ON A GENERAL DIVISOR PROBLEM ASSOCIATED TO DEDEKIND ZETA FUNCTION OVER A CERTAIN SPARSE SEQUENCE OF POSITIVE INTEGERS

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Let K_3 be a non-normal cubic extension over $\mathbb{Q}$. And let $\tau_k^{K_3}(n)$ denote the k-dimensional divisor function in the number field $K_3/\mathbb{Q}$. In this paper, we investigate the asymptotic behaviour of higher power moments of $\tau_k^{K_3}(n)$ over a certain sparse sequence of positive integers. In a more explicit manner, we consider the asymptotic formula of the following type

$$\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \left(\tau_k^{K_3}(n)\right)^\ell,$$

where $k\geqslant 2, \ell\geqslant 2$ are any given positive integers. Furthermore, as an application, we also establish the asymptotic formula of the variance of $(\tau_k^{K_3}(n))^\ell$. These results generalize the recent works in this direction.

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1. INTRODUCTION

Arithmetic functions play a prominent role in number theory, and it is customary to investigate the average behaviour of arithmetic functions by establishing the corresponding asymptotic formulae. The average behaviour of the coefficients of the Dedekind zeta function is an interesting and important topic in modern number theory, which plays a significant role in algebraic number theory.

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Let $K/\mathbb{Q}$ be a number field of degree d. The Dedekind zeta function is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} (N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \big(1-(N\mathfrak{p})^{-s}\big)^{-1}, \quad \Re(s)>1, \tag{1}$$

where $N\mathfrak{a}$ is the norm of the integral ideals $\mathfrak{a}$, and the sum ranges over all the non-zero ideals in the ring $\mathcal{O}_K$. We can rewrite the Dedekind zeta function as a Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \quad \Re(s)>1, \tag{2}$$

where $a_K(n)$ denotes the number of integral ideal in K with norm n, which is called the coefficients of the Dedekind zeta function. It is obvious that $a_K(n) \geqslant 0$ for all $n \geqslant 1$. It is well known that $a_K(n)$ is a real multiplicative function and for any $\varepsilon > 0$,

$$a_K(n) \leqslant \tau(n)^d \ll n^{\varepsilon}, \tag{3}$$

here $\tau(n)$ is the classical divisor function and $d = [K:\mathbb{Q}]$. From the result in Chandrasekharan and Narasimhan [3], we know the tighter upper bound for $a_K(n)$ that

$$a_K(n) \leqslant \big(\tau(n)\big)^{d-1}. \tag{4}$$

Clearly, we can expand the expression (2) as an Euler product

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s} = \prod_p \Big(1 + \frac{a_K(p)}{p^s} + \cdots + \frac{a_K(p^k)}{p^{ks}} + \cdots\Big), \quad \Re(s)>1.$$

The investigation of coefficients of the Dedekind zeta function has a rich history, and it has also attracted the attentions of plenty of scholars. Landau [31] proved the asymptotic formula

$$\sum_{n\leqslant x} a_K(n) = cx + O(x^{1-\frac{2}{d+1}+\varepsilon}) \tag{5}$$

for arbitrary algebraic number fields of degree $d \geqslant 2$, where $c > 0$ is some suitable constant depending on K. Let $K/\mathbb{Q}$ be a number field with $d = [K:\mathbb{Q}]$, for h being the class number of K, and set r_1 and $2r_2$ the number of real and complex conjugate field embeddings, respectively, one has

$$\sum_{n\leqslant x} a_K(n) = h\lambda x + E(x), \tag{6}$$

where

$$\lambda := \frac{2^{r_1+r_2}\pi^{r_2}R}{w|\Delta|^{\frac{1}{2}}},$$

Here, the symbols w, R and Δ denote the number of roots of unity in K, the regulator of K, and the discriminant of K, respectively. In 2020, Paul and Sankaranarayanan [50] made some improvement for $E(x)$ appearing in (6) in comparison with (5) for general number fields and cyclotomic fields under some mild conditions.

Let $K/\mathbb{Q}$ be a Galois extension of degree d. In [3], Chandraseknaran and Narasimhan considered the second moment of $a_K(n)$, and they proved that

$$\sum_{n\leqslant x} a_K^2(n) \ll x\log^{d-1} x.$$

Later, for $K/\mathbb{Q}$ being a Galois extension of degree d, Chandraseknaran and Good [2] investigated the higher moments of $a_K(n)$ and established the asymptotic formulas

$$\sum_{n\leqslant x} a_K^\ell(n) = xP_K(\log x) + O(x^{1-\frac{2}{d^\ell}+\varepsilon}),$$

where $\ell \geqslant 2$ is a positive integer and $P_K(t)$ is a polynomial of t with degree $d^{\ell-1}-1$. In 2010, Lü and Wang [45] improved the results of Chandraseknaran and Good, by establishing the asymptotic formula that

$$\sum_{n\leqslant x} a_K^\ell(n) = xP_K(\log x) + O(x^{1-\frac{3}{d^\ell+6}+\varepsilon})$$

for any fixed integer $\ell \geqslant 2$, where $P_K(t)$ is a polynomial of t with degree $d^{\ell-1}-1$.

A significant and important problem in number theory is to consider the quantity

$$S(x) := \sum_{n\leqslant x} \tau(n),$$

where $x > 0$ is a sufficiently large number. Indeed, one can establish the following asymptotic formula

$$S(x) = x\log x + (2\gamma-1)x + O(x^\vartheta),$$

where γ is the Euler's constant and ϑ is a real number with $0 < \vartheta < 1$. The precise determination of the exponent ϑ is the celebrated Dirichlet divisor problem. In an analogous manner, we can also study the k-dimensional divisor problem, which investigates the average behaviour of $\tau_k(n)$, generalizing the classical Dirichlet divisor problem. Here, as usual, $\tau_k(n)$ denotes the number of the representations of n as a product of k positive integers. Let $k \geqslant 2$ be any fixed integer, and let $\Delta_k(x)$ denotes the error terms of the asymptotic formula for

$$\sum_{n\leqslant x} \tau_k(n),$$

where $\zeta(s)^k = \sum_{n\geqslant 1} \tau_k(n)n^{-s}$. Using elementary argument, one can show that

$$\Delta_k(x) \ll x^{\frac{k-1}{k}} \log^{k-2} x.$$

Define $\alpha_k > 0$ as the least number of $\Delta_k(x)$ such that $\Delta_k(x) \ll x^{\alpha_k}$, the exact order of α_k is famously known as the general divisor problem which is still widely open up to-date. For a more comprehensive treatment, the interested readers may refer to Ivić [25, Chapter 13] and the references therein. The estimations of the average behaviour of classical divisor function over certain interesting sparse arithmetic sequences has also been investigated extensively in the literature, by appealing to some deep analytic methods from this fascinating area. For a historical line of developments, the readers may refer to [54] and the references therein.

For a number field K and positive integers k and n, it is a natural problem to consider the number of representations we can write n as a product of norms of k ideals in the ring of integers $\mathcal{O}_K$ of K. A number of authors are interested in investigating the average behaviour of the arithmetic function

$$\tau_k^K(n) = \sum_{N(\mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_k)=n} 1 = \sum_{n=n_1n_2\ldots n_k} a_K(n_1)a_K(n_2)\ldots a_K(n_k), \tag{7}$$

which is known as the k-dimensional divisor problem in the number field K. Namely, we are interested in the asymptotic behaviour of the sum

$$\sum_{n\leqslant x} \tau_k^K(n) = \sum_{N(\mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_k)\leqslant x} 1.$$

Clearly, using (7), one can infer that $\tau_k^K(n)$ is a multiplicative function of n. In 1988, Panteleeva [49] considered the divisor problem for quadratic and cyclotomic fields, establishing the corresponding asymptotic formulae for both cases. Let $K = \mathbb{Q}(\sqrt{D})$ for some square-free integer D, with $|D| \leqslant \log^2 x$. Then, for any integer $k \geqslant 1$, she showed that

$$\sum_{n\leqslant x} \tau_k^K(n) = xP_k(\log x) + \widetilde{\theta} x^{1-\frac{10}{133}k^{-\frac{2}{3}}} (C\log x)^{2k},$$

where P_k is a polynomial of degree $k-1$, $|\widetilde{\theta}| \leqslant 1$, and $C > 0$ is an absolute constant. For the cyclotomic field $K = \mathbb{Q}(\zeta_k)$, where ζ_k is a k-root of unity. Then, among other things, she successfully proved that

$$\sum_{n\leqslant x} \tau_k^K(n) = xP_k(\log x) + \widetilde{\theta} x^{1-\frac{1}{12}(\varphi(t)k)^{-\frac{2}{3}}} (C\log x)^{\varphi(t)k},$$

where P_k is a polynomial of degree $k-1$, $|\widetilde{\theta}| \leqslant 1$, and $C > 0$ is an absolute constant, and $\varphi(t)$ denotes the Euler's totient function. Afterwards, a number of authors extended the above divisor problems to several number fields in various settings (see, for example, [4, 10, 44]). For general divisor problem associated to holomorphic cusp forms, a number of authors investigated this profound topic and obtained some enlightening results (see, for instance, [17, 41, 42]).

Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D. In 2008, Fomenko [12] considered the second and third moments of $a_{K_3}(n)$ under the condition $D < 0$ and he proved that

$$\sum_{n\leqslant x} a_{K_3}^2(n) = c_1 x\log x + c_2 x + O(x^{\frac{9}{11}+\varepsilon}), \tag{8}$$

and

$$\sum_{n\leqslant x} a_{K_3}^3(n) = xP(\log x) + O(x^{\frac{73}{79}+\varepsilon}), \tag{9}$$

where c_1 and c_2 are some suitable constants, and $P(t)$ is a polynomial of t with degree 4. In 2013, Lü [43] refined the exponents in the error terms of (8) and (9) to $\frac{23}{31}$ and $\frac{235}{259}$, respectively. Recently, Liu [35] made further improvement concerning the error term of (9) to $\frac{1361}{1501}$, and he also considered the general divisor problem for non-normal cubic extension $K_3/\mathbb{Q}$. Very recently, the author and his collaborator [22] generalized the above results to

$$S_{K_3,\ell}(x) := \sum_{n\leqslant x} a_{K_3}^{\ell}(n)$$

for any fixed integer $\ell \geqslant 4$, by adopting the recent breakthrough of Newton and Thorne [47, 48], establishing the corresponding asymptotic formulae. More accurately, for any given integer $\ell \geqslant 4$, we successfully proved that

$$S_{K_3,\ell}(x) = xP_{K_3,\ell}(\log x) + O(x^{\alpha_\ell+\varepsilon}), \tag{10}$$

where $P_{K_3,\ell}(t)$ denotes a polynomial in t of degree η_ℓ, with η_ℓ determined by

$$\eta_\ell = \begin{cases} \kappa_{\ell,1} + \kappa_{\ell,2} - 1, & \ell = 2\ell_1 \geqslant 4, \\ \nu_{\ell,1} + \nu_{\ell,2} - 1, & \ell = 2\ell_2 + 1 \geqslant 5, \end{cases} \tag{11}$$

and $\alpha_\ell = 1 - \frac{2}{3^\ell}$, here, the constants $\kappa_{\ell,i}, \nu_{\ell,i}, i = 1, 2$ are given by

$$\kappa_{\ell,1} = 1 + \sum_{j=1}^{\ell_1} \binom{\ell}{2j} A_j, \quad \kappa_{\ell,2} = \sum_{j=1}^{\ell_1-1} \binom{\ell}{2j+1} D_j, \tag{12}$$

and

$$\nu_{\ell,1} = 1 + \sum_{j=1}^{\ell_2} \binom{\ell}{2j} A_j, \quad \nu_{\ell,2} = \sum_{j=1}^{\ell_2} \binom{\ell}{2j+1} D_j, \tag{13}$$

respectively, and the constants $A_j, D_j, j \geqslant 1$ are defined as

$$A_j = \frac{(2j)!}{j!(j+1)!}, \quad D_j = \frac{4 \cdot (2j+1)!}{(j-1)!(j+3)!}, \quad j \geqslant 1.$$

In order to understand the arithmetic functions in different aspects, an important direction is to consider the average behaviour of the arithmetic functions over certain sparse sequences. In 2017, Yang [57] derived the asymptotic formula for the sum

$$\widetilde{S}_{K_3,\ell}(x) := \sum_{n_1^2+n_2^2 \leqslant x} a_{K_3}^{\ell}(n_1^2+n_2^2)$$

with $\ell = 1$. In 2020, Hu and Wang [15] considered the average behaviour of higher moments of the arithmetic function $a_{K_3}(n)$ over sum of two squares by using the Rankin–Selberg theory, and, moreover, they proved that

$$\widetilde{S}_{K_3,\ell}(x) = xP_\ell(\log x) + O(x^{\theta_\ell+\varepsilon}) \tag{14}$$

for any $\varepsilon > 0$, where $2 \leqslant \ell \leqslant 8$ and $P_\ell(t)$ is a polynomial of t with degree η_ℓ,

$$\begin{array}{llll} \eta_2 = 1, & \eta_3 = 4, & \eta_4 = 12, & \eta_5 = 33, \\ \eta_6 = 88, & \eta_7 = 232, & \eta_8 = 609, & \end{array}$$

the exponents in the error terms are given by

$$\begin{array}{llll} \theta_2 = \dfrac{51}{59}, & \theta_3 = \dfrac{70}{73}, & \theta_4 = \dfrac{71}{72}, & \theta_5 = \dfrac{217}{218}, \\ \theta_6 = \dfrac{1987}{1990}, & \theta_7 = \dfrac{6047}{6050}, & \theta_8 = \dfrac{18356}{18359}. & \end{array}$$

Very recently, the author [20] generalized the result of Hu and Wang [15] to the cases for any fixed integer $\ell \geqslant 9$, by adopting the recent breakthrough of Newton and Thorne [47, 48], together with the nice analytic properties of the associated L-functions. In a more explicit expression, for any given integer $\ell \geqslant 9$, we are able to show that

$$\widetilde{S}_{K_3,\ell}(x) = xP^*_{K_3,\ell}(\log x) + O(x^{\beta_\ell+\varepsilon}),$$

where $P^*_{K_3,\ell}(t)$ denotes a polynomial in t of degree η_ℓ as defined in (11), and

$$\beta_\ell = \begin{cases} 1 - \frac{21}{7\cdot 3^{\ell+1}-4\kappa_{\ell,1}+3}, & \ell = 2\ell_1 \geqslant 10, \\ 1 - \frac{21}{7\cdot 3^{\ell+1}-4\nu_{\ell,1}+3}, & \ell = 2\ell_2+1 \geqslant 9, \end{cases}$$

where $\kappa_{\ell,1}$ and $\nu_{\ell,1}$ are defined as (12) and (13), respectively. In the meanwhile, Liu and Lao [38] independently refined the results of (14), and also generalized it to the cases for all $\ell \geqslant 2$, via a different approach in comparison with the author, by using the recent work of Newton and Thorne [47, 48], along with a different shifting the line of integration and better subconvexity bounds for the associated L-functions.

More recently, motivated by the above results, the author [21] established the asymptotic formulae for the summatory function

$$\sum_{n_1^2+n_2^2\leqslant x} \left(\tau_k^{K_3}\left(n_1^2+n_2^2\right)\right)^{\ell},$$

where $n_1, n_2 \in \mathbb{Z}$, and $k \geqslant 2, \ell \geqslant 2$ are any fixed positive integers. More precisely, for $k \geqslant 2, \ell \geqslant 2$ being any given integers and for any $\varepsilon > 0$, the author established the following

$$S_{K_3,k,\ell}(x) = x\overline{P}_{K_3,k,\ell}(\log x) + O(x^{\vartheta_{k,\ell}+\varepsilon}),$$

where $\overline{P}_{K_3,k,\ell}(t)$ is a polynomial in t of degree

$$\deg \overline{P}_{K_3,k,\ell} = \begin{cases} k^{\ell}(\kappa_{\ell,1}+\kappa_{\ell,2})-1, & \ell = 2\ell_1 \geqslant 2, \\ k^{\ell}(\nu_{\ell,1}+\nu_{\ell,2})-1, & \ell = 2\ell_2+1 \geqslant 3, \end{cases}$$

and the exponents are given by

$$\vartheta_{k,\ell} = \begin{cases} 1-\frac{210}{1537k^2+30}, & \ell = 2, \\ 1-\frac{210}{4831k^3+30}, & \ell = 3, \\ 1-\frac{21}{21\cdot(3k)^{\ell}-4\kappa_{k,\ell,1}+3}, & \ell = 2\ell_1 \geqslant 4, \\ 1-\frac{21}{21\cdot(3k)^{\ell}-4\nu_{k,\ell,1}+3}, & \ell = 2\ell_2+1 \geqslant 5, \end{cases}$$

where $\kappa_{k,\ell,1} = k^{\ell}\kappa_{\ell,1}$ and $\nu_{k,\ell,1} = k^{\ell}\nu_{\ell,1}$, and the constants $\kappa_{\ell,1}, \kappa_{\ell,2}$ and $\nu_{\ell,1}, \nu_{\ell,2}$ are defined as (12) and (13), respectively. For the results related to the Fourier coefficients of holomorphic cusp forms over certain sparse sequence of positive integers, the interested readers can refer to [16, 18, 19, 36, 56] and various illuminating references therein.

