LIMIT RESOLVABILITY OF A QUASI-LINEAR SYSTEM ON A NANOLAYER THROUGH INPUT STABILITY ANALYSIS

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Communicated by Gabriela Marinoschi

This paper aims to discuss a stabilization problem for quasi-linear systems and to study the asymptotic behavior of a distributed system on an evolution domain with a p-Laplace operator in a containing structure of a nanolayer. The epi-convergence method is considered to find the limit problem with interface conditions. This approach consists of studying the stability of the approximate problem associated with our initial problem, then studying the limit behavior in order to determine the stability of the limit problem. The obtained results are numerically tested.

AMS 2020 Subject Classification: 35J92, 35B40, 76M50, 93B52, 82B24.

Key words: quasi-linear, limit behavior, limit problems, feedback control, epi-convergence method.

1. INTRODUCTION

A dynamical system is always affected by perturbations. Therefore, the system needs to function well to remedy these perturbations by considering the notion of stabilization, which entails returning the perturbed system to its equilibrium state.

The problem of linear and nonlinear system stabilization in finite dimensions is now well established and has given rise to a rich literature (Whonham [20], Quin [18], Azzo–Houpis [9], Lasalle [13], etc.). In the case of infinite-dimensional systems, works on strong and/or weak stabilization of distributed systems are mainly due to Ball–Slemrod [5], Triggiani [19], Benchimol [6] etc.

The link between the asymptotic behavior in time of a system, the spectral properties of its dynamics, and the existence of a Lyapunov functional are explored in [15]. The exponential stability is studied in [19] via an appropriate state space decomposition. The asymptotic and exponential stability are studied in [4], using the Riccati equation. Pritchard [16, 17], uses a dynamic programming approach to reduce the quadratic cost minimization problem to a solution of the Riccati equation, which is differential in the finite horizon case and algebraic in the infinite horizon case. This study was completed by Zabczyk [22], then developed by other authors: Lions [14], Curtain and Pritchard [7, 8], Quinn [18] etc., the researchers that study stability are essentially concentrated on the global domain with the macroscopic structure in which the system evolves. We then return to the works of Quinn [18], and Ball–Slemrod [5], concerning finite and infinite-dimensional nonlinear systems, to characterize the regional stabilization feedbacks, particularly the one with minimal cost. Koshkin [12] generalizes Wonham's theorem on the solvability of Riccati equations with algebraic operators in Banach spaces. Artamonov [3] shows that for an X reflexive Banach space, a Riccati integral equation with a non-autonomous operator has a unique, strongly continuous, self-adjoint, and non-negative solution.

To do so, let us considers the problem of quasi-linear evolution in a body that occupies a bonded domain, $\Omega \subset \mathbb{R}^3$, with a Lipschitz border $\partial\Omega$, composed of a nanolayer B_ε, with an oscillating border $\Sigma_\varepsilon^\pm$, and a remaining region of Ω_ε (see Figure 1). The body occupying the Ω domain is bounded, and the operator $L \in C_s(\mathcal{I}; \mathcal{L}(X^*, X))$ is linear and bounded, $L(t) = L^*(t)$ and positive ($\langle L(t)x, x\rangle \geq 0$, $\forall x \in X^*$), for all $t \in \mathcal{I}$. With the set of admissible controls $U_{ad} = \{u(t) \in X^* : \|u\|_{X^*} \leq C\}, \forall t \in \mathcal{I}$. The problem is modeled with the following equations

$$(\mathscr{P}) \begin{cases} \dot{z} - \operatorname{div}(|\nabla z|^{p-2}\nabla z) = 0, \text{ in } \Omega_\varepsilon^\infty; \\ \dot{z} = \frac{1}{\varepsilon^\alpha} \operatorname{div}(|\nabla z|^{p-2}\nabla z) + L(t)u, \text{ in } B_\varepsilon^\infty; \\ z(t,x) = 0, \text{ on } \Gamma^\infty =]0,\infty[\times\partial\Omega; \\ z(0,x) = z_0, \text{ in } \Omega; \\ [z(t,x)] = 0, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm; \\ |\nabla z|^{p-2}\frac{\partial z}{\partial n}|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}|\nabla z|^{p-2}\frac{\partial z}{\partial n}|_{B_\varepsilon}, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases}$$

The conductivity is expressed by $\frac{1}{\varepsilon^\alpha}$, a triple

$$X = W^{1,p}(B_\varepsilon) \hookrightarrow \mathcal{H} = L^2(B_\varepsilon) \hookrightarrow X^* = (W^{1,p})'(B_\varepsilon)$$

of spaces with dense embeddings for a reflexive Banach space X for $p > 1$, where $\mathcal{H}$ is a Hilbert space. Now, the aim of the present work is to study the stability of a quasi-linear problem via a feedback control with a p-Laplace operator and interface conditions. In our case, we work with a B_ε region of nanostructure, which can cause problems during the numerical resolution with the finite element method and, more precisely, during the creation of the mesh of the domain, which is very fine and can cause numerical explosions.

In this article, we focus on establishing the following main result, which demonstrates the limit behavior presented in the theorem below.

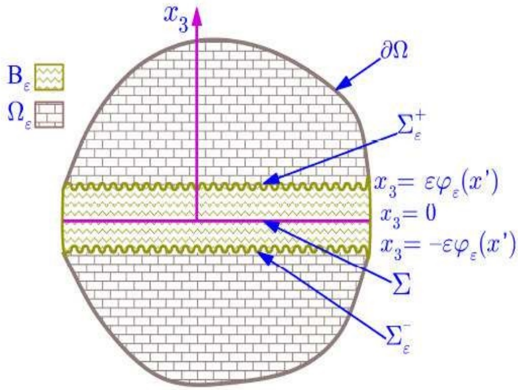

Figure 1 – Domain Ω

Consider the following energy operator:

$$F_\varepsilon(z_\varepsilon) = \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p - \int_{B_\varepsilon^\infty} L(t)u_\varepsilon z_\varepsilon.$$

One denotes by τ_f the weak topology on $L^p(0,\infty; W_0^{1,p}(\Omega))$.

Theorem 1.1. *According to the values of α, there exists a functional F^α defined on $L^p(0,\infty; W_0^{1,p}(\Omega))$ with a value in $\mathbb{R}\cup\{+\infty\}$ such that $\tau_f - \lim_e F_\varepsilon = F^\alpha$ in $L^p(0,\infty; W_0^{1,p}(\Omega))$, where the functional F^α is given by*

(1) *If $0 \le \alpha < 1$:*

$$F^\alpha(z) = \frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p,$$

for all $z \in L^p(0,\infty; W_0^{1,p}(\Omega))$.

