta - t_0}, \tag{2.7}$$

$$p'(t_0) := \lim_{r\to 1} p'(rt_0) = \lim_{r\to 1}\int_{\mathbb{T}} \frac{2\zeta d\mu(\zeta)}{(\zeta - rt_0)^2} = \int_{\mathbb{T}} \frac{2\zeta d\mu(\zeta)}{(\zeta - t_0)^2}, \tag{2.8}$$

where the leftmost equalities hold due to (2.1), while the rightmost are justified by Lebesgue's dominated convergence theorem. Since $\Re(\frac{\zeta}{\zeta - t_0}) = 1$, it is immediate from (2.7) that $\Re p(t_0) = 0$. In addition, we see from (2.8) that

$$t_0 p'(t_0) = \int_{\mathbb{T}} \frac{2\zeta t_0 d\mu(\zeta)}{(\zeta - t_0)^2} = -\int_{\mathbb{T}} \frac{2d\mu(\zeta)}{|\zeta - t_0|^2} = -2\Delta_p(t_0) < 0. \tag{2.9}$$

Assuming (4), we use the boundary asymptotic expansion

$$p(rt_0) = p(t_0) + (rt_0 - t_0)p'(t_0) + o(1-r)$$

to compute the similar one for $\Delta_p(rt_0)$:

$$\Delta_p(rt_0) = \frac{\Re(p(t_0) - (1-r)\Re(t_0 p'(t_0)) + o(1-r)}{1-r^2} = -\frac{t_0 p'(t_0)}{1+r} + o(1),$$

which implies (1). Finally, it follows from (2.9) that $p'(t_0) = 0$ implies $d\mu = 0$ and hence $p(z) \equiv i\Im(p(0))$, by (2.1). □

Via the Cayley transform accompanied by straightforward manipulations, Theorem 2.1 translates to its original Schur-class version [14, 5].

Theorem 2.2. *For $s \in \mathcal{S}$ and $t_0 \in \mathbb{T}$, the following are equivalent:*

$$\text{(1)} \quad d = \liminf_{z\to t_0} d_s(z) < \infty, \quad \textit{where } d_s(z) := \frac{1 - |s(z)|^2}{1 - |z|^2}. \tag{2.10}$$

$$\text{(2)} \quad d_s(t_0) := \lim_{r\to 1} d_s(rt_0) < \infty.$$

$$\text{(3)} \quad \textit{The limits } s(t_0),\ s'(t_0) \textit{ exist and } |s(t_0)| = 1. \tag{2.11}$$

Moreover, if these conditions hold, then

$$d = d_s(t_0) = t_0 s'(t_0)\overline{s(t_0)} = |s'(t_0)|. \tag{2.12}$$

Finally, if $s'(t_0) = 0$, then s is a unimodular constant.

3. FIXED POINTS OF SCHUR-CLASS FUNCTIONS

Our presentation relies on a linear fractional representation of $f \in \mathcal{S}$ with a given Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$, that is, on a single-point Schur-class interpolation result. The interior and the boundary cases were considered separately by Schur [18] and Julia [14]; curiously, in the fixed-point case, both cases are settled by the same formula.

THEOREM 3.1. *A point $\zeta_0 \in \overline{\mathbb{D}}$ is the Denjoy–Wolff point of a function $f \in \mathcal{S}\backslash\{\mathrm{id}\}$ if and only if f admits a representation*

$$f(z) = z - \frac{(z-\zeta_0)(1-z\overline{\zeta}_0)}{p(z) + (\frac{1+|\zeta_0|^2}{2} - z\overline{\zeta}_0)} \quad \text{for some } p \in \mathcal{C}. \tag{3.1}$$

If $|\zeta_0| = 1$, then (3.1) simplifies to

$$f(z) = z + \frac{(z-\zeta_0)^2}{\zeta_0 p(z) - (z-\zeta_0)}, \qquad p \in \mathcal{C}. \tag{3.2}$$

Furthermore, f of the form (3.2) is hyperbolic (i.e., $f'(\zeta_0) < 1$) if

$$p(\zeta_0) = 0 \quad \text{and} \quad |p'(\zeta_0)| < \infty, \tag{3.3}$$

and is parabolic ($f'(\zeta_0) = 1$) otherwise.

Proof. It is verified by straightforward computations that f of the form (3.1) belongs to $\mathcal{S}$ and satisfies $f(\zeta_0) = \zeta_0$. The "only if" part is verified separately for the interior and the boundary cases as follows.

If $|\zeta_0| < 1$ and $f(\zeta_0) = \zeta_0$, then the function

$$\mathcal{E}(z) = \frac{f(z)-\zeta_0}{1-f(z)\overline{\zeta}_0} \cdot \frac{1-z\overline{\zeta}_0}{z-\zeta_0} \tag{3.4}$$

belongs to the Schur class $\mathcal{S}$, by the Schwarz–Pick lemma. Therefore, the Cayley transform of $\mathcal{E}$,

$$p(z) = \frac{1-|\zeta_0|^2}{2} \cdot \frac{1+\mathcal{E}(z)}{1-\mathcal{E}(z)} = \frac{(z-\zeta_0)(1-z\overline{\zeta}_0)}{z-f(z)} - \frac{1+|\zeta_0|^2}{2} + z\overline{\zeta}_0$$

belongs to the Carathéodory class $\mathcal{C}$. Solving the latter equation for f gives equation (3.1). Note that by (3.4), $f'(\zeta_0) = \mathcal{E}(\zeta_0)$ and hence, $|f'(\zeta_0)| \le 1$ (with equality if and only if f is an automorphism of $\mathbb{D}$). Furthermore, $f'(\zeta_0) \neq 1$, unless $f(z) = \mathrm{id}(z) \equiv z$.