Recently, there are a flurry of activities towards the average behaviour of coefficients of Dedekind zeta function for $K_3/\mathbb{Q}$ over certain sparse sequence of positive integers. Before proceeding further, we introduce Hypothesis μ concerning the subconvexity of $\zeta(s)$ as follows.

Hypothesis μ. *There exists a least real number μ such that*

$$\zeta\Big(\frac{1}{2}+it\Big) \ll (1+|t|)^{\mu+\varepsilon} \tag{15}$$

for any $\varepsilon > 0$, where $\mu = \mu(\frac{1}{2})$.

It is well known that the Phragmén–Lindelöf principle leads to

$$\zeta(\sigma+it) \ll (1+|t|)^{2\mu(1-\sigma)+\varepsilon}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$. Very recently, Hiary, Patel and Yang [23] derived the explicit subconvexity bound for $\zeta(s)$ which yields that

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \leqslant 0.618 t^{\frac{1}{6}} \log t,$$

by employing a new version of Kusmin–Landau bound. The best record up to-date towards Hypothesis μ is due to Bourgain [1] with $\mu = \frac{13}{84}$, hence, one has

$$\zeta(\sigma + it) \ll (1 + |t|)^{\frac{13}{42}(1-\sigma)+\varepsilon}$$

unconditionally for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $|t| \geqslant 1$. The celebrated Lindelöf Hypothesis asserts that $\mu(\frac{1}{2}) = 0$, and this remains one of the most challenging open problems in modern number theory.

In 2023, for any given integer $k \geqslant 1$, Sharma and Sankaranarayanan [54] considered the asymptotic formula of the summatory function

$$\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \tau_k^{K_3}(n).$$

In fact, they successfully established the following

$$\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \tau_k^{K_3}(n) = x^3 P_{k-1}(\log x) + \widetilde{E}_{K_3,k}(x),$$

where $P_{k-1}(t)$ is a polynomial in t of degree $k-1$, and the error term $\widetilde{E}_{K_3,k}(x)$ can be evaluated explicitly for which it gives a non-trivial upper bound. Here, the error term $\widetilde{E}_{K_3,k}(x)$ is closely linked with Hypothesis μ.

Inspired by the above enlightening results, in this paper the principal purpose is to consider the higher power moments analogue of the result obtained by Sharma and Sankaranarayanan [54]. More precisely, we consider the asymptotic behaviour of the following summatory function

$$S_{K_3,k,\ell}(x) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2 \leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6) \in \mathbb{Z}^6}} \left(\tau_k^{K_3}(n)\right)^\ell, \tag{16}$$

where $k \geqslant 2, \ell \geqslant 2$ are any fixed positive integers. More accurately, we are able to establish the result as follows.

Theorem 1.1. *Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D with $D < 0$. Let $k \geqslant 2, \ell \geqslant 2$ be any given integers, and let $S_{K_3,k,\ell}(x)$ be defined as (16). Then, for any $\varepsilon > 0$, we have*

$$S_{K_3,k,\ell}(x) = x^3 P_{K_3,k,\ell}(\log x) + O\big(x^{\alpha_{k,\ell}+\varepsilon}\big),$$

where $P_{K_3,k,\ell}(t)$ *denotes a polynomial in* t *with degree* $\nu_{k,\ell,1}+\nu_{k,\ell,4}-1$*, and the constants* $\nu_{k,\ell,i}, 1 \leqslant i \leqslant 4$ *are defined as* (35)*. Here, the exponents* $\alpha_{k,\ell}$ *are given by* $\alpha_{k,\ell}=3-\frac{2}{7\theta_{k,\ell}}$*, and* $\theta_{k,\ell}$ *are determined by*

$$\theta_{k,\ell}=\begin{cases}\frac{1}{7}(3k)^{\ell}+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu+\frac{29}{126}, & \\ \frac{1}{7}(3k)^{\ell}-\frac{8}{147}\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}+\frac{47}{882}, & \text{if } \mu=\frac{13}{84}\\ \frac{1}{7}(3k)^{\ell}-\frac{1}{7}\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}+\frac{29}{126}, & \text{if } \mu=0.\end{cases} \tag{17}$$

The interplay of number theory and statistics revealed fruitful results in the literature, and has also received considerable attentions in recent decades, see, for example, [36, 37, 46]. For a random variable X defined on a countable sample space $\mathbb{V}$, we denote by $E(X)$ and $\mathrm{Var}(X)$ the mathematical expectation and variance of X, respectively. As a direct application of Theorem 1.1, we can obtain the asymptotic formulae of the variances of $(\tau_k^{K_3}(n))^{\ell}, k \geqslant 2, \ell \geqslant 2$ for

$$\mathcal{D}:=\left\{n\in\mathbb{Z}^{+}:1\leqslant n\leqslant x,\quad n=\sum_{j=1}^{6}a_j^2,\quad \boldsymbol{a}\in\mathbb{Z}^6\right\}, \tag{18}$$

denoted by $\mathrm{Var}((\tau_k^{K_3}(n))^{\ell})_{\mathcal{D}}$. Here, we set $\boldsymbol{a}=(a_1,a_2,a_3,a_4,a_5,a_6)$. More precisely, we have the following result.

Theorem 1.2. *Let* $k \geqslant 2, \ell \geqslant 2$ *be any given integers, and let* $K_3/\mathbb{Q}$ *be a non-normal cubic extension, which is given by an irreducible polynomial* $h(x)=x^3+Ax^2+Bx+C$ *of discriminant* D *with* $D<0$*. Let* $\mathcal{D}$ *be defined as* (18)*. Then,*

$$\mathrm{Var}\big((\tau_k^{K_3}(n))^{\ell}\big)_{\mathcal{D}}=\widetilde{P}_{K_3,k,2\ell}(\log x)+O(x^{-\frac{2}{7\theta_{k,2\ell}}+\varepsilon}),$$

where $\widetilde{P}_{K_3,k,2\ell}(t)$ *is a polynomial in* t *with degree* $\nu_{k,2\ell,1}+\nu_{k,2\ell,4}-1$*. Here,* $\theta_{k,2\ell}$ *are given by* (17)*, and the constants* $\nu_{k,\ell,i}, 1 \leqslant i \leqslant 4$ *are defined as* (35)*.*

The organization of this paper is arranged as follows. In Section 2, we introduce some preliminaries and also give some useful lemmas. In Section 3, we are devoted to the proofs of two main propositions, which are crucial to the proofs of the main results in this paper. In Section 4, we complete the proofs of Theorems 1.1 and 1.2.

Throughout the paper, we assume that $\varepsilon > 0$ is an arbitrarily small number which may vary in different occurrences. Let $\binom{\ell}{j}$ be the binomial coefficient, with the convention that $\binom{\ell}{j}=0$ provided $j<0$. And we always denote by p a prime number.

2. PRELIMINARIES

In this section, we review some analytic properties of automorphic L-functions and introduce some useful lemmas, which play an essential role in the proof of the main results in this paper.