(2) *If $\alpha \ge 1$:*

$$F^\alpha(z) = \frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_{]0,\infty[\times\Sigma} |\nabla' z_{|\Sigma}|^p,$$

for all $z \in \mathbb{G} \subset L^p(0,\infty; W_0^{1,p}(\Omega))$.

The idea would be to look for another equivalent approximation model to work with the finite element method in an accurate way in order to obtain the limit problem and reach the object of this article, which is organized as follows: Section 3 serves to show what we call feedback-controlled stability using Riccati's equations with algebraic operators in Banach spaces, which makes the quasi-linear system stable for the approximate problem associated with the initial problem, and we prove a priori estimates. Then, we proceed to the limit, using preliminary results, definitions, and some properties for the minimization problem. The epi-convergence method is considered to find the limit problem with interface conditions. Finally, in Section 4, we give a numerical test illustrating the theoretical results obtained.

2. PRELIMINARIES

2.1. Notations

In this section, we give the notations that are used throughout this paper:

- $\mathcal{Q} =]0,\infty[\times\Omega$, $\Omega_\varepsilon^\infty =]0,\infty[\times\Omega_\varepsilon$, $B_\varepsilon^\infty =]0,\infty[\times B_\varepsilon$.
- $L^p\left(0,\infty,X\right)$ has the norm,

$$\|z\|_{L^p(0,\infty,X)} = \left(\int_0^\infty \|z(t)\|_X^p dt\right)^{\frac{1}{p}}$$

with X, a Banach space.

- $[z]_{\Sigma_\varepsilon^\pm} = z|_{\overline{\Omega_\varepsilon}}|_{\Sigma_\varepsilon^\pm} - z|_{\overline{B_\varepsilon}}|_{\Sigma_\varepsilon^\pm}$.
- We have

$$\mathbb{G} = \begin{cases} z \in L^p(0,\infty; W_0^{1,p}(\Omega)) : \eta(\alpha)z(t)\mid_\Sigma \in L^p(0,\infty; W^{1,p}(\Sigma)) & \text{if } \alpha \le 1; \\ z \in L^p(0,\infty; W_0^{1,p}(\Omega)) : z(t)\mid_\Sigma = C & \text{if } \alpha > 1. \end{cases}$$

$$\mathbb{D} = \begin{cases} \mathcal{D}(]0,\infty[\times\Omega) & \text{if } \alpha \le 1; \\ \{z \in \mathcal{D}(]0,\infty[\times\Omega) : z(t)\mid_\Sigma = C\} & \text{if } \alpha > 1, \end{cases}$$

- Let us define the operator m^ε which transforms functions defined z on B_ε into functions defined on Σ by

$$m^\varepsilon z(t,x_1,x_2) = \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z(t,x_1,x_2,x_3)dx_3.$$

$(t,x) = (t,x',x_3)$, where $x' = (x_1,x_2)$, $\lambda = 1,2$, $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, the space $Y =]0,1[\times]0,1[$, the function $\varphi : \mathbb{R}^2 \to [a_1,a_2]$ where φ is Y-periodic and $a_2 \ge a_1 > 0$, $\varphi_\varepsilon(x') = \varphi(\frac{x'}{\varepsilon})$, $\frac{\partial\varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma)\cap L^\infty(\Sigma), m(\varphi) = \int_Y \varphi(x')dx'$, $\eta(\alpha) = \lim_{\varepsilon\to 0}\varepsilon^{1-\alpha}$, with $\alpha \ge 0$.

- If X_1, X_2 are Banach spaces, then by $\mathcal{L}\left(X_1,X_2\right)$ we denote the space of bounded operators acting from X_1 to X_2.
- By $C_s(\mathcal{I};\mathcal{L}(X_1,X_2))$ we denote the space of strongly continuous operator functions on interval $\mathcal{I} =]0,T[$ ranging in $\mathcal{L}(X_1,X_2)$.

In the following, C denotes any constant with respect to ε.

2.2. **Functional Framework**

In this section, we propose a concept of operator's sequence convergence known as epi-convergence, which is a subset of the Γ-convergence introduced by De Giorgi and Spagnolo [10]. It is well suited to the asymptotic analysis of sequences of minimization problems.

Definition 2.1 ([2], Definition 1.9). Let (X, τ) be a reflexive Banach space, $F_\varepsilon : X \to \mathbb{R} \cup +\infty$ a family of convex functionals, and $F : X \to \mathbb{R} \cup +\infty$ a convex functional. Suppose that

1. $\liminf_{\varepsilon \to 0} F_\varepsilon(x) \geq F(x)$ for all $x \in X$.

2. For any sequence $(x_\varepsilon) \subset X$ such that $x_\varepsilon \rightharpoonup x$ weakly in X, we have $\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \leq F(x)$. Then, we have $F_\varepsilon \xrightarrow{\tau-\text{epi}} F$.

Note the following fundamental result of epi-convergence.

Theorem 2.2 ([2], Theorem 1.10). *Suppose that*

(1) *F_ε admits a minimizer on X,*

(2) *the sequence $(\bar{z}_\varepsilon)$ is τ -relatively compact,*

(3) *The sequence F_ε epi-converges to F in this topology τ. Then each cluster point $\bar{z}$ of the sequence $(\bar{z}_\varepsilon)$ minimizes F on $\mathbb{X}$ and*

$$\lim_{\varepsilon' \to 0} F_{\varepsilon'}(\bar{z}_{\varepsilon'}) = F(\bar{z})$$

if $(\bar{z}_{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\bar{z}_\varepsilon)_\varepsilon$ that converges to $\bar{z}$.

This theorem shows that if X is a reflexive Banach space, then Riccati equation has a unique strongly continuous self-adjoint non-negative solution $P(t)$.

Theorem 2.3 ([3], Theorem 2). *Let X be a reflexive Banach space and the following assumptions hold:*

1. *$\{S\}_{0 \leq s \leq t \leq T}$ is strongly continuous and uniformly bounded forward evolution family in $\mathcal{L}(X)$. Then S^* is strongly continuous and uniformly bounded backward evolution family in $\mathcal{L}(X^*)$.*

2. *Operator functions $C \in C_s(\mathcal{I}; \mathcal{L}(X, X^*))$ and $L \in C_s(\mathcal{I}; \mathcal{L}(X^*, X))$.*

3. *$C(t) = C^*(t) \geq 0$ and $L(t) = L^*(t) \geq 0$ for all $t \in \mathcal{I}$. Then for all self-adjoint non-negative $P_0 \in \mathcal{L}(X, X^*)$ the (backward) integral Riccati equation*

$$P(t) = S^*(t)P_0S(t) + \int_0^t S^*(t-s)\big(C^*(s)C(s) \\ - P(s)L(s)L^*(s)P(s)\big)S(t-s)ds, \qquad t \geq 0$$

has a unique self-adjoint non-negative solution $P \in C_s(\mathcal{I}; \mathcal{L}(X, X^*))$.