If $|\zeta_0| = 1$ and $f(\zeta_0) = \zeta_0$, we write (3.2) (or equivalently, (3.1)) as

$$\frac{\zeta_0 + f(z)}{\zeta_0 - f(z)} - \frac{\zeta_0 + z}{\zeta_0 - z} = \frac{2}{p(z)}. \tag{3.5}$$

By (3.4), the Herglotz measure of the Carathéodory-class function $\frac{\zeta_0+f(z)}{\zeta_0-f(z)}$ has an atom $\frac{1}{f'(\zeta_0)}$ at ζ_0, and therefore, the function on the left side of (3.5) still belongs to $\mathcal{C}$ if and only if $f'(\zeta_0) \le 1$ and its Herglotz measure has an atom at ζ_0 if and only if $f'(\zeta_0) < 1$. Passing to the reciprocals in (3.5) shows that $p(z)$ belongs to $\mathcal{C}$ if and only if $f'(\zeta_0) \le 1$ and that $p(z)$ satisfies conditions (3.3) if and only if $f'(\zeta_0) < 1$. $\square$

Remark 3.2. Let f be of the form (3.1) and let μ be the Herglotz measure of p. Then for any $\zeta \in \mathbb{T}\backslash\{\zeta_0\}$,

$$f(\zeta) = \zeta \quad \text{if and only if} \quad p(\zeta) = \infty. \tag{3.6}$$

Furthermore, for each fixed point $\zeta \in \mathbb{T}\backslash\{\zeta_0\}$ of f,

$$f'(\zeta) - 1 = \frac{|\zeta - \zeta_0|^2}{2\mu(\{\zeta\})}. \tag{3.7}$$

The equivalence (3.6) is clear from (3.1). Combining (3.1) and (3.4) we have

$$\begin{aligned}
f'(\zeta) = \lim_{r\to 1} \frac{f(r\zeta) - f(\zeta)}{r\zeta - \zeta} &= \lim_{r\to 1} \frac{r\zeta - \dfrac{(r\zeta - \zeta_0)(1 - r\zeta\overline{\zeta}_0)}{p(r\zeta) + (\frac{1+|\zeta_0|^2}{2} - r\zeta\overline{\zeta}_0)} - \zeta}{(r-1)\zeta} \\
&= 1 + \lim_{r\to 1} \frac{(r\zeta - \zeta_0)(1 - r\zeta\overline{\zeta}_0)}{\zeta(1-r)p(r\zeta) + (1-r)\zeta(\frac{1+|\zeta_0|^2}{2} - r\zeta\overline{\zeta}_0)} \\
&= 1 + \frac{(\zeta - \zeta_0)(1 - \zeta\overline{\zeta}_0)}{\zeta \lim_{r\to 1}(1-r)p(r\zeta)} = \frac{|\zeta - \zeta_0|^2|}{2\mu(\{\zeta\})}.
\end{aligned}$$

In fact, the equivalence (3.6) holds true for any point $\zeta \in \overline{\mathbb{D}}\backslash\{\zeta_0\}$, but the interior case $|\zeta| < 1$ is irrelevant unless $p \equiv \infty$ (i.e., $f = \text{id}$). Excluding this exceptional case, we see that f of the form (3.2) indeed has no interior fixed points and that ζ_0 is a unique interior fixed point for f of the form (3.1). Furthermore, if $p \not\equiv \infty$, then $f'(\zeta) > 1$ for every fixed point of f other than ζ_0.

Remark 3.3. By (3.7), the finite measure μ has atom at any point from the set K_f defined in (1.2), and besides, μ may have atom at ζ_0. Therefore, the set K_f is at most countable.

3.1. **Cowen–Pommerenke inequalities**

We now get to inequalities (3.8).

THEOREM 3.4. *Let $f \neq \text{id}$ be a Schur-class function with the Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$ and the set K_f (1.2) of all other regular boundary fixed points.*

1. *If $|\zeta_0| < 1$, then*

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \leq \frac{1 - |f'(\zeta_0)|^2}{|1 - f'(\zeta_0)|^2}. \tag{3.8}$$

2. *If* $|\zeta_0| = 1$ *and* $f'(\zeta_0) < 1$, *then*

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \le \frac{f'(\zeta_0)}{1 - f'(\zeta_0)}. \tag{3.9}$$

3. *If* $|\zeta_0| = f'(\zeta_0) = 1$, $\Re(\zeta_0 f''(\zeta_0)) = 0$ *and* $f''(\zeta_0) \neq 0$, *then*

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \le \frac{2 f'''(\zeta_0)}{3 f''(\zeta_0)^2} - 1. \tag{3.10}$$

4. *If* $|\zeta_0| = f'(\zeta_0) = 1$, $f''(\zeta_0) = 0$, *and* $\Re(\zeta_0^3 f^{(4)}(\zeta_0)) = 2|f'''(\zeta_0)| < \infty$, *then*