Let K_3 be a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D. If $D < 0$, from the paper of Fomenko [12] we know that

$$\zeta_{K_3}(s) = \zeta(s)L(f, s), \tag{19}$$

where f is a holomorphic cusp form of weight 1 with respect to the congruence group $\Gamma_0(|D|)$. If f is a holomorphic cusp form of integral weight κ for the congruence group $\Gamma_0(N)$, then f admits a Fourier expansion at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz), \qquad \Im(z) > 0,$$

where $e(z) = e^{2\pi iz}$, and the Fourier coefficients $\lambda_f(n) \in \mathbb{R}$ are Hecke eigenvalues of the Hecke operators T_n with $\lambda_f(1) = 1$. It is well known that for each p, there exist two complex numbers $\alpha_f(p), \beta_f(p)$ such that

$$\lambda_f(p^\nu) = \frac{\alpha_f(p)^{\nu+1} - \beta_f(p)^{\nu+1}}{\alpha_f(p) - \beta_f(p)} \qquad (\nu \geqslant 1)$$

and

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p) p^{-\frac{1}{2}}, \quad \beta_f(p) = 0, & \text{if } p \| D|, \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1, & \text{if } p \nmid |D|, \end{cases}$$

with $\varepsilon_f(p) = \pm 1$. By Deligne's bound [9], we also have

$$|\lambda_f(n)| \ll n^{\varepsilon}$$

for any $\varepsilon > 0$. For more background and details, the interested reader can refer to [33, Section 1].

From (19), we have the convolution

$$a_{K_3}(n) = \sum_{d|n} \lambda_f(d).$$

In particular, we have

$$a_{K_3}(p) = 1 + \lambda_f(p). \tag{20}$$

Let K be a number field of degree d over the rational field $\mathbb{Q}$, and let $\zeta_K(s)$ be the Dedekind zeta function of the field K. Then from the definition of $\tau_k^K(n)$ in (7), we have

$$\zeta_K(s)^k = \sum_{\mathfrak{a}_1} \sum_{\mathfrak{a}_2} \cdots \sum_{\mathfrak{a}_k} \big(N_{K/\mathbb{Q}}(\mathfrak{a}_1) N_{K/\mathbb{Q}}(\mathfrak{a}_2) \ldots N_{K/\mathbb{Q}}(\mathfrak{a}_k)\big)^{-s}$$

$$= \sum_{n=1}^{\infty} \frac{\tau_k^K(n)}{n^s}, \qquad \Re(s) > 1. \tag{21}$$

Since $\tau_k^K(n)$ is also multiplicative, by (4), then

$$\begin{aligned} \tau_k^K(n) &= \sum_{n=n_1\ldots n_k} a_K(n_1)\ldots a_K(n_k) \\ &\ll \sum_{n=n_1\ldots n_k} \big(\tau(n_1)\ldots\tau(n_k)\big)^{d-1} \ll n^{\varepsilon}. \end{aligned} \tag{22}$$

The j-th symmetric power L-function attached to f is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \big(1 - \alpha_f(p)^{j-m}\beta_f(p)^m p^{-s}\big)^{-1}$$

for $\Re(s) > 1$. We may expand it into a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} \\ &= \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right), \quad \Re(s) > 1. \end{aligned}$$

In particular, we have $L(\mathrm{sym}^0 f, s) = \zeta(s)$, and $L(\mathrm{sym}^1 f, s) = L(f, s)$. Apparently, $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function of n.

Let χ be a Dirichlet character modulo q. Similarly, for $j \geqslant 1$, the j-th twisted symmetric L-function is defined as

$$\begin{aligned} L(\mathrm{sym}^j f \otimes \chi, s) &= \prod_p \prod_{m=0}^{j} \big(1 - \alpha_f(p)^{j-m}\beta_f(p)^m \chi(p) p^{-s}\big)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}, \qquad \Re(s) > 1. \end{aligned}$$

For $(p, |D|) = 1$, it is standard (cf. [33]) to find that

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m,$$

which can be written as

$$\lambda_f(p^j) = \lambda_{\mathrm{sym}^j f}(p) = U_j\big(\lambda_f(p)/2\big), \tag{23}$$

where $U_j(x)$ is the j-th Chebyshev polynomial of the second kind.

Associated to a primitive cusp form f, there is an automorphic cuspidal representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$ and hence, an automorphic L-function $L(\pi_f, s)$

which coincides with $L(f,s)$, namely

$$L(\pi_f,s)=L(f,s).$$

It is predicted by the Langlands functoriality conjecture that π_f gives rise to a symmetric power lift $\mathrm{sym}^j\pi_f$-an automorphic representation whose L-function is the symmetric power L-function attached to f,

$$L(\mathrm{sym}^j\pi_f,s)=L(\mathrm{sym}^j f,s).$$

For the known cases, the lifts are cuspidal, namely, there exists an automorphic cuspidal self-dual representation, denote by $\mathrm{sym}^j\pi_f$ of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function is the same as $L(\mathrm{sym}^j f,s)$. For $j=1,2,3,4$, this special Langlands functoriality conjecture that $\mathrm{sym}^j f$ is automorphic is shown by a series of important works. See, for example, Gelbert and Jacquet [13], Kim [28], Kim and Shahidi [29,30], and Shahidi [53]. Later, Dieulefait [11] and Clozel and Thorne [6–8] investigated the cases $j\leqslant 8$. Very recently, Newton and Thorne [47,48] proved that $\mathrm{sym}^j f$ corresponds with a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j\geqslant 1$. In particular, in [48, Theorem A.1] the authors established the existence of the symmetric power liftings $\mathrm{sym}^n\pi_f$ for all $n\geqslant 1$ regarding the automorphy of the symmetric power lifting for cuspidal Hecke eigenforms of weight 1. Hence, the symmetric power L-function $L(\mathrm{sym}^j f,s)$ for any $j\geqslant 1$ is an entire function and satisfies certain functional equation. For the primitive holomorphic cusp forms f considered in our occurrence, the result of Fomenko [12] gives that $L(\mathrm{sym}^3 f,s)$ can be analytically extended to the whole complex plane except for a simple pole at $s=1$.

Let

$$r_k(n):=\#\big\{(a_1,a_2,\ldots,a_k)\in\mathbb{Z}^k : n=a_1^2+a_2^2+\cdots+a_k^2\big\}, \tag{24}$$

allowing zeros, distinguishing signs, and orders. For $k=6$ in (24), we learn from [55, Lemma 2.1] that

$$r_6(n)=16\sum_{d|n}\chi(n/d)d^2-4\sum_{d|n}\chi(d)d^2,$$

where χ is the non-principal Dirichlet character modulo 4, i.e.,

$$\chi(n)=\begin{cases}1, & \text{if } n\equiv 1(\mathrm{mod}4),\\ -1, & \text{if } n\equiv -1(\mathrm{mod}4),\\ 0, & \text{if } n\equiv 0(\mathrm{mod}2).\end{cases} \tag{25}$$

We can also rewrite $r_6(n)$ as

$$r_6(n)=16\sum_{d|n}\chi(d)\frac{n^2}{d^2}-4\sum_{d|n}\chi(d)d^2$$

$$:= 16l(n) - 4v(n) := l_1(n) - v_1(n).$$

Clearly, the functions $l(n)$ and $v(n)$ are both multiplicative, due to the fact that the non-principal character χ is multiplicative.

Note that

$$\begin{aligned} l(p) &= p^2 + \chi(p), \\ l(p^2) &= p^4 + p^2\chi(p) + \chi(p^2), \end{aligned} \tag{26}$$

and

$$\begin{aligned} v(p) &= 1 + p^2\chi(p), \\ v(p^2) &= 1 + p^2\chi(p) + p^4\chi(p^2). \end{aligned} \tag{27}$$

For $S_{K_3,k,\ell}(x)$ defined as (16), we can reinterpret it as

$$\begin{aligned} S_{K_3,k,\ell}(x) &= \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \big(\tau_k^{K_3}(n)\big)^\ell \\ &= \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x \\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} 1 \\ &= \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell r_6(n) \\ &= 16\sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell l(n) - 4\sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell v(n) \\ &= \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell l_1(n) - \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell v_1(n) \\ &:= S_{K_3,k,\ell,1}(x) - S_{K_3,k,\ell,2}(x), \end{aligned} \tag{28}$$

where

$$S_{K_3,k,\ell,1}(x) = \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell l_1(n), \tag{29}$$

and

$$S_{K_3,k,\ell,2}(x) = \sum_{n\leqslant x}\big(\tau_k^{K_3}(n)\big)^\ell v_1(n), \tag{30}$$

and $k \geqslant 2, \ell \geqslant 2$ are any given integers. Then, for $\Re(s) > 3$, we can define the associated L-series by

$$L_{K_3,k,\ell}(s) = \sum_{n=1}^{\infty}\frac{(\tau_k^{K_3}(n))^\ell l_1(n)}{n^s} = 16\sum_{n=1}^{\infty}\frac{(\tau_k^{K_3}(n))^\ell l(n)}{n^s}, \tag{31}$$

and

$$\mathcal{L}_{K_3,k,\ell}(s)=\sum_{n=1}^{\infty}\frac{(\tau_k^{K_3}(n))^{\ell}v_1(n)}{n^s}=4\sum_{n=1}^{\infty}\frac{(\tau_k^{K_3}(n))^{\ell}v(n)}{n^s}, \tag{32}$$

respectively. By the multiplicative property of $(\tau_k^{K_3}(n))^{\ell}, l(n), v(n)$, for $\Re(s)>1$, one has

$$L_{K_3,k,\ell}(s)=16\prod_p\left(1+\sum_{m\geqslant 1}\frac{(\tau_k^{K_3}(p^m))^{\ell}l(p^m)}{p^{ms}}\right), \tag{33}$$

and

$$\mathcal{L}_{K_3,k,\ell}(s)=4\prod_p\left(1+\sum_{m\geqslant 1}\frac{(\tau_k^{K_3}(p^m))^{\ell}v(p^m)}{p^{ms}}\right). \tag{34}$$

Now, we state an important result due to Lü and Ma [44].