3. MAIN RESULTS

3.1. Stability Study

Consider the following approximate problem:

$$(\mathscr{P}_\varepsilon)\begin{cases} \dot{z}_\varepsilon(t,x) - \operatorname{div}(|\nabla z_\varepsilon(t,x)|^{p-2}\nabla z_\varepsilon(t,x)) = 0, \text{ in } \Omega_\varepsilon^\infty; \\ \dot{z}_\varepsilon(t,x) = \frac{1}{\varepsilon^\alpha}\operatorname{div}(|\nabla z_\varepsilon(t,x)|^{p-2}\nabla z_\varepsilon(t,x)) + L(t)u_\varepsilon(t), \text{ in } B_\varepsilon^\infty; \\ z_\varepsilon(t,x) = 0, \text{ on } \Gamma^\infty =]0,\infty[\times\partial\Omega; \\ z_\varepsilon(0,x) = z_{0,\varepsilon}, \text{ in } \Omega; \\ [z_\varepsilon(t,x)] = 0, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm; \\ |\nabla z_\varepsilon(t,x)|^{p-2}\frac{\partial z_\varepsilon(t,x)}{\partial n}|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}|\nabla z_\varepsilon(t,x)|^{p-2}\frac{\partial z_\varepsilon(t,x)}{\partial n}|_{B_\varepsilon}, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases}$$

We are interested in stabilizing the following equation:

$$\dot{z}_\varepsilon = \frac{1}{\varepsilon^\alpha}\operatorname{div}\big(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon\big) + L(t)u_\varepsilon \text{ in } B_\varepsilon^\infty, \tag{1}$$

with interface conditions.

Based on successive linearizations, the Newton method replaces the discrete nonlinear problem (1) with an iterative sequence of linear problems, which can be directly solved by standard methods of linear algebra.

The term $A(z_{\varepsilon,k+1}) = \frac{1}{\varepsilon^\alpha}\operatorname{div}(|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k+1})$ can be linearized using the following expression:

$$A(z_{\varepsilon,k+1}) = A(z_{\varepsilon,k}) + (z_{\varepsilon,k+1} - z_{\varepsilon,k})A'(z_{\varepsilon,k}) \tag{2}$$

which is obtained from Taylor's series, with second and higher orders neglected. The term $A'(z_{\varepsilon,k})$ represents the differentiation of $A(z_{\varepsilon,k})$ with respect to $z_{\varepsilon,k}$. If we assume $z_{\varepsilon,k}$ is the known value and $z_{\varepsilon,k+1}$ is unknown, the right-hand side of equation (2) is always linear. Equation (1) now becomes

$$\dot{z}_{\varepsilon,k+1} = \frac{1}{\varepsilon^\alpha}\operatorname{div}\big(|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k+1}\big) + L(t)u_{\varepsilon,k+1} \text{ in } B_\varepsilon^\infty. \tag{3}$$

The z's in the boundary conditions now become z_{k+1}'s. It should be noted that if $z_{k+1} = z_k$, then equation (3) reduces to (1). It is obvious that in applying equation (2) to linearize (1) or any nonlinear equation, only the nonlinear terms need to be considered.

According to the Hille–Yosida theorem, we have the existence of a solution, denoted $z_{\varepsilon,k=0}$, then computing $z_{\varepsilon,k=1}, k = 1, 2, \ldots$, from equation (3), with the values for $z_{\varepsilon,k=1}$ known, the values for $z_{\varepsilon,k=2}$ are obtained. This process is repeated until the required accuracy is obtained.

Let ε be the maximum error allowed, and the required accuracy can be defined by the following equation:

$$\|z_{\varepsilon,k+1} - z_{\varepsilon,k}\| < \varepsilon, \quad k = 0, 1, 2, \ldots$$

We show the convergence of the method. By contradiction, let us suppose that $z_{\varepsilon,k+1} - z_{\varepsilon,k}$ is not bounded for k big enough and take

$$z'_{\varepsilon,k} = \frac{z_{\varepsilon,k+1} - z_{\varepsilon,k}}{\|z_{\varepsilon,k+1} - z_{\varepsilon,k}\|_{L^p(0,\infty;X)}},$$

replacing $z'_{\varepsilon,k}$ in the problem (3), with $X = W^{1,p}(B_\varepsilon)$, and using the variational formulation, we obtain:

$$<\dot{z}'_{\varepsilon,k}, z'_{\varepsilon,k}>_{B_\varepsilon} = <\frac{1}{\varepsilon^\alpha}\operatorname{div}(|\nabla z'_{\varepsilon,k-1}|^{p-2}\nabla z'_{\varepsilon,k}), z'_{\varepsilon,k}>_{B_\varepsilon} + <L(t)u'_{\varepsilon,k}, z'_{\varepsilon,k}>_{B_\varepsilon}$$

and

$$<\dot{z}'_{\varepsilon,k}, z'_{\varepsilon,k}>_{B_\varepsilon} + <\frac{1}{\varepsilon^\alpha}|\nabla z'_{\varepsilon,k-1}|^{p-2}\nabla z'_{\varepsilon,k}, \nabla z'_{\varepsilon,k}>_{B_\varepsilon} = - <Pz'_{\varepsilon,k}(t), z'_{\varepsilon,k}>_{B_\varepsilon}.$$

Furthermore

$$<\dot{z}'_{\varepsilon,k}, z'_{\varepsilon,k}>_{]0,T[\times B_\varepsilon} + <\frac{1}{\varepsilon^\alpha}|\nabla z'_{\varepsilon,k-1}|^{p-2}\nabla z'_{\varepsilon,k}, \nabla z'_{\varepsilon,k}>_{]0,T[\times B_\varepsilon} \leq 0$$

$$\int_0^T \frac{d}{dt}\|z'_{\varepsilon,k}\|^2_{L^2} + \frac{1}{\varepsilon^\alpha} \inf_{\substack{\int_0^T \||\nabla z'_{\varepsilon,k-1}|^{p-2}\|_{\frac{p}{p-2}} \\ = \|z'_{\varepsilon,k-1}\|^p_{L^p(0,T;X)} = 1}} <|\nabla z'_{\varepsilon,k-1}|^{p-2}, |\nabla z'_{\varepsilon,k}|^2>_{]0,T[\times B_\varepsilon} \leq 0$$

$$\int_0^T \frac{d}{dt}\|z'_{\varepsilon,k}\|^2_{L^2} + \int_0^T \frac{1}{\varepsilon^\alpha}\|z'_{\varepsilon,k}\|^p_{1,p} = \int_0^T \frac{d}{dt}\|z'_{\varepsilon,k}\|^2_{L^2} + \int_0^T \frac{1}{\varepsilon^\alpha}\||\nabla z'_{\varepsilon,k}|^2\|_{\frac{p}{2}} \leq 0$$

$$\|z'_{\varepsilon,k}(T,x)\|^2_{L^2} - \|z'_{\varepsilon,k}(0,x)\|^2_{L^2} + \int_0^T \frac{1}{\varepsilon^\alpha}\|z'_{\varepsilon,k}\|^p_{1,p} \leq 0,$$

such that when $T \to \infty$, we get:

$$\|z'_{\varepsilon,k}\|^p_{L^p(0,\infty,X)} = \int_0^\infty \|z'_{\varepsilon,k}\|^p_{1,p} \leq C\varepsilon^\alpha.$$

Contradiction as the norm $\|z'_{\varepsilon,k}\|^p_{L^p(0,\infty,X)} = 1$, hence $\|z_{\varepsilon,k+1} - z_{\varepsilon,k}\| < \varepsilon$.