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \le \frac{3 f^{(5)}(\zeta_0)}{10 f'''(\zeta_0)^2} - \frac{3 f^{(4)}(\zeta_0)^2}{8 f'''(\zeta_0)^3} - 1. \tag{3.11}$$

Furthermore, equality holds in each case if and only if the Herglotz measure of the parameter $p \in \mathcal{C}$ *in the representation formula* (3.1) *of* f *is discrete.*

Proof. By Theorem 3.1, f can be represented in the form (3.1). Let μ be the Herglotz measure of the parameter $p \in \mathcal{C}$ in the representation formula (3.1). If $\mu(\{\zeta_0\}) = 0$, then due to (3.7), we have

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} = \sum_{\zeta \in K_f} \frac{2\mu(\{\zeta\})}{|\zeta - \zeta_0|^2} = \int_{\mathbb{T}} \frac{2 d\mu_d(\zeta)}{|\zeta - \zeta_0|^2} \le \int_{\mathbb{T}} \frac{2 d\mu(\zeta)}{|\zeta - \zeta_0|^2}, \tag{3.12}$$

where μ_d is the discrete part of the measure μ. Thus, (3.12) holds with equality if and only if μ is discrete. Writing the rightmost integral in (3.12) in terms of $f^{(j)}(\zeta_0)$ settles parts (1)–(3). Details are given below.

Elliptic case. If $|\zeta_0| < 1$, then we see from (3.1) that

$$f'(\zeta_0) = 1 - \frac{1 - |\zeta_0|^2}{p(\zeta_0) + \frac{1 - |\zeta_0|^2}{2}}, \quad \text{that is,} \quad p(\zeta_0) = \frac{1 - |\zeta_0|^2}{2} \cdot \frac{1 + f'(\zeta_0)}{1 - f'(\zeta_0)}. \tag{3.13}$$

Therefore,

$$\int_{\mathbb{T}} \frac{2 d\mu(\zeta)}{|\zeta - \zeta_0|^2} = \Delta_p(z_0) = \frac{p(\zeta_0) + \overline{p(\zeta_0)}}{1 - |\zeta_0|^2} = \frac{1 - |f'(\zeta_0)|^2}{|1 - f'(\zeta_0)|^2},$$

by (2.3) and (3.13). Combining the latter equality with (3.12) implies (3.8).

Hyperbolic case. If $|\zeta_0|=1$ and $f'(\zeta_0)<1$, then $p(\zeta_0)=0$, by (3.3). In particular, $\mu(\{\zeta_0\})=0$, and relations (3.12) hold. We see from (3.2) that

$$f'(\zeta_0)=1+\frac{1}{\zeta_0 p'(\zeta_0)-1}, \quad \text{that is,} \quad \zeta_0 p'(\zeta_0)=\frac{f'(\zeta_0)}{f'(\zeta_0)-1}<0.$$

Thus, the conditions in part (4) of Theorem 2.1 are met and hence,

$$\int_{\mathbb{T}}\frac{2d\mu(\zeta)}{|\zeta-\zeta_0|^2}=\lim_{r\to 1}\frac{p(r\zeta_0)+\overline{p(r\zeta_0)}}{1-r^2}=-\zeta_0 p'(\zeta_0)=\frac{f'(\zeta_0)}{1-f'(\zeta_0)}, \tag{3.14}$$

by (2.6). Combining the latter with (3.12) leads us to (3.9).

Parabolic case. If $|\zeta_0|=1$, $f'(\zeta_0)=1$, and $\mu(\{\zeta_0\})=0$, we still have relations (3.12), which produce a meaningful estimate (in terms of ζ_0) for the series on the left side only if the integral on the right side converges. By Theorem 2.1, this is the case if and only if the boundary limits $p(\zeta_0)$ and $p'(\zeta_0)$ exist, and $\Re(p(\zeta_0))=0$. Note that since f is parabolic, $p(\zeta_0)\neq 0$, by (3.3). The existence of the latter limits is equivalent to the existence of $f''(\zeta_0)$ and $f'''(\zeta_0)$. Solving (3.2) for p and taking the radial limits gives

$$p(\zeta_0)=\frac{2}{\zeta_0 f''(\zeta_0)}, \quad \zeta_0 p'(\zeta_0)=1-\frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2}. \tag{3.15}$$

In particular, the requested condition $\Re(p(\zeta_0))=0$ translates to

$$\Re\big(\zeta_0 f''(\zeta_0)\big)=0.$$

If this condition is met, then (3.14) modifies to

$$\int_{\mathbb{T}}\frac{2d\mu(\zeta)}{|\zeta-\zeta_0|^2}=\lim_{r\to 1}\frac{p(r\zeta_0)+\overline{p(r\zeta_0)}}{1-r^2}=-\zeta_0 p'(\zeta_0)=\frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2}-1,$$

on account of (3.15). Combining the latter with (3.12) leads us to (3.10).