LEMMA 2.1 ([44, Lemma 2.1]). *Let K be an algebraic number field of finite degree d over the rational field $\mathbb{Q}$, and the function $\tau_k^K(n)$ is defined as* (7)*. Then, for any prime p, we have*

$$\tau_k^K(p)=ka_K(p).$$

For simplicity, we write

$$\prod_j{}^{\flat}L(\mathrm{sym}^{n_j}f,s)^{r_j}:=\prod_j\big(L(\mathrm{sym}^j f,s-2)L(\mathrm{sym}^{n_j}f\otimes\chi,s)\big)^{r_j},$$

and

$$\prod_j{}^{\dagger}L(\mathrm{sym}^{n_j}f,s)^{r_j}:=\prod_j\big(L(\mathrm{sym}^j f,s)L(\mathrm{sym}^{n_j}f\otimes\chi,s-2)\big)^{r_j},$$

where $n_j\geqslant 0, r_j\geqslant 1, j\geqslant 0$ are any given integers, and χ is a non-principal character modulo 4. We assume the convention that $\prod_0{}^{\flat},\prod_0{}^{\dagger}$ denote the products with 1.

In order to obtain the asymptotic behaviour of $S_{K_3,k,\ell}(x)$, we need to decompose the associated L-series $L_{K_3,k,\ell}(s)$ and $\mathcal{L}_{K_3,k,\ell}(s)$ given by (33) and (34), respectively, into the product of lower degree irreducible L-functions, along with a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s)\geqslant\frac{5}{2}+\varepsilon$ for any $\varepsilon>0$.

LEMMA 2.2. *Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is determined by an irreducible polynomial $h(x)=x^3+Ax^2+Bx+C$ of discriminant D, with $D<0$. Let $k\geqslant 2,\ell\geqslant 2$ be any fixed integers, and let $L_{K_3,k,\ell}(s)$ be defined by* (31)*. Then*

$$L_{K_3,k,\ell}(s)=G_{K_3,k,\ell}(s)U_{k,\ell}(s),$$

where $G_{K_3,k,\ell}(s)$ *is an* L*-function of degree* $(3k)^\ell$ *of the type*

$$G_{K_3,k,\ell}(s) = \Big({\prod_{1\leqslant j\leqslant 4}}^{\flat} L(\mathrm{sym}^{j-1} f, s)^{\nu_{k,\ell,j}} \Big) {\prod}^{\flat} \widetilde{G}_{K_3,k,\ell}(s),$$

where $\widetilde{G}_{K_3,k,\ell}(s)$ *is an* L*-function of degree* $(3k)^\ell - \sum_{j=1}^4 j\nu_{k,\ell,j}$*, which can be represented as*

$$\widetilde{G}_{K_3,k,\ell}(s) = \prod_j L(\mathrm{sym}^{r_j} f, s)^{\omega_j}$$

for suitable constants $r_j \geqslant 4, \omega_j \geqslant 0, j \geqslant 0$*, and the exponents* $\nu_{k,\ell,j}, 1 \leqslant j \leqslant 4$ *are defined as*

$$\nu_{k,\ell,j} = \begin{cases} \kappa_{k,\ell,j}, & \ell = 2\ell_1 \geqslant 2, \\ \widetilde{\kappa}_{k,\ell,j}, & \ell = 2\ell_2 + 1 \geqslant 3, \end{cases} \tag{35}$$

and the exponents $\kappa_{k,\ell,j}, 1 \leqslant j \leqslant 4$ *are given by*

$$\kappa_{k,\ell,1} = k^\ell \Big(1 + \sum_{i=1}^{\ell_1} \binom{\ell}{2i} A_i\Big), \quad \kappa_{k,\ell,2} = k^\ell \Big(\ell + \sum_{i=1}^{\ell_1 - 1} \binom{\ell}{2i+1} B_i\Big),$$

$$\kappa_{k,\ell,3} = k^\ell \sum_{i=1}^{\ell_1} \binom{\ell}{2i} C_i, \qquad \kappa_{k,\ell,4} = k^\ell \sum_{i=1}^{\ell_1 - 1} \binom{\ell}{2i+1} D_i,$$

and the exponents $\widetilde{\kappa}_{k,\ell,j}, 1 \leqslant j \leqslant 4$ *are given by*

$$\widetilde{\kappa}_{k,\ell,1} = k^\ell \Big(1 + \sum_{j=1}^{\ell_2} \binom{\ell}{2j} A_j\Big), \quad \widetilde{\kappa}_{k,\ell,2} = k^\ell \Big(\ell + \sum_{j=1}^{\ell_2} \binom{\ell}{2j+1} B_j\Big),$$

$$\widetilde{\kappa}_{k,\ell,3} = k^\ell \sum_{j=1}^{\ell_2} \binom{\ell}{2j} C_j, \qquad \widetilde{\kappa}_{k,\ell,4} = k^\ell \sum_{j=1}^{\ell_2} \binom{\ell}{2j+1} D_j.$$

The constants A_j, B_j, C_j *and* D_j *with* $j \geqslant 1$ *are given by*

$$A_j = \frac{(2j)!}{j!(j+1)!}, \qquad B_j = 2\frac{(2j+1)!}{j!(j+2)!},$$

$$C_j = \frac{3\cdot(2j)!}{(j-1)!(j+2)!}, \qquad D_j = \frac{4\cdot(2j+1)!}{(j-1)!(j+3)!}.$$

Here, the Dirichlet series $U_{k,\ell}(s)$ *converges absolutely and uniformly in the half-plane* $\Re(s) \geqslant \frac{5}{2} + \varepsilon$ *and* $U_{k,\ell}(s) \neq 0$ *with* $\Re(s) = 3$.

Proof. From (20), (26) and Lemma 2.1, along with the binomial expansion, we learn that

$$\big(ka_{K_3}(p)\big)^\ell l(p) = k^\ell \big(1 + \lambda_f(p)\big)^\ell \big(p^2 + \chi(p)\big)$$

$$= k^\ell \Big(\sum_{n=0}^{\ell} \binom{\ell}{n} \lambda_f^n(p) \Big) \big(p^2 + \chi(p) \big). \tag{36}$$

From the result of Lau and Lü [32, Lemma 7.1], for $j = 2m, m \geqslant 1$, we have

$$\lambda_f^j(p) = A_m + \sum_{1 \leqslant u \leqslant m-1} C_m(u) \lambda_{\mathrm{sym}^{2u} f}(p) + \lambda_{\mathrm{sym}^{2m} f}(p), \tag{37}$$

and, for $j' = 2m' + 1, m' \geqslant 1$,

$$\lambda_f^{j'}(p) = B_{m'} \lambda_f(p) + \sum_{1 \leqslant u' \leqslant m'-1} D_{m'}(u') \lambda_{\mathrm{sym}^{2u'+1} f}(p) + \lambda_{\mathrm{sym}^{2m'+1} f}(p), \tag{38}$$

where $A_m, B_{m'}, C_m(u)$ and $D_{m'}(u')$ are suitable constants which can be evaluated explicitly as in [32, Lemma 7.1]. In the right half-plane $\Re(s) > 3$, the p-th coefficient of the Euler product determines the analytic properties of the L-series $L_{K_3,k,\ell}(s)$.