As we get $A_{\varepsilon,k}$, it approaches A_ε, which allows us to have a semigroup $S_{\varepsilon,k}$ associated to the operator $A_{\varepsilon,k}$.

By the following Riccati equation [21]:

$$\dot{P}(t) = A^*_{\varepsilon,k}P(t) + P(t)A_{\varepsilon,k} + C^*C - P(t)L(t)L^*(t)P(t), \quad P(0) = P_0.$$

Note that $A^*_{\varepsilon,k}$ is the adjoint operator.

Since $L^p(0,\infty; W_0^{1,p}(B_\varepsilon))$ is a reflexive Banach space, the Riccati equation has a unique strongly continuous, self-adjoint, and non-negative solution $P(t)$ (see Theorem 2.3).

Formally, it can be demonstrated by analogy to the Hilbert case [20, 21] that the optimal control is found in the feedback form $u_{\varepsilon,k}(t) = -L^*(t)Pz_{\varepsilon,k}(t)$, where P is a bounded symmetric positive definite operator that solves the algebraic operator Riccati equation (see [12]).

3.2. **Limit Behavior of Solution**

3.2.1 Approximate problem

The set $V = W_0^{1,p}(\Omega)$ is a Banach and reflexive space, where $W_0^{1,p}(\Omega)$ has the norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$, according to the separability of V, hence it admits a countable basis $\{w_1, w_2, w_3, \ldots, w_m, \ldots\}$, with $w_i \in V$, for all m, $\{w_1, w_2, w_3, \ldots, w_m\}$ is a free family, $H = \mathrm{Vect}\{w_1, w_2, w_3, \ldots, w_m, \ldots\}$ is dense in V.

Let us consider in the spaces $V_m = \mathrm{Vect}\{w_1, w_2, w_3, \ldots, w_m\}$ the following approximate problem. We put

$$z_\varepsilon(t) = \sum_{i=1}^{m} h_{i\varepsilon}(t)w_i \in V_m.$$

$$(\mathscr{P}_{m,\varepsilon}) \begin{cases} <\dot{z_\varepsilon}, w_i>_{\Omega_\varepsilon} - <\mathrm{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon), w_i>_{\Omega_\varepsilon} = 0, \text{ in }]0,\infty[\\ <\dot{z_\varepsilon}, w_i>_{B_\varepsilon} = <\frac{1}{\varepsilon^\alpha}\,\mathrm{div}(|\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon), w_i>_{B_\varepsilon} + <L(t)u_\varepsilon, w_i>_{B_\varepsilon}, \\ \hfill \text{in }]0,\infty[\\ z_\varepsilon(t,x) = 0, \text{ on } \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z_\varepsilon(0,x) = z_{0,\varepsilon}, \text{ in } \Omega \\ [z_\varepsilon(t,x)] = 0, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ |\nabla z_\varepsilon|^{p-2}\frac{\partial z_\varepsilon}{\partial n}|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}|\nabla z_\varepsilon|^{p-2}\frac{\partial z_\varepsilon}{\partial n}|_{B_\varepsilon}, \text{ on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases}$$

With $<\cdot,\cdot>$ is a duality bracket.

From the results on systems of differential equations, we are sure that the problem $(\mathscr{P}_{m,\varepsilon})$ has a solution $z_\varepsilon(t)$ in an interval $]0,T[$, such that $T \to \infty$ for $\varepsilon \to 0$.

Lemma 3.1. *The family $(z_{\varepsilon,k})_{\varepsilon>0,k\in\mathbb{N}}$ satisfies:*

$$\int_0^\infty \|\nabla z_{\varepsilon,k}\|^p_{L^p(B_\varepsilon)} \le C\varepsilon^\alpha. \tag{4}$$

$$\int_0^\infty \|\nabla z_{\varepsilon,k}\|^p_{L^p(\Omega_\varepsilon)} \leq C. \tag{5}$$

Moreover, $z_{\varepsilon,k}$ *is bounded in* $L^p(0,\infty,W_0^{1,p}(\Omega))$.

Proof. Consider problem $(\mathscr{P}_{m,\varepsilon})$; multiply the equations defined on B_ε^∞ and $\Omega_\varepsilon^\infty$ by $h_{i\varepsilon}(t)$ and sum from $i=1$ to m, for a fixed k. We get

$$\begin{aligned}<\dot{z}_{\varepsilon,k}, z_{\varepsilon,k}>_\Omega =&< \operatorname{div}\big(|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}\big), z_{\varepsilon,k}>_{\Omega_\varepsilon}\\ &+<\frac{1}{\varepsilon^\alpha}\operatorname{div}\big(|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}\big), z_{\varepsilon,k}>_{B_\varepsilon}+<L(t)u_{\varepsilon,k}, z_{\varepsilon,k}>_{B_\varepsilon},\end{aligned}$$

and furthermore

$$\begin{aligned}<\dot{z}_{\varepsilon,k}, z_{\varepsilon,k}>_\Omega+<|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}, \nabla z_{\varepsilon,k}>_{\Omega_\varepsilon}+<\frac{1}{\varepsilon^\alpha}|\nabla z_{\varepsilon,k}|^{p-2}\nabla z_{\varepsilon,k}, \nabla z_{\varepsilon,k}>_{B_\varepsilon}\\ =-<Pz_{\varepsilon,k}(t), z_{\varepsilon,k}>_{B_\varepsilon}.\end{aligned}$$