Special parabolic case. If $f'(\zeta_0)=1$ and $f''(\zeta_0)=0$, then it follows from (3.2) and (2.2) that

$$f'''(\zeta_0)=\lim_{r\to 1}\frac{6}{\zeta_0^2(r-1)p(r\zeta_0)-\zeta_0^2(r-1)^2}=-\frac{3}{\zeta_0^2\mu(\{\zeta_0\})}.$$

Also, if $f'''(\zeta_0)=0$, then necessarily $f=\mathrm{id}$. Hence, in the current case, the measure μ has atom at ζ_0,

$$\mu(\{\zeta_0\})=-\frac{3}{\zeta_0^2 f'''(\zeta_0)}=\frac{3}{|f'''(\zeta_0)|}.$$

If we let $\widetilde{\mu}=\mu-\mu(\{\zeta_0\})\delta_{\zeta_0}=\mu+\frac{3}{\zeta_0^2 f'''(\zeta_0)}\delta_{\zeta_0}$ and

$$\widetilde{p}(z):=p(z)+\frac{3}{\zeta_0^2 f'''(\zeta_0)}\cdot\frac{\zeta_0+z}{\zeta_0-z}=i\gamma+\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}d\widetilde{\mu}(\zeta), \tag{3.16}$$

then the formula (3.2) can be written in terms of $\widetilde{p}$ as

$$f(z) = z + \frac{(z-\zeta_0)^2}{\zeta_0\widetilde{p}(z) + \zeta_0 - z - \frac{3}{\zeta_0 f'''(\zeta_0)} \cdot \frac{\zeta_0+z}{\zeta_0-z}}, \tag{3.17}$$

while relations (3.12) modify to

$$\sum_{\zeta\in K_f} \frac{1}{f'(\zeta)-1} = \sum_{\zeta\in K_f} \frac{2\mu(\{\zeta\})}{|\zeta-\zeta_0|^2} \le \int_{\mathbb{T}} \frac{2d\widetilde{\mu}(\zeta)}{|\zeta-\zeta_0|^2} = \lim_{r\to 1} \frac{\widetilde{p}(r\zeta_0)+\overline{\widetilde{p}(r\zeta_0)}}{1-r^2}. \tag{3.18}$$

Similarly to the previous case, the inequality (3.18) leads to an estimate of the series on the left side only if the limit on the right side is finite. By Theorem 2.1, this is the case if and only if the boundary limits $\widetilde{p}(\zeta_0)$, $\widetilde{p}'(\zeta_0)$ exist and $\Re(\widetilde{p}(\zeta_0)) = 0$. The existence of the latter limits is equivalent to the existence of $f^{(4)}(\zeta_0)$, $f^{(5)}(\zeta_0)$. Solving (3.17) for $\widetilde{p}$ and taking the radial limits gives

$$\widetilde{p}(\zeta_0) = -\frac{3f^{(4)}(\zeta_0)}{2\zeta_0 f'''(\zeta_0)^2} - \frac{3}{\zeta_0^2 f'''(\zeta_0)} = -\frac{3\zeta_0^3 f^{(4)}(\zeta_0)}{2|f'''(\zeta_0)|^2} + \frac{3}{|f'''(\zeta_0)|}, \tag{3.19}$$

$$\zeta_0\widetilde{p}'(\zeta_0) = -\frac{3f^{(5)}(\zeta_0)}{10 f'''(\zeta_0)^2} + \frac{3f^{(4)}(\zeta_0)^2}{8f'''(\zeta_0)^3} + 1. \tag{3.20}$$

Due to (3.19), the condition $\Re(\widetilde{p}(\zeta_0)) = 0$ translates to

$$\Re\big(\zeta_0^3 f^{(4)}(\zeta_0)\big) = 2|f'''(\zeta_0)|.$$

If this condition is satisfied, then

$$\lim_{r\to 1} \frac{\widetilde{p}(r\zeta_0)+\overline{\widetilde{p}(r\zeta_0)}}{1-r^2} = -\zeta_0\widetilde{p}'(\zeta_0),$$

by Theorem 2.1, which being combined with (3.20), leads us to (3.11).

The last statement of the theorem is obvious, as (3.12) and (3.18) hold with equality if and only if the measure μ is discrete. □

Example 3.5. The inner function

$$f(z) = ze^{\frac{z+1}{z-1}}$$

has the Denjoy–Wolff point $\zeta_0 = 0$ and regular boundary fixed points

$$K_f = \Big\{\xi_n = \frac{2\pi i n+1}{2\pi i n-1} : n\in\mathbb{Z}\Big\}, \quad f'(\xi_n) = \frac{4\pi^2 n^2+3}{2}, \quad f'(0) = \frac{1}{e} \tag{3.21}$$

We have

$$\sum_{\zeta\in K_f} \frac{1}{f'(\zeta)-1} = \sum_{n=-\infty}^{\infty} \frac{2}{4\pi^2n^2+1} = \coth(1/2) = \frac{e+1}{e-1} = \frac{1-|f'(0)|^2}{|1-f'(0)|^2},$$

which directly verifies that (3.8) holds for this function with equality. The same conclusion follows by Theorem 3.4, since f can be represented in the form (3.1) with $\zeta_0 = 0$,

$$p(z) = \frac{1}{2} \cdot \frac{1 + e^{\frac{z+1}{z-1}}}{1 - e^{\frac{z+1}{z-1}}} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta),$$

and since, furthermore, μ is the discrete measure supported by K_f with atoms

$$\mu\big(\{\xi_n\}\big) = \frac{1}{4\pi n^2 + 1} \quad \text{for } n \in \mathbb{Z}.$$

Example 3.6. For any $\zeta_0 \in \mathbb{T}$, the (inner) function

$$f(z) = z + \frac{(z-\zeta_0)^2}{\zeta_0 p(z) - (z - \zeta_0)}, \quad \text{where } p(z) = \frac{1}{2} \cdot \frac{1 + e^{\frac{z+1}{z-1}}}{1 - e^{\frac{z+1}{z-1}}}, \tag{3.22}$$

has the Denjoy–Wolff point ζ_0 and regular boundary fixed points $\xi_n = \frac{2\pi i n+1}{2\pi i n-1}$ as in (3.21). We next spell out several specific choices of ζ_0.