Therefore, the term $(k a_{K_3}(p))^\ell l(p)$ in (36) can be rephrased as

$$\big(k a_{K_3}(p) \big)^\ell l(p) = \sum_{j \geqslant 1} \omega_j \big(p^2 \lambda_{\mathrm{sym}^{n_j} f}(p) + \lambda_{\mathrm{sym}^{n_j} f}(p) \chi(p) \big), \tag{39}$$

where $\omega_j, n_j \geqslant 0$ are some suitable constants. Set

$$b(p) = \sum_{j \geqslant 1} \omega_j \big(p^2 \lambda_{\mathrm{sym}^{n_j} f}(p) + \lambda_{\mathrm{sym}^{n_j} f}(p) \chi(p) \big) \tag{40}$$

and extends to all integers n for $b(n)$ using multiplicativity, by defining the automorphic L-function

$$G_{K_3,k,\ell}(s) := \prod_j{}^{\flat} L(\mathrm{sym}^{n_j} f, s)^{\omega_j} := \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{41}$$

From (33), (36), (39)–(41), we can obtain

$$\begin{aligned} L_{K_3,k,\ell}(s) &= G_{K_3,k,\ell}(s) \cdot 16 \bigg(1 + \frac{(k a_{K_3}(p^2))^\ell l(p^2) - b(p^2)}{p^{2s}} + \cdots \bigg) \\ &:= G_{K_3,k,\ell}(s) U_{k,\ell}(s), \end{aligned} \tag{42}$$

where $U_{k,\ell}(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geqslant \frac{5}{2} + \varepsilon$.

Now, it remains to determine the exponents of the associated L-functions in the decomposition of $G_{K_3,k,\ell}(s)$. In the half-plane $\Re(s) > \frac{5}{2}$, the coefficients of p^{-s} determine the analytic property of $L_{K_3,k,\ell}(s)$. On combining (36)–(38), and comparing the p-th coefficients of both sides of (42), we can calculate the exponents of associated L-functions as depicted in the next lemma. $\square$

LEMMA 2.3. *Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is determined by an irreducible polynomial $h(x) = x^3 + Ax^2 + Bx + C$ of discriminant D, with $D < 0$. Let $k \geqslant 2, \ell \geqslant 2$ be any fixed integers, and let $\mathcal{L}_{K_3,k,\ell}(s)$ be defined by* (32)*. Then*

$$\mathcal{L}_{K_3,k,\ell}(s) = H_{K_3,k,\ell}(s)\widetilde{U}_{k,\ell}(s),$$

where $H_{K_3,k,\ell}(s)$ is an L-function of degree $(3k)^\ell$ of the type

$$H_{K_3,k,\ell}(s) = \Bigg(\prod_{1\leqslant j\leqslant 4}{}^{\dagger} L(\mathrm{sym}^{j-1}f, s)^{\nu_{k,\ell,j}}\Bigg)\prod{}^{\dagger}\widetilde{H}_{K_3,k,\ell}(s),$$

where $\widetilde{H}_{K_3,k,\ell}(s)$ is an L-function of degree $(3k)^\ell - \sum_{j=1}^4 j\nu_{k,\ell,j}$, which can be represented as

$$\widetilde{H}_{K_3,k,\ell}(s) = \prod_j L(sym^{r_j}f, s)^{\omega_j} \tag{43}$$

for suitable constants $r_j \geqslant 4, \omega_j \geqslant 0, j \geqslant 0$, and the exponents $\nu_{k,\ell,j}, 1 \leqslant j \leqslant 4$ are defined as in (35)*. Here, the Dirichlet series $\widetilde{U}_{k,\ell}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geqslant \frac{5}{2} + \varepsilon$ and $\widetilde{U}_{k,\ell}(s) \neq 0$ with $\Re(s) = 3$.*

Proof. This can be handled in a similar approach along the line of argument as that of Lemma 2.2, on noting

$$\begin{aligned}\big(ka_{K_3}(p)\big)^\ell v(p) &= k^\ell\big(1+\lambda_f(p)\big)^\ell\big(1+p^2\chi(p)\big)\\ &= k^\ell\Bigg(\sum_{n=0}^{\ell}\binom{\ell}{n}\lambda_f^n(p)\Bigg)\big(1+p^2\chi(p)\big).\end{aligned}$$

This completes the proof of Lemma 2.3. $\square$

Remark 2.4. In the half-plane $\Re(s) > \frac{1}{2}$, we learn from Fomenko [12] that $L(\mathrm{sym}^3 f, s)$ has an analytic continuation to that half-plane except for a simple pole at $s = 1$ though in most cases it is entire. Then, we learn from Lemmas 2.2 and 2.3 that the factorizations of $L_{K_3,k,\ell}(s)$ and $\mathcal{L}_{K_3,k,\ell}(s)$ in the same half-plane have a pole of finite order at $s = 1$ which comes from the factors $\zeta(s)$ and $L(\mathrm{sym}^3 f, s)$.

To prove the main result, we also need the following individual or average subconvexity bounds for the associated automorphic L-functions.

LEMMA 2.5. *For any $\varepsilon > 0$, one has*

$$\int_1^T \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt \ll T^{1+\varepsilon} \tag{44}$$

uniformly for $T \geqslant 1$.

Proof. The result follows from the work of Ivić [24]. □

LEMMA 2.6 ([52, Lemma 1]). *For $\frac{1}{2} \leqslant \sigma \leqslant 2$, and T sufficiently large, there exists a $T^* \in [T, T+T^{1/3}]$ such that the bound*

$$\log \zeta(\sigma + iT^*) \ll (\log\log T^*)^2 \ll (\log\log T)^2$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$, and hence

$$|\zeta(\sigma + iT^*)| \ll \exp\big((\log\log T^*)^2\big) \ll T^{\varepsilon} \tag{45}$$

on the horizontal line with $t = T^*$ for $\frac{1}{2} \leqslant \sigma \leqslant 2$.

Remark 2.7. By adopting a similar argument as that of [39, Lemma 2.6], we know that (45) also holds for Dirichlet L-function $L(s,\chi)$ in the t-aspect. Here, as in the context, χ is a non-principal character with modulus 4.

LEMMA 2.8. *For any $\varepsilon > 0$, we have*

$$L(f, \sigma + it) \ll (1+|t|)^{\max\{\frac{2}{3}(1-\sigma),0\}+\varepsilon} \tag{46}$$

and

$$L(\mathrm{sym}^2 f, \sigma + it) \ll (1+|t|)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon} \tag{47}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$.

Proof. The results follow from Phragmén–Lindelöf convexity principle for a strip, along with the impressive work of Good [14] and Lin, Nunes and Qi [34, Corollary 1.2], respectively. □

LEMMA 2.9. *Let χ be a non-principal character modulo 4, for any $\varepsilon > 0$, we have*

$$\int_1^T \left|L\left(\frac{5}{7} + it, \chi\right)\right|^{12} dt \ll T^{1+\varepsilon}$$

uniformly for $T \geqslant 1$, and

$$L(f \otimes \chi, \sigma + it) \ll (1+|t|)^{\max\{\frac{2}{3}(1-\sigma),0\}+\varepsilon},$$

$$L(\mathrm{sym}^2 f \otimes \chi, \sigma + it) \ll (1+|t|)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon},$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$ and $|t| \geqslant 1$.

Proof. By adopting a similar argument as that of [39, Lemma 2.6], it can be seen that twisting L-functions by a character χ does not affect the subconvexity bounds, convexity bounds and integral mean estimates of the corresponding L-functions in the t's aspect. For a detailed description, the interested readers are invited to refer to [36, Lemma 2.7]. □

Remark 2.10. By following a similar argument as in [39, Lemma 2.6], one can also show that Hypothesis μ also holds for $L(s,\chi)$ in the t-aspect, where χ is the non-principal character modulo 4.

Let $\boldsymbol{d} := (d_1,\dots,d_J)$, $\boldsymbol{m} = (m_1,\dots,m_J)$ with $d_j, m_j \in (\mathbb{N}\cup\{0\})$, and set

$$A(\boldsymbol{d},\boldsymbol{m}) = \frac{1}{2}\sum_{j=1}^{J} d_j(m_j+1).$$

Let χ be a primitive character modulo q, and define

$$\mathfrak{L}_{\boldsymbol{m}}^{\boldsymbol{d}}(f,\chi,s) := \prod_{j=1}^{J} L(\mathrm{sym}^{m_j} f\otimes\chi, s)^{d_j}, \tag{48}$$

this general L-function is in the sense of Perelli [51] due to the recent deep works of Newton and Thorne [47, 48]. The following lemma follows plainly from Perelli [51, Theorem 4].