We have $u_{\varepsilon,k}(t)=-L^*(t)Pz_{\varepsilon,k}(t)$, where P is a bounded symmetric positive definite operator, so $L(t)u_{\varepsilon,k}=-Pz_{\varepsilon,k}(t)$, and since P is a bounded symmetric positive so $-<Pz_{\varepsilon,k}(t), z_{\varepsilon,k}>_{B_\varepsilon}\leq 0$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |z_{\varepsilon,k}|^2+\int_{\Omega_\varepsilon}|\nabla z_{\varepsilon,k}|^p+\frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon}|\nabla z_{\varepsilon,k}|^p\leq 0, \tag{6}$$

and by integration from 0 to T:

$$\frac{1}{2}\|z_{\varepsilon,k}(T,x)\|^2_{L^2(\Omega)}-\frac{1}{2}\|z_{0,\varepsilon}\|^2_{L^2(\Omega)}\leq-\frac{1}{\varepsilon^\alpha}\int_0^T\|\nabla z_{\varepsilon,k}\|^p_{L^p(B_\varepsilon)}-\int_0^T\|\nabla z_{\varepsilon,k}\|^p_{L^p(\Omega_\varepsilon)}.$$

So,

$$\frac{1}{\varepsilon^\alpha}\int_0^T\|\nabla z_{\varepsilon,k}\|^p_{L^p(B_\varepsilon)}+\int_0^T\|\nabla z_{\varepsilon,k}\|^p_{L^p(\Omega_\varepsilon)}\leq-\frac{1}{2}\|z_{\varepsilon,k}(T,x)\|^2_{L^2(\Omega)}+C.$$

Then, let us reduce $-\frac{1}{2}\|z_{\varepsilon,k}(T,x)\|^2_{L^2(\Omega)}$ by 0, such that $T\to\infty$, we get

$$\frac{1}{\varepsilon^\alpha}\int_0^\infty\|\nabla z_{\varepsilon,k}\|^p_{L^p(B_\varepsilon)}+\int_0^\infty\|\nabla z_{\varepsilon,k}\|^p_{L^p(\Omega_\varepsilon)}\leq C.$$

So

$$\int_0^\infty\|\nabla z_{\varepsilon,k}\|^p_{L^p(B_\varepsilon)}\leq\varepsilon^\alpha C. \tag{7}$$

and

$$\int_0^\infty\|\nabla z_{\varepsilon,k}\|^p_{L^p(\Omega_\varepsilon)}\leq C. \tag{8}$$

It is clear that for a small enough ε, the solution $(z_{\varepsilon,k})$ is bounded in the space $L^p(0,\infty,W_0^{1,p}(\Omega))$. □

Since $L^p(0,\infty;W_0^{1,p}(\Omega))$ is a reflexive space, then there exists a subsequence of $(z_{\varepsilon,k})_{\varepsilon>0,k\in\mathbb{N}}$, always denoted by $z_{\varepsilon,k}$, such that $z_{\varepsilon,k}\rightharpoonup z^*$ in $L^p(0,\infty;W_0^{1,p}(\Omega))$.

3.3. **Proof of Theorem 1.1**

We recall the energy operator of our problem

$$\inf_{z\in L^p(0,\infty;W_0^{1,p}(\Omega))}\left\{\frac{1}{p}\int_{\Omega_\varepsilon^\infty}|\nabla z|^p+\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty}|\nabla z|^p-\int_{B_\varepsilon^\infty}L(t)uz\right\} \qquad (\mathscr{P}_1)$$

Remark 3.2 ([11]). According to the Hille–Yosida theorem, we have the existence of a solution. Moreover, z is given by the formula

$$z(t)=\mathbf{S}_{\Delta_{\mathrm{p}}}(t)z_0-\int_0^t\mathbf{S}_{\mathrm{A}}(t-s)Pz(s)ds,\quad t\geq 0$$

where $S_{\Delta_p}(t)$ denotes the semigroup associated to Δ_p.

To prove our theorem, we need to establish Lemmas 3.3 and 3.4 and Proposition 3.5.

LEMMA 3.3. *The operator m^ε is linear and bounded in $L^p(0,\infty;L^p(B_\varepsilon))$ (resp. $L^p(0,\infty;W_0^{1,p}(B_\varepsilon))$) and in $L^p(0,\infty;L^p(\Sigma))$ (resp. $L^p(0,\infty;W_0^{1,p}(\Sigma))$). Moreover, for all $z\in L^p(0,\infty;W_0^{1,p}(B_\varepsilon))$, we have*

$$\left\|m^\varepsilon z-z_{|\Sigma}\right\|^p_{L^p(]0,\infty[\times\Sigma)}\leq C\varepsilon^{p-1}\int_0^\infty\int_{B_\varepsilon}|\nabla z|^p. \tag{9}$$

Proof. We have

$$\int_\Sigma|m^\varepsilon z|^p\,dx_1dx_2=\int_\Sigma\left(\frac{1}{2\varepsilon\varphi_\varepsilon}\right)^p\left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}zdx_3\right|^p dx_1dx_2.$$

Since $0<a_1\leq\varphi_\varepsilon\leq a_2$, and according to the Hölder inequality,

$$\begin{aligned}\int_\Sigma|m^\varepsilon z|^p\,dx_1dx_2&\leq\int_\Sigma\frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|z|^pdx_3\right)dx_1dx_2\\&\leq\frac{1}{2\varepsilon a_1}\int_\Sigma\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|z|^pdx_3\right)dx_1dx_2.\end{aligned}\tag{10}$$

Since $z\in L^p\left(]0,\infty[\times B_\varepsilon\right)$ and (10), it follows that $m^\varepsilon z\in L^p(]0,\infty[\times\Sigma)$. Let $z\in\overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$, we have

$$\begin{aligned}\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)(t,x_1,x_2)&=\frac{1}{2}\frac{\partial}{\partial x_\lambda}\left(\int_{-1}^1 z\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right)dx_3\right)\\&=\frac{1}{2}\left(\int_{-1}^1\frac{\partial z}{\partial x_\lambda}(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon)\right.\\&\qquad\left.+\,\varepsilon x_3\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\frac{\partial z}{\partial x_3}(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon)dx_3\right)\\&=\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left(\frac{\partial z}{\partial x_\lambda}+\frac{x_3}{\varepsilon\varphi_\varepsilon}\cdot\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\cdot\frac{\partial z}{\partial x_3}\right)dx_3.\end{aligned}$$

So that,

$$\begin{aligned}\int_{\Sigma}\left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^p &= \int_{\Sigma}\left|\frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\frac{\partial z}{\partial x_\lambda}+\left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3}dx_3\right)\right|^p\\ &\leq \left(\frac{1}{2\varepsilon a_1}\right)^p\int_{\Sigma}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\frac{\partial z}{\partial x_\lambda}+\left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3}\right|^p dx_3\right).\end{aligned}$$