(a) If $\zeta_0 = \frac{\pi i+1}{\pi i-1}$, then

$$f'(\zeta_0) = \frac{\pi^2+1}{\pi^2+5}, \quad f'(\xi_n) = 1 + \frac{\pi^2(2n-1)^2}{\pi^2+1} \quad \text{for } \ n \in \mathbb{Z}.$$

Since $f'(\zeta_0) < 1$, the point ζ_0 is hyperbolic. Furthermore,

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} = \frac{\pi^2+1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2+1}{\pi^2} \cdot \frac{\pi^2}{4} = \frac{\pi^2+1}{4} = \frac{f'(\zeta_0)}{1 - f'(\zeta_0)},$$

which confirms that (3.9) holds for f with equality.

(b) If $\zeta_0 = \frac{\pi i+2}{\pi i-2}$, then

$$f'(\zeta_0) = 1, \quad f''(\zeta_0) = -4i\frac{\pi i - 2}{\pi i + 2}, \quad f'''(\zeta_0) = -\frac{3}{2}(\pi^2 + 20) \cdot \left(\frac{\pi i - 2}{\pi i + 2}\right)^2.$$

Therefore, ζ_0 is parabolic, $\Re(\zeta_0 f''(\zeta_0)) = 0$, and

$$\frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2} - 1 = \frac{\pi^2+4}{16}.$$

On the other hand,

$$f'(\xi_n) = 1 + \frac{2\pi^2(4n-1)^2}{\pi^2+4} \quad \text{for } n \in \mathbb{Z}.$$

Then we have

$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} = \frac{\pi^2+4}{2\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(4n-1)^2} = \frac{\pi^2+4}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$= \frac{\pi^2+4}{2\pi^2} \cdot \frac{\pi^2}{8} = \frac{\pi^2+4}{16} = \frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2} - 1,$$

which confirms that (3.10) holds for f with equality.

(c) If $\zeta_0 = 1$ (the only limit point of K_f), then $f'(1) = 1$, $f''(1) = 4$, $f'''(1) = 24$. Hence, $\zeta_0 = 1$ is parabolic. Furthermore, $f'(\xi_n) = 3$ for all $n \in \mathbb{Z}$. Hence, the series on the left side of (3.11) diverges, while the expression on the right side equals zero. Therefore, inequality (3.10) does not hold in this case.

Example 3.7. The function

$$f(z) = \frac{5z+1}{6+z-z^2}$$

belongs to the Schur class and has two boundary fixed points: $\zeta_0 = 1$ (the parabolic Denjoy–Wolff point) and $\zeta_1 = -1$. Simple calculations reveal

$$f'(-1) = \frac{3}{2}, \quad f'(1) = 1, \quad f''(1) = \frac{2}{3}, \quad f'''(1) = \frac{4}{3}.$$

Therefore, $\frac{1}{f'(-1)-1} = 2$, $\frac{2f'''(1)}{3f''(1)^2} - 1 = 1$ and the inequality (3.10) does not hold.

Note that the extra condition $\Re(\zeta_0 f''(\zeta_0)) = 0$ from part (3) in Theorem 3.4 is satisfied in Example 3.6 (b) and is not met in Example 3.6 (c) and Example 3.7.

The following result was established in [8] for the elliptic and hyperbolic cases and in [2] for the parabolic case.

Proposition 3.8. *Let us assume that $f \in \mathcal{S}$ has finitely many regular fixed points: $K_f = \{\zeta_1, \ldots, \zeta_n\}$. Then* (3.8)–(3.11) *hold with equality if and only if f is a finite Blaschke product: of degree n in the hyperbolic case, of degree $n+1$ in the elliptic or parabolic case* (3.10), *and of degree $n+2$ in the special parabolic case* (3.11).