LEMMA 2.11. *Let* $\mathfrak{L}_{\boldsymbol{m}}^{\boldsymbol{d}}(f,\chi,s)$ *be defined as in* (48), *for any* $\varepsilon > 0$, *we have*

$$\int_T^{2T} |\mathfrak{L}_{\boldsymbol{m}}^{\boldsymbol{d}}(f,\chi,\sigma+it)|^2 dt \ll_{f,\varepsilon,\boldsymbol{d},\boldsymbol{m}} \big(q(1+T)\big)^{2A(\boldsymbol{d},\boldsymbol{m})(1-\sigma)+\varepsilon} \tag{49}$$

uniformly for $\frac{1}{2}\leqslant\sigma\leqslant 1+\varepsilon$ *and* $T\geqslant 1$, *and*

$$\mathfrak{L}_{\boldsymbol{m}}^{\boldsymbol{d}}(f,\chi,\sigma+it) \ll_{f,\varepsilon,\boldsymbol{d},\boldsymbol{m}} \big(q(1+|t|)\big)^{A(\boldsymbol{d},\boldsymbol{m})(1-\sigma)+\varepsilon} \tag{50}$$

uniformly for $\frac{1}{2}\leqslant\sigma\leqslant 1+\varepsilon$ *and* $|t|\geqslant 1$.

Proof. This can be derived by following an argument similar to that of Zou et al. [58, Lemmas 8], which was originally deduced from Jiang and Lü [27, Lemma 2.4]. □

3. THE MAIN PROPOSITIONS

In order to obtain the asymptotic formula for $S_{K_3,k,\ell}(x)$, one needs to demonstrate the following two propositions concerning the asymptotic behaviour of $S_{K_3,k,\ell,1}(x)$ and $S_{K_3,k,\ell,2}(x)$ as defined in (29) and (30), respectively. Now, we are in the stage to prove the following main propositions.

PROPOSITION 3.1. *Let* $K_3/\mathbb{Q}$ *be a non-normal cubic extension, which is given by an irreducible polynomial* $h(x) = x^3 + Ax^2 + Bx + C$ *of discriminant* D *with* $D < 0$. *Let* $k \geqslant 2, \ell \geqslant 2$ *be any given integers, and let* $S_{K_3,k,\ell,1}(x)$ *be defined as* (29). *Then, for any* $\varepsilon > 0$, *we have*

$$S_{K_3,k,\ell,1}(x) = x^3 P_{K_3,k,\ell}(\log x) + O(x^{3-\frac{2}{7\theta_{k,\ell}}+\varepsilon}),$$

where $P_{K_3,k,\ell}(t)$ denotes a polynomial in t with degree $\nu_{k,\ell,1}+\nu_{k,\ell,4}-1$, and the constant $\theta_{k,\ell}$ is given by

$$\theta_{k,\ell} = \frac{1}{7}(3k)^{\ell} + \frac{1}{7}(4\mu-1)\nu_{k,\ell,1} - \frac{2}{21}\nu_{k,\ell,2} - \frac{8}{7}\mu + \frac{29}{126}, \tag{51}$$

and the constants $\nu_{k,\ell,i}, 1 \leqslant i \leqslant 4$ are defined as (35). *Here, μ is the least non-negative number appearing in* (15).

Proof. Applying Perron's formula (see, e.g., [40, Theorem 2.1]) and invoking Lemma 2.2, we have

$$\begin{aligned} S_{K_3,k,\ell,1}(x) &= \sum_{n\leqslant x} \big(\tau_k^{K_3}(n)\big)^{\ell} l_1(n) \\ &= \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT} L_{K_3,k,\ell}(s)\frac{x^s}{s}ds + O\Big(\frac{x^{3+\varepsilon}}{T}\Big), \end{aligned} \tag{52}$$

where $\eta = 3+\varepsilon$, and we make the special choice $T = T^*$ which satisfies (45), here $2 \leqslant T \leqslant x$ is some suitable parameter to be determined later. We note that the utilization of Lemma 2.6 in handling the integrals over the horizontal segments can be ensured by the choice of the parameter T, which is taken as a positive power of sufficiently large x.

By shifting the line of integration in (52) to the parallel line with $\Re(s) = \alpha := \frac{19}{7}$, together with Cauchy's residue theorem, we get

$$\begin{aligned} S_{K_3,k,\ell,1}(x) &= \operatorname*{Res}_{s=3}\Big\{L_{K_3,k,\ell}(s)\frac{x^s}{s}\Big\} \\ &\quad + \frac{1}{2\pi i}\Big\{\int_{\alpha-iT}^{\alpha+iT} + \int_{\alpha+iT}^{\eta+iT} + \int_{\eta-iT}^{\alpha-iT}\Big\} L_{K_3,k,\ell}(s)\frac{x^s}{s}ds \\ &\quad + O\Big(\frac{x^{3+\varepsilon}}{T}\Big) \\ &:= x^3 P_{K_3,k,\ell}(\log x) + J_1 + J_2 + J_3 + O\Big(\frac{x^{3+\varepsilon}}{T}\Big), \end{aligned} \tag{53}$$

where $P_{K_3,k,\ell}(t)$ denotes a polynomial in t with degree $\nu_{k,\ell,1}+\nu_{k,\ell,4}-1$, and the constant $\nu_{k,\ell,1}$ and $\nu_{k,\ell,4}$ are defined as (35). In the region

$$\alpha \leqslant \Re(s) \leqslant \eta, |\Im(s)| \leqslant T,$$

by Lemma 2.2, we note that the L-series $L_{K_3,k,\ell}(s)$ is a meromorphic function having a pole at $s = 3$ of order $\nu_{k,\ell,1}+\nu_{k,\ell,4}$ coming from the factors $\zeta(s)^{\nu_{k,\ell,1}}$ and $L(\mathrm{sym}^3 f, s)^{\nu_{k,\ell,4}}$, which contributes the main term $x^3 P_{K_3,k,\ell}(\log x)$.

Now, it remains to handle the three integrals J_1, J_2 and J_3, by exploiting the aforementioned nice analytic properties of the associated L-functions. We begin to handle the integral J_1. For simplicity, using Lemma 2.2, we can rewrite $G_{K_3,k,\ell}(s)$ as

$$G_{K_3,k,\ell}(s) := \big(\zeta(s-2)L(s,\chi)\big)^{\nu_{k,\ell,1}}\big(L(f,s-2)L(f\otimes\chi,s)\big)^{\nu_{k,\ell,2}}\prod\nolimits^{\flat} G^*_{K_3,k,\ell}(s),$$

where $G^*_{K_3,k,\ell}(s)$ is an L-function of degree $\nu^*_{k,\ell} := (3k)^\ell - \sum_{i=1}^2 i\nu_{k,\ell,i}$, which takes the shape

$$G^*_{K_3,k,\ell}(s) := \prod_j L(\mathrm{sym}^{n_j} f, s)^{\omega_j},$$

for some suitable constants $n_j \geqslant 2, \omega_j \geqslant 1, j \geqslant 1$, and the constants $\nu_{k,\ell,i}, i = 1,2$ are defined as (35). For convenience, by (44), (46), and (49), we have

$$R_1(T_1) := \int_{T_1}^{2T_1} \left|\zeta\left(\frac{5}{7}+it\right)\right|^{12} dt \ll T_1^{1+\varepsilon}, \tag{54}$$

$$\begin{aligned} R_2(T_1) &:= \int_{T_1}^{2T_1} \left|L\left(f,\frac{5}{7}+it\right)\right|^3 dt \\ &\ll \sup_{T_1\leqslant t\leqslant 2T_1}\left|L\left(f,\frac{5}{7}+it\right)\right| \int_{T_1}^{2T_1}\left|L\left(f,\frac{5}{7}+it\right)\right|^2 dt \\ &\ll T_1^{\frac{2}{3}\cdot\frac{2}{7}+2\cdot\frac{2}{7}+\varepsilon} \ll T_1^{\frac{16}{21}+\varepsilon}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} R_3(T_1) &:= \int_{T_1}^{2T_1}\left|G^*_{K_3,k,\ell}\left(\frac{5}{7}+it\right)\right|^2 dt \\ &\ll T_1^{\frac{2}{7}\nu^*_{k,\ell}+\varepsilon} \ll T_1^{\frac{2}{7}(3k)^\ell - \frac{2}{7}\nu_{k,\ell,1} - \frac{4}{7}\nu_{k,\ell,2}+\varepsilon}. \end{aligned} \tag{56}$$