However, $\frac{\partial\varphi}{\partial x_\lambda}\in\mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$, then $\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}$ is bounded, and therefore

$$\int_{\Sigma}\left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^p\leq\frac{C}{\varepsilon}\int_{B_\varepsilon}\left(\left|\frac{\partial z}{\partial x_\lambda}\right|^p+\left|\frac{\partial z}{\partial x_3}\right|^p\right)dx_3\leq\frac{C}{\varepsilon}\int_{B_\varepsilon}|\nabla z|^p.$$

By density arguments, for any $z\in L^p(0,\infty;W_0^{1,p}(B_\varepsilon))$, we have

$$\int_0^\infty\int_{\Sigma}\left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^p\leq\frac{C}{\varepsilon}\int_0^\infty\int_{B_\varepsilon}|\nabla z|^p.$$

Let $z\in\overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$, so that

$$\|m^\varepsilon z-z_{|\Sigma}\|^p_{L^p(\Sigma)}=\int_{\Sigma}\left|\left(\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}z(t,x_1,x_2,x_3)dx_3\right)-z(t,x_1,x_2,0)\right|^p dx_1dx_2.$$

Using the Hölder inequality,

$$\begin{aligned}\|m^\varepsilon z-z_{|\Sigma}\|^p_{L^p(\Sigma)}&\leq\frac{1}{2\varepsilon a_1}\int_{\Sigma}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|z(t,x_1,x_2,x_3)-z(t,x_1,x_2,0)\right|^p dx_3\right)dx_1dx_2\\ &\leq\frac{C}{\varepsilon}\int_{\Sigma}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\int_0^{x_3}\frac{\partial z}{\partial x_3}(t,x_1,x_2,w)dw\right|^p dx_3\right)dx_1dx_2\\ &\leq\frac{C}{\varepsilon}\int_{\Sigma}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|x_3|^{p-1}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\frac{\partial z}{\partial x_3}(t,x_1,x_2,w)\right|^p dw\right)dx_3\right)dx_1dx_2\\ &\leq C\varepsilon^{p-1}\int_{\Sigma}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\frac{\partial z}{\partial x_3}\right|^p dx_3\right)dx_1dx_2\\ &\leq C\varepsilon^{p-1}\int_{B_\varepsilon}|\nabla z|^p dx.\end{aligned}$$

By density arguments, we have for all $z\in L^p(0,\infty;W_0^{1,p}(B_\varepsilon))$

$$\|m^\varepsilon z-z_{|\Sigma}\|^p_{L^p(]0,\infty[\times\Sigma)}\leq C\varepsilon^{p-1}\int_0^\infty\int_{B_\varepsilon}|\nabla z|^p dxdt.$$

Hence the result. $\square$

Lemma 3.4. *Let $(z_\varepsilon)_{\varepsilon>0}\subset L^p(0,\infty;W_0^{1,p}(\Omega))$ which satisfies (4) and (5). Then*

$$\|\nabla'\left(m^\varepsilon z_\varepsilon\right)\|^p_{(L^p(]0,\infty[\times\Sigma))^p}\leq C\varepsilon^{\alpha-1}. \tag{11}$$

In addition, $m^\varepsilon z_\varepsilon$ have a bounded sub-sequence in $L^p(]0,\infty[\times\Sigma)$.

Proof. According to a result of Lemma 3.3, we have

$$\int_0^\infty \left\| \frac{\partial\left(m^\varepsilon z_\varepsilon\right)}{\partial x_\lambda} \right\|^p_{L^p(\Sigma)^p} \leq C\varepsilon^{-1} \int_0^\infty \int_{B_\varepsilon} |\nabla z_\varepsilon|^p \, dx.$$

According to (4), one has

$$\int_0^\infty \left\| \frac{\partial\left(m^\varepsilon z_\varepsilon\right)}{\partial x_\lambda} \right\|^p_{L^p(\Sigma)^p} \leq C\varepsilon^{\alpha-1}.$$

Then from Lemma 3.3, we get

$$\|m^\varepsilon z - z_{|\Sigma}\|^p_{L^p(]0,\infty[\times\Sigma)} \leq C\varepsilon^{p-1} \int_0^\infty \int_{B_\varepsilon} |\nabla z|^p \leq C\varepsilon^{\alpha+p-1}.$$

The sequence $(z_\varepsilon)_\varepsilon$ is bounded in $L^p\left(0,\infty; W_0^{1,p}(\Omega)\right)$, it follows that there exists $z^* \in L^p(0,\infty; W_0^{1,p}(\Omega))$ and a sub-sequence z_ε, always noted z_ε, such as $z_\varepsilon \rightharpoonup z^*$ in $L^p(0,\infty; W_0^{1,p}(\Omega))$, then $z_{\varepsilon|\Sigma}$ is a bounded sequence in $L^p(]0,\infty[\times\Sigma)$.

Since,

$$\|m^\varepsilon z_\varepsilon\|_{L^p(]0,\infty[\times\Sigma)} \leq \|m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma}\|_{L^p(]0,\infty[\times\Sigma)} + \|z_{\varepsilon|\Sigma}\|_{L^p(]0,\infty[\times\Sigma)},$$

then there exists C such that $\|m^\varepsilon z_\varepsilon\|^p_{L^p(]0,\infty[\times\Sigma)} \leq C$. $\square$

Proposition 3.5. *$(z_\varepsilon)_\varepsilon$ has a weakly convergent sub-sequence to an element z^* in $L^p(0,\infty; W_0^{1,p}(\Omega))$ satisfactory*

(1) *If $\alpha = 1$, $z^*|_\Sigma \in L^p(0,\infty; W_0^{1,p}(\Sigma))$.*

(2) *If $\alpha > 1$, $z^*|_\Sigma = C$.*

Proof. The sequence z_ε is bounded in $L^p(0,\infty; W_0^{1,p}(\Omega))$. It follows that there is an element $z^* \in L^p(0,\infty; W_0^{1,p}(\Omega))$ and a sub-sequence of z_ε, always designated by z_ε such as $z_\varepsilon \rightharpoonup z^*$ in $L^p(0,\infty; W_0^{1,p}(\Omega))$. We have

$$\|m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma}\|^p_{L^p(]0,\infty[\times\Sigma)} \leq C\varepsilon^{\alpha+p-1} \text{ and } z_{\varepsilon|\Sigma} \rightharpoonup z^*_{|\Sigma} \text{ in } L^p(]0,\infty[\times\Sigma).$$

For $\alpha = 1$, according to the evaluation (11), the sequence $\nabla' m^\varepsilon z_\varepsilon$ has a sub-sequence, always denoted by $\nabla' m^\varepsilon z_\varepsilon$ weakly convergent to an element z_2 in $L^p(0,\infty; L^p(\Sigma))^2$, as $m^\varepsilon z_\varepsilon \rightharpoonup z^*_{|\Sigma}$ in $L^p(0,\infty; W_0^{1,p}(\Sigma))$ and $\nabla' z^*_{|\Sigma} = z_2$. Hence $z^*_{|\Sigma} \in L^p(0,\infty; W_0^{1,p}(\Sigma))$. For $\alpha > 1$, one shows, as in the case $\alpha = 1$ and taking $z_2 = 0$, that $z^*_{|\Sigma} = C$. Hence the results. $\square$

The previous results have helped us to highlight our fundamental result (Theorem 1.1).