Proof. We take f in the form (3.1) and write it as

$$f(z) = \frac{zp(z) - z\frac{1+|\zeta_0|^2}{2} + \zeta_0}{p(z) + \frac{1+|\zeta_0|^2}{2} - z\overline{\zeta_0}}. \tag{3.23}$$

By (3.12), equality holds in (3.8)–(3.10) if and only if the Herglotz measure μ of $p \in \mathcal{C}$ is supported by K_f, i.e., if and only if p is a rational function

$$p(z) = \sum_{i=1}^{n} \mu\big(\{\zeta_i\}\big) \cdot \frac{\zeta_i - z}{\zeta_i + z} \tag{3.24}$$

of degree n and with real part vanishing on $\mathbb{T}\backslash K_f$, in which case, f of the form (3.1) is unimodular on $\mathbb{T}$ and therefore, f is a finite Blaschke product. Upon substituting (3.24) into (3.1), we can write f as the ratio of two polynomials of degree $n+1$. Zero cancellation cannot occur at the poles $\zeta_1,\dots,\zeta_n$ of p. To examine other points, we solve the system

$$zp(z) - z\frac{1+|\zeta_0|^2}{2} + \zeta_0 = 0, \qquad p(z) + \frac{1+|\zeta_0|^2}{2} - z\overline{\zeta}_0 = 0 \tag{3.25}$$

and conclude that the numerator and the denominator in (3.23) vanish simultaneously if and only if either

$$(1)\ z=\zeta_0 \ \text{ and } \ p(\zeta_0) = -\frac{1-|\zeta_0|^2}{2} \quad \text{or} \quad (2)\ z=\frac{1}{\overline{\zeta}_0} \ \text{ and } \ p\Big(\frac{1}{\overline{\zeta}_0}\Big) = \frac{1-|\zeta_0|^2}{2}.$$

Since $\Re(p(z))$ is nonnegative for $|z|<1$ and nonpositive for $|z|>1$, the latter is possible if and only if $|\zeta_0|=1$ and $p(\zeta_0)=0$, that is, in the hyperbolic case. Note that ζ_0 is a multiple common zero in (3.25) if and only if $p'(\zeta_0)=\overline{\zeta}_0$. But in this case, $\zeta_0 p'(\zeta_0) = |\zeta_0|^2 = 1 > 0$ which cannot happen, by (2.6). Therefore, in the hyperbolic case there is one zero cancellation in (3.23) and therefore, $\deg f = n$. Zero cancellations do not occur in the elliptic and parabolic cases and therefore, in these cases, $\deg f = n+1$.

If (3.11) holds with equality, then μ is supported by $K_f\cup\{\zeta_0\}$. Therefore, p is of the form

$$p(z) = \sum_{i=0}^{n} \mu\big(\{\zeta_i\}\big)\cdot \frac{\zeta_i - z}{\zeta_i + z}$$

and since the zero cancellation does not occur in (3.23), f is a Blaschke product of degree $n+2$. This completes the proof of the "only if" part. The "if part" is verified directly. □

We are not aware of an intrinsic characterization of $f\in\mathcal{S}$ for which one of the relations (3.8)–(3.11) holds with equality. Such f is necessarily inner since the Herglotz measure μ of the parameter $p\in\mathcal{C}$ is discrete and hence, p has imaginary boundary values almost everywhere on $\mathbb{T}$. On the other hand, if μ is a continuous singular measure, then still $\Re(p(\zeta))=0$ a.e. on $\mathbb{T}$, and hence, f of the form (3.1) is an inner function for which equalities in (3.8)–(3.11) cannot occur, by the last statement in Theorem 3.4.

Remark 3.9. Let $f\neq \mathrm{id}$ be a Schur-class function with the Denjoy–Wolff point $\zeta_0\neq 0$ and the set K_f (1.2) of all other regular boundary fixed points. Then

$$\sum_{\zeta\in K_f} \frac{|\zeta-\zeta_0|^2}{f'(\zeta)-1} \le 2\Re\Big(\frac{\zeta_0}{f(0)}\Big) - 1 - |\zeta_0|^2. \tag{3.26}$$

Proof. Since $\zeta_0 \neq 0$, we have $f(0) \neq 0$. As in the proof of Theorem 3.4, we take f in the form (3.1). Then

$$f(0) = \frac{\zeta_0}{p(0) + \frac{1+|\zeta_0|^2}{2}}, \quad \text{that is,} \quad p(0) = \frac{\zeta_0}{f(0)} - \frac{1+|\zeta_0|^2}{2}. \tag{3.27}$$

Let μ be the Herglotz measure of p. By (3.7) and similarly to (3.12),

$$\sum_{\zeta \in K_f} \frac{|\zeta - \zeta_0|^2}{f'(\zeta) - 1} = \sum_{\zeta \in K_f} 2\mu(\{\zeta\}) \leq 2 \int_{\mathbb{T}} d\mu(\zeta) = p(0) + \overline{p(0)},$$

which together with (3.27) implies (3.26). □

Inequality (3.26) was established in [8] for the boundary case $|\zeta_0| = 1$.

4. FIXED POINTS OF HOLOMORPHIC PSEUDO-CONTRACTIONS

The Schwarz–Pick lemma tells us that any Schur-class function is a contraction with respect to the hyperbolic metric on $\mathbb{D}$. Holomorphic *pseudo-contractions* were introduced in [10] (see also [11, Section 3.8]) as a specialization of the general *pseudo-contraction* (introduced in the Banach-space context by Browder in [4]; see also [15]) to the hyperbolic metric on $\mathbb{D}$. Alternatively, the class $\mathcal{PC}$ of holomorphic pseudo-contractions can be introduced in terms of Schur-class functions as follows.

Let us observe that given $f \in \mathcal{S}$, for each $t \in [0, 1)$ and any point $w \in \mathbb{D}$, the boundary (radial) limits of the function

$$G(z) = tf(z) + (1-t)w$$

are less than one in modulus. Therefore, F has no boundary fixed points and its Denjoy–Wolff point $z := J_t(w)$ belongs to $\mathbb{D}$. Note that for each $t \in [0, 1)$ the function J_t belongs to $\mathcal{S}$ as a function of w and that the curve $J_t(w)$ converges to the Denjoy–Wolff point of f (as $t \to 1$) for each fixed $w \in \mathbb{D}$. Dropping the assumption that f is bounded in $\mathbb{D}$, we arrive to the alternative definition of a holomorphic pseudo-contraction.

Definition 4.1. A function f analytic on $\mathbb{D}$ is called *pseudo-contractive* if for each $t \in [0, 1)$ and each $w \in \mathbb{D}$, the equation

$$z = tf(z) + (1-t)w \tag{4.1}$$

has a unique solution $z = J_t(w) \in \mathbb{D}$ which is analytic in w.

Similarly to the Schur-class case, given an $f \in \mathcal{PC}$, the curve $J_t(w)$ converges to the same fixed point $\zeta_0 \in \overline{\mathbb{D}}$ of f for each $w \in \mathbb{D}$ (see [1, Theorem 3.3]) which suggests to call ζ_0 the *Denjoy–Wolff point* of f. Another result from [1] is parallel to Theorem 3.1, although the proof is quite different.

THEOREM 4.2. *A point $\zeta_0 \in \overline{\mathbb{D}}$ is the Denjoy–Wolff point of a function $f \in \mathcal{PC}\backslash\{\mathrm{id}\}$ if and only if f admits a representation*

$$f(z) = z - \frac{(z-\zeta_0)(1-z\overline{\zeta}_0)}{p(z)} \quad \textit{for some} \quad p \in \mathcal{C}. \tag{4.2}$$

Immediate consequences of the representation formula (4.2) are listed below.

• If $|\zeta_0| < 1$, then $f'(\zeta_0) = 1 - \frac{1-|\zeta_0|^2}{p(\zeta_0)}$ and hence, $\Re(f'(\zeta_0)) < 1$.

• If $|\zeta_0| = 1$, then $f'(\zeta_0) = 1$ unless p satisfies conditions (3.3) in which case,

$$f'(\zeta_0) = 1 + \frac{1}{\zeta_0 p'(\zeta_0)} = 1 - \frac{1}{|p'(\zeta_0)|} < 1.$$

As in the Schur-class case, $f'(\zeta_0)$ is a real number, but now it does not have to be positive.

• A point $\zeta \in \overline{\mathbb{D}}$ is a fixed point of $f \in \mathcal{PC}$ of the form (4.2) if and only if $p(\zeta) = \infty$. Therefore, f has no fixed points in $\mathbb{D}$ other than ζ_0.

• If $\zeta \in \mathbb{T}\backslash\{\zeta_0\}$ is a fixed point of $f \in \mathcal{PC}$ of the form (4.2) and μ is the Herglotz measure of p, then the boundary derivative $f'(\zeta)$ exists and

$$f'(\zeta) - 1 = \frac{|\zeta-\zeta_0|^2}{2\mu(\{\zeta\})} \tag{4.3}$$

(the proof is the same as in the Schur-class case, see Remark 3.2). Hence, the Denjoy–Wolff point of $f \in \mathcal{PC}$ can be defined as a unique fixed point ζ_0 of f such that $f'(\zeta_0) \not> 1$.

Due to (4.3), relations (3.12) hold for any $f \in \mathcal{PC}$ which leads to inequalities for the series (1.2) in the present setting with the sharp upper bound

$$\lim_{r\to 1} \frac{p(r\zeta_0) + \overline{p(r\zeta_0)}}{1 - r^2|\zeta_0|^2}.$$

The only difference with the Schur-class case is that in the course of evaluating the latter limit in terms of the original f, we use the formula (4.2) rather than equation (3.1). As a result, each right-hand side expression in (3.8)–(3.11) gets the extra term

$$\lim_{r\to 1} \frac{2\Re(\frac{1+|\zeta_0|^2}{2} - r|\zeta_0|^2)}{1-r^2|\zeta_0|^2} = 1 + \lim_{r\to 1} \frac{|\zeta_0|^2(1-r)^2}{1-r^2|\zeta_0|^2} = 1.$$

We arrive at the following result established in [1].

THEOREM 4.3. *Let $f \neq \mathrm{id}$ be a Schur-class function with the Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$ and the set K_f (1.2) of all other regular boundary fixed points.*

1. *If $|\zeta_0| < 1$, then*
$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \leq \frac{1 - |f'(\zeta_0)|^2}{|1 - f'(\zeta_0)|^2} + 1.$$

2. *If $|\zeta_0| = 1$ and $f'(\zeta_0) < 1$, then*
$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \leq \frac{1}{1 - f'(\zeta_0)}$$

3. *If $|\zeta_0| = f'(\zeta_0) = 1$, $\Re(\zeta_0 f''(\zeta_0)) = 0$ and $f''(\zeta_0) \neq 0$, then*
$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \leq \frac{2f'''(\zeta_0)}{3f''(\zeta_0)^2}.$$

4. *If $|\zeta_0| = f'(\zeta_0) = 1$, $f''(\zeta_0) = 0$, and $\Re(\zeta_0^3 f^{(4)}(\zeta_0)) = 2|f'''(\zeta_0)| < \infty$, then*
$$\sum_{\zeta \in K_f} \frac{1}{f'(\zeta) - 1} \leq \frac{3f^{(5)}(\zeta_0)}{10f'''(\zeta_0)^2} - \frac{3f^{(4)}(\zeta_0)^2}{8f'''(\zeta_0)^3}.$$

Furthermore, equality holds in each case if and only if the Herglotz measure of the parameter $p \in \mathcal{C}$ in the representation formula (4.2) of f is discrete.