Proof. The limit behavior of the problem $(\mathscr{P}_1)$, is derived with the epi-convergence method. Let

$$\begin{aligned} F_\varepsilon(z_\varepsilon) &= \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p - \int_{B_\varepsilon^\infty} L(t)u_\varepsilon z_\varepsilon \\ &= \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p - \int_{B_\varepsilon^\infty} L(t)(-L^*(t)Pz_\varepsilon(t))z_\varepsilon \\ &= \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^p + \int_{B_\varepsilon^\infty} Pz_\varepsilon(t)z_\varepsilon. \end{aligned}$$

(a) We determine the upper epi-limit.

Let $z \in \mathbb{G} \subset L^p(0,\infty;W_0^{1,p}(\Omega))$, and (z_ε^k) a sequence in $L^p(0,\infty;W_0^{1,p}(\Omega))$. From the fact that Δ_p is of type S^+, we conclude the convergence $z_\varepsilon^k \to z$ in the space $L^p(0,\infty;W_0^{1,p}(\Omega))$, when $k \to +\infty$.

Since $z_\varepsilon^k \to z$ in $L^p(0,\infty,W_0^{1,p}(\Omega))$, when $k \to +\infty$. According to the classical result, the diagonalization lemma [2, Lemma 1.15], there is a function $k(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$, such as $z_\varepsilon^{k(\varepsilon)} \to z$ in $L^p(0,\infty,W_0^{1,p}(\Omega))$, when $\varepsilon \to 0$.

$$F_\varepsilon(z_\varepsilon^k) = \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \int_{B_\varepsilon^\infty} Pz_\varepsilon^k(t)z_\varepsilon^k.$$

So that

$$\begin{aligned} F_\varepsilon(z_\varepsilon^k) &= \frac{1}{p}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p + \frac{1}{p}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p \\ &\quad + \frac{1}{p\varepsilon^\alpha}\int_{]0,\infty[\times B_\varepsilon} |\nabla z_\varepsilon^k|^p + \int_{B_\varepsilon^\infty} Pz_\varepsilon^k(t)z_\varepsilon^k \\ &= \frac{1}{p}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p + \frac{1}{p}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p \\ &\quad + \frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon|\nabla' z_{\varepsilon|\Sigma}^k|^p + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon Pz_{\varepsilon|\Sigma}^k(t)z_{\varepsilon|\Sigma}^k. \end{aligned}$$

Since φ_ε is bounded, we can easily verify that

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{p}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p\right\} = 0.$$

Since $Pz_\varepsilon^k(t) \in \mathcal{L}(X,X^*)$, the boundedness of $Pz_\varepsilon^k(t)$ in X^* yields the convergences $Pz_\varepsilon^k(t) \rightharpoonup Pz(t)$ in X^*, and $z_\varepsilon^k \to z$ in $L^p(0,\infty;W_0^{1,p}(\Omega))$:

1. If $\alpha = 1$: Since $\varphi_\varepsilon \overset{*}{\rightharpoonup} m(\varphi)$ in $L^\infty(\Sigma)$ and $\varepsilon^{1-\alpha} \to \eta(\alpha)$, it follows that

$$\lim_{\varepsilon\to 0}\frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon|\nabla' z_{\varepsilon|\Sigma}^k|^p = \frac{2m(\varphi)\eta(\alpha)}{p}\int_{]0,\infty[\times\Sigma} |\nabla' z_{|\Sigma}|^p.$$

By passing to the upper limit, we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) = \limsup_{\varepsilon\to 0}\Big(\frac{1}{p}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon|\nabla' z_{\varepsilon|\Sigma}^k|^p + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon P z_{\varepsilon|\Sigma}^k(t) z_{\varepsilon|\Sigma}^k\Big)$$
$$\leq \frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_{]0,\infty[\times\Sigma} |\nabla' z_{|\Sigma}|^p.$$

2. If $\alpha \neq 1$: By passing to the upper limit, we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) = \limsup_{\varepsilon\to 0}\Big(\frac{1}{p}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_\varepsilon^k|^p + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon P z_{\varepsilon|\Sigma}^k(t) z_{\varepsilon|\Sigma}^k\Big)$$
$$\leq \frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p.$$

(b) We determine the lower epi-limit.

Let $z \in \mathbb{G}$ and (z_ε^k) a sequence in $L^p(0,\infty; W_0^{1,p}(\Omega))$ such as $z_\varepsilon^k \rightharpoonup z$ in $L^p(0,\infty; W_0^{1,p}(\Omega))$, so that

$$\chi_{\Omega_\varepsilon^\infty}\nabla z_\varepsilon^k \rightharpoonup \nabla z \quad \text{in } L^p(0,\infty, L^p(\Omega))^3. \tag{12}$$

1. If $\alpha \neq 1$: Since

$$F_\varepsilon(z_\varepsilon^k) \geq \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \int_{B_\varepsilon^\infty} P z_\varepsilon^k(t) z_\varepsilon^k.$$

According to (12) and by passage to the lower limit, one obtains

$$\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) \geq \frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p.$$

2. If $\alpha = 1$: If $\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) = +\infty$, there is nothing to prove, because

$$\frac{1}{p}\int_{]0,\infty[\times\Omega} |\nabla z|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_{]0,\infty[\times\Sigma} |\nabla' z_{|\Sigma}|^p \leq +\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) < +\infty$, there is a sub-sequence of $F_\varepsilon(z_\varepsilon^k)$ still designated by $F_\varepsilon(z_\varepsilon^k)$ and a constant $C > 0$, such as $F_\varepsilon(z_\varepsilon^k) \leq C$, which implies that

$$\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \int_{B_\varepsilon^\infty} P z_\varepsilon^k(t) z_\varepsilon^k \leq C. \tag{13}$$

Therefore, z_ε^k satisfies the hypothesis of Lemma 3.4, and according to this last inequality, $\nabla' m^\varepsilon z_\varepsilon^k$ is bounded in $L^p(0,\infty;L^p(\Sigma))^2$, so there is an element $z_1 \in L^p\ (0,\infty;L^p(\Sigma))^2$ and a sub-sequence of $\nabla' m^\varepsilon z_\varepsilon^k$, always designated by $\nabla' m^\varepsilon z_\varepsilon^k$, such as $\nabla' m^\varepsilon z_\varepsilon^k \rightharpoonup z_1$ in $L^p(0,\infty;L^p(\Sigma))^2$, since $z_{\varepsilon|\Sigma} \rightharpoonup z_{|\Sigma}$ in $L^p(]0,\infty[\times\Sigma)$, and thanks to (9) and (13), one has $m^\varepsilon z_\varepsilon^k \rightharpoonup z_{|\Sigma}$ in $L^p(]0,\infty[\times\Sigma)$, then $m^\varepsilon z_\varepsilon^k \rightharpoonup z_{|\Sigma}$ in $L^p(0,\infty;W_0^{1,p}(\Sigma))$, so $z_1 = \nabla' z_{|\Sigma}$, so that $\nabla' m^\varepsilon z_\varepsilon^k \rightharpoonup \nabla' z_{|\Sigma}$ in $L^p(0,\infty;L^p(\Sigma))^2$,

$$
\begin{aligned}
F_\varepsilon(z_\varepsilon^k) &\geq \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \int_{B_\varepsilon^\infty} P z_\varepsilon^k(t) z_\varepsilon^k \\
&\geq \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon |\nabla' m^\varepsilon z_\varepsilon^k|^p \\
&\quad + 2\varepsilon \int_{]0,\infty[\times\Sigma} \varphi_\varepsilon P z_\varepsilon^k(t) m^\varepsilon z_\varepsilon^k.
\end{aligned}
$$

Using the subdifferential inequality, we have

$$
\begin{aligned}
F_\varepsilon(z_\varepsilon^k) \geq & \frac{1}{p}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon^k|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon |\nabla' z_{|\Sigma}|^p \\
& + \frac{2\varepsilon^{1-\alpha}}{p}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon |\nabla' z_{|\Sigma}|^{p-2}\nabla' z_{|\Sigma}(\nabla' m^\varepsilon z_\varepsilon^k - \nabla' z_{|\Sigma}) \\
& + 2\varepsilon \int_{]0,\infty[\times\Sigma} \varphi_\varepsilon P z_\varepsilon^k(t) m^\varepsilon z_\varepsilon^k.
\end{aligned}
$$

By [1, Lemma 7.1], we have $\varphi_\varepsilon \to m(\varphi)$ in $L^2(\Sigma)$, so according to (12) and by passing to the lower limit, we obtain

$$
\liminf_{\varepsilon\to 0} F_\varepsilon(z_\varepsilon^k) \geq \frac{1}{p}\int_{]0,\infty\times\Omega} |\nabla z|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_{]0,\infty[\times\Sigma} |\nabla' z_{|\Sigma}|^p.
$$

Hence the result. $\square$

To complete our main theoretical result, we have the following proposition.

Proposition 3.6. *According to the values of parameter α, there exists (z^*,u^*) satisfying*

$$
\begin{aligned}
z_\varepsilon^k &\rightharpoonup z^* \ \text{in}\ L^p(0,\infty;W_0^{1,p}(\Omega)) \\
P z_\varepsilon^k(t) &\rightharpoonup P z(t) \ \text{in}\ X^* \\
F^\alpha(z^*) &= \inf_{v\in\mathbb{G}} \{F^\alpha(v)\}.
\end{aligned}
$$

Proof. Firstly, (z_ε^k) is bounded in $L^p(0,\infty;W_0^{1,p}(\Omega))$, so it has a τ-cluster point z^* in $L^p(0,\infty;W_0^{1,p}(\Omega))$. As a consequence of a classical result of epi-convergence (see Theorem 2.2), we have z^* is a solution of the problem

$$\inf_{v\in\mathbb{G}}\{F^\alpha(v)\}. \qquad (\mathscr{P}_{lim}) \quad \square$$

3.4. **Conclusion**

In this paper, we worked on a class of quasi-linear evolution systems and showed that this approach is stabilizable by a control for the approximate equivalent problem on a three-dimensional bung. We also learned the limiting behavior of this type of problem, finding that the effect of the nanolayer does not exist and the control disappears, i.e., the nanolayer behaves as a part of Ω and the limiting problem becomes an autonomous problem.

4. **NUMERICAL TESTS**

For a sufficiently small value of ε, the solution z_ε of the approaching problem approaches the solution z^* of the limit problem. We are interested in the numerical treatment in this section and we concentrate on the impact of the control on the B_ε^∞ domain, with

$$T = 10$$
$$\Omega = \{(x,y)| \ \ x\in]0,1[, y\in]-1,1[, z\in]0,1[\}$$
$$B_\varepsilon =]0,1[\times]-\varphi_\varepsilon(x),\varphi_\varepsilon(x)[\times]0,1[$$

$L(t)u$ is linear and bounded
$$u_\varepsilon(t) = -L^*(t)Pz_\varepsilon(t)$$
$$\varphi_\varepsilon(x) = 1.2+\sin(\pi\tfrac{x}{\varepsilon}).$$

Using the Python programming language, with the finite element method and the Newton method, with $p = 2.1$ and $\varepsilon = 1e-10$, one has the results shown in the table.

The solution of the approximation problem converges to that of the limit problem.

Initially, u^* does not stabilize the state on all of Ω, which is normal because the control is defined only on B_ε, so the control stabilizes the state only on a sub-region, so $\alpha = 1$ is all that is of interest.

t	$\|z_\varepsilon\|$			$\|z^*\|$		
	$\alpha=0.1$	$\alpha=1$	$\alpha=3$	$\alpha=0.1$	$\alpha=1$	$\alpha=3$
t=0	6.19263e+10	6.19263e+10	6.19263e+10	14.8623	14.8623	14.8623
t=3.3	4.57764e-05	3.8147e-05	6.10352e-05	9.11214e-06	1.95399e-13	1.59872e-14
t=5	1.88712e-05	1.87712e-13	1.00003e-15	9.11214e-06	1.9255e-13	1e-15
t=6.6	1.79815e-05	1.41572e-13	6.19263e+10	9.11214e-06	1.9255e-13	14.8623
t=8.3	1.8238e-05	1.16754e-13	6.19263e+10	9.11214e-06	1.9255e-13	14.8623
t=10	1.78143e-05	1.16425e-13	6.10352e-05	9.11214e-06	1.9255e-13	14.8623

Table 1 – System stability for different α values

Table 1 shows that the solution of the approximation problem converges to that of the limit problem and demonstrates that u_ε stabilizes the state z_ε, and u^* stabilizes the state z^* on the nanolayer. Specifically, when $\alpha = 1$, we found that the system is stabilized, and even when α is close to 1, stability can still be observed. However, as we move further away from $\alpha = 1$, we begin to lose stability, which is the desired outcome. This illustrates that the model is suitable for control specialists working on the nanolayer.

Acknowledgments. The authors would like to thank the referee(s) for his/her comments and suggestions on the manuscript.

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Received 12 November 2023

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