5. FIXED POINTS OF INFINITESIMAL GENERATORS

Let us consider a one-parameter semigroup $(\phi_t)_{t\geq 0} \subset \mathcal{S}$, i.e., a collection of Schur-class functions $\phi_t(z)$ such that $\phi_0 = \mathrm{id}$,
$$\phi_s \circ \phi_t = \phi_{s+t} \text{ for all } s, t \geq 0 \quad \text{and} \quad \lim_{t\to 0} \phi_t(z) = z \text{ for all } z \in \mathbb{D}.$$
If a $(\phi_t)_{t\geq 0}$ is non-trivial in the sense that $\phi_t \neq \mathrm{id}$ for some $t > 0$, then all elements ϕ_t $(t > 0)$ have the same Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$. Furthermore, the function $t \to \phi_t(z)$ is differentiable for any $z \in \mathbb{D}$, and there exists a unique function $G(z)$ analytic on $\mathbb{D}$ such that
$$\frac{d\phi_t(z)}{dt} = G\big(\phi_t(z)\big) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

This function G is called the *infinitesimal generator* of the semigroup $(\phi_t)_{t\geq 0}$, and the point ζ_0 is referred to as to the Denjoy–Wolff point of G. A result due to Berkson and Porta [3] states that any infinitesimal generator with the Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$ admits the representation

$$G(z) = \frac{(z-\zeta_0)(1-z\overline{\zeta}_0)}{p(z)} \quad \text{for some } p \in \mathcal{C}. \tag{5.1}$$

Comparing (5.1) and (4.2), we see that G is an infinitesimal generator if and only if $f(z) := z - G(z)$ is a holomorphic pseudo-contraction. Therefore, we have

- If $|\zeta_0| < 1$, then $G'(\zeta_0) = \frac{1-|\zeta_0|^2}{p(\zeta_0)}$ and hence, $\Re(G'(\zeta_0)) > 0$.
- If $|\zeta_0| = 1$, then $G'(\zeta_0) = 0$ unless p satisfies conditions (3.3) in which case,

$$G'(\zeta_0) = -\frac{1}{\zeta_0 p'(\zeta_0)} = \frac{1}{|p'(\zeta_0)|} > 0.$$

- A point $\zeta \in \overline{\mathbb{D}}$ is a fixed point of G of the form (4.2) if and only if $p(\zeta) = \infty$. Therefore, G has no fixed points in $\mathbb{D}$ other than ζ_0.
- If $\zeta \in \mathbb{T}\backslash\{\zeta_0\}$ is a fixed point of G of the form (4.2) and μ is the Herglotz measure of p, then the boundary derivative $G'(\zeta)$ exists and

$$G'(\zeta) = -\frac{|\zeta-\zeta_0|^2}{2\mu(\{\zeta\})} < 0. \tag{5.2}$$

By letting $f(z) = z - G(z)$ in Theorem 4.2, we arrive at the following result.

THEOREM 5.1. *Let $G \not\equiv 0$ be an infinitesimal generator with the Denjoy–Wolff point $\zeta_0 \in \overline{\mathbb{D}}$ and let K_G be the set of all other regular boundary fixed points of G.*

1. *If $|\zeta_0| < 1$, then*

$$\sum_{\zeta\in K_G} \frac{1}{G'(\zeta)} \geq -\frac{2\Re(G'(\zeta_0))}{|G'(\zeta_0|^2}$$

2. *If $|\zeta_0| = 1$ and $G'(\zeta_0) > 0$, then*

$$\sum_{\zeta\in K_G} \frac{1}{G'(\zeta)} \geq -\frac{1}{G'(\zeta_0)}$$

3. *If $|\zeta_0| = G'(\zeta_0) = 0$, $\Re(\zeta_0 G''(\zeta_0)) = 0$ and $G''(\zeta_0) \neq 0$, then*

$$\sum_{\zeta\in K_G} \frac{1}{G'(\zeta)} \geq \frac{2G'''(\zeta_0)}{3G''(\zeta_0)^2}.$$

4. *If* $|\zeta_0| = G'(\zeta_0) = 0$, $G''(\zeta_0) = 0$, *and* $\Re(\zeta_0^3 G^{(4)}(\zeta_0)) = 2|G'''(\zeta_0)| < \infty$, *then*

$$\sum_{\zeta\in K_G} \frac{1}{G'(\zeta)-1} \geq \frac{3G^{(5)}(\zeta_0)}{10G'''(\zeta_0)^2} - \frac{3G^{(4)}(\zeta_0)^2}{8G'''(\zeta_0)^3}.$$

Furthermore, equality holds in each case if and only if the Herglotz measure of the parameter $p \in \mathcal{C}$ *in the representation formula* (5.1) *of* f *is discrete.*

We refer to [7, 12] for several related results involving fixed points of infinitesimal generators.

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