From Hypothesis μ, (46), (54)–(56), along with Hölder's inequality, we get

$$\begin{aligned} J_1 &= \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} L_{K_3,k,\ell}(s)\frac{x^s}{s}dt \\ &\ll x^{\frac{19}{7}+\varepsilon}\int_1^T \left|\zeta\left(\frac{5}{7}+it\right)^{\nu_{k,\ell,1}} L\left(f,\frac{5}{7}+it\right)^{\nu_{k,\ell,2}} G^*_{K_3,k,\ell}\left(\frac{5}{7}+it\right)t^{-1}\right|dt \\ &\quad + x^{\frac{19}{7}+\varepsilon} \\ &\ll x^{\frac{19}{7}+\varepsilon}\sup_{1\leqslant T_1\leqslant T/2}\sup_{T_1\leqslant t\leqslant 2T_1} T_1^{-1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{\nu_{k,\ell,1}-2}\left|L\left(f,\frac{5}{7}+it\right)\right|^{\nu_{k,\ell,2}-1} \\ &\quad\cdot R_1(T_1)^{\frac{1}{6}}R_2(T_1)^{\frac{1}{3}}R_3(T_1)^{\frac{1}{2}} \\ &\ll x^{\frac{19}{7}+\varepsilon}T^{\frac{2}{7}(2\mu(\nu_{k,\ell,1}-2)+\frac{2}{3}(\nu_{k,\ell,2}-1))+\frac{1}{6}+\frac{1}{3}\cdot\frac{16}{21}+\frac{1}{2}\cdot\frac{2}{7}\cdot\nu^*_{k,\ell}-1+\varepsilon} \\ &\ll x^{\frac{19}{7}+\varepsilon}T^{\frac{1}{7}(3k)^\ell+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu-\frac{97}{126}+\varepsilon}. \end{aligned} \tag{57}$$

Here, we appeal to the fact that $U_{k,\ell}(s) \ll 1$ for $\Re(s) \geqslant \frac{5}{2}+\varepsilon$.

Now, we turn to handle the contributions from the horizontal line integrals J_2 and J_3. By appealing to Lemmas 2.2, 2.6, 2.8, and (50), we have

$$\begin{aligned}
J_2+J_3 &\ll x^{2+\varepsilon}\int_{\frac{5}{7}}^{1+\varepsilon} |G_{K_3,k,\ell}(\sigma+iT)x^{\sigma}T^{-1}|d\sigma \\
&\ll x^{2+\varepsilon}\sup_{\frac{5}{7}\leqslant\sigma\leqslant 1+\varepsilon} x^{\sigma}|\zeta(\sigma+iT)|^{\nu_{k,\ell,1}}|L(f,\sigma+iT)|^{\nu_{k,\ell,2}} \\
&\quad\cdot|L(\mathrm{sym}^2 f,\sigma+iT)|^{\nu_{k,\ell,3}}|L(\mathrm{sym}^3 f,\sigma+iT)|^{\nu_{k,\ell,4}} \\
&\quad\cdot|\widetilde{G}_{K_3,k,\ell}(\sigma+iT)|T^{-1} \\
&\ll x^{2+\varepsilon}\max_{\frac{5}{7}\leqslant\sigma\leqslant 1+\varepsilon} x^{\sigma}T^{(\nu_{k,\ell,1}\varepsilon+\frac{2}{3}\nu_{k,\ell,2}+\frac{6}{5}\nu_{k,\ell,3}+\frac{1}{2}((3k)^{\ell}-\sum_{j=1}^{3}j\nu_{k,\ell,j}))(1-\sigma)+\varepsilon} \\
&\quad\cdot T^{-1} \\
&\ll \frac{x^{3+\varepsilon}}{T}+x^{\frac{19}{7}+\varepsilon}T^{\frac{1}{7}(3k)^{\ell}-\frac{1}{7}\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{3}{35}\nu_{k,\ell,3}-1+\varepsilon}. \qquad (58)
\end{aligned}$$

Combining (53), (57) and (58), it leads to

$$\begin{aligned}
S_{K_3,k,\ell,1}(x) &= x^3 P_{K_3,k,\ell}(\log x) \\
&\quad + O(x^{\frac{19}{7}+\varepsilon}T^{\frac{1}{7}(3k)^{\ell}+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu-\frac{97}{126}+\varepsilon}) \qquad (59)\\
&\quad + O\Big(\frac{x^{3+\varepsilon}}{T}\Big).
\end{aligned}$$

Set

$$\theta_{k,\ell} := \frac{1}{7}(3k)^{\ell}+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu+\frac{29}{126}.$$

On taking $T = x^{\frac{2}{7\theta_{k,\ell}}}$ in (59), we obtain

$$S_{K_3,k,\ell,1}(x) = x^3 P_{K_3,k,\ell}(\log x) + O(x^{3-\frac{2}{7\theta_{k,\ell}}+\varepsilon}).$$

This completes the proof of Proposition 3.1. $\square$

In what follows, we deal with the sum $S_{K_3,k,\ell,2}(x)$, by following the argument similar to the proof of Proposition 3.1 with slight modifications. We have the following result.

PROPOSITION 3.2. *Let $K_3/\mathbb{Q}$ be a non-normal cubic extension, which is given by an irreducible polynomial $h(x) = x^3+Ax^2+Bx+C$ of discriminant D with $D<0$. Let $k \geqslant 2, \ell \geqslant 2$ be any given integers, and let $S_{K_3,k,\ell,2}(x)$ be defined as* (30). *Then, for any $\varepsilon > 0$, we have*

$$S_{K_3,k,\ell,2}(x) = O(x^{3-\frac{2}{7\theta_{k,\ell}}+\varepsilon}),$$

where the constant $\theta_{k,\ell}$ is defined the same as in (51).

Proof. By applying Perron's formula to Lemma 2.3, we have

$$S_{K_3,k,\ell,2}(x) = \sum_{n\leqslant x}(\tau_k^{K_3}(n))^\ell v_1(n)$$

$$= \frac{1}{2\pi i}\int_{\eta-iT}^{\eta+iT}\mathcal{L}_{K_3,k,\ell}(s)\frac{x^s}{s}ds + O\Big(\frac{x^{3+\varepsilon}}{T}\Big), \tag{60}$$

where $\eta = 3+\varepsilon$, and we make the special choice $T = T^*$ satisfying (45), here $2 \leqslant T \leqslant x$ is some suitable parameter to be determined later.

Now, we shift the line of integration in (60) to the parallel line with $\Re(s) = \alpha := \frac{19}{7}$, in combination with Cauchy's residue theorem, then

$$S_{K_3,k,\ell,2}(x) = \frac{1}{2\pi i}\Big\{\int_{\alpha-iT}^{\alpha+iT} + \int_{\alpha+iT}^{\eta+iT} + \int_{\eta-iT}^{\alpha-iT}\Big\}\mathcal{L}_{K_3,k,\ell}(s)\frac{x^s}{s}ds + O\Big(\frac{x^{3+\varepsilon}}{T}\Big)$$

$$:= I_1 + I_2 + I_3 + O\Big(\frac{x^{3+\varepsilon}}{T}\Big), \tag{61}$$

In the region $\alpha \leqslant \Re(s) \leqslant \eta, |\Im(s)| \leqslant T$, using Lemma 2.3, it can be found that the L-series $\mathcal{L}_{K_3,k,\ell}(s)$ is analytic and has no singularity inside.

Now, the main goal is to estimate the three integrals I_1, I_2 and I_3 suitably, with recourse to the aforementioned analytic properties of the associated L-functions. For brevity, by using Lemma 2.3, we can rewrite $H_{K_3,k,\ell}(s)$ as

$$H_{K_3,k,\ell}(s) := \big(\zeta(s)L(s-2,\chi)\big)^{\nu_{k,\ell,1}}\big(L(f,s)L(f\otimes\chi, s-2)\big)^{\nu_{k,\ell,2}}\prod\nolimits^{\dagger} H^*_{K_3,k,\ell}(s),$$

where $H^*_{K_3,k,\ell}(s)$ is an L-function of degree $\nu^*_{k,\ell} := (3k)^\ell - \sum_{i=1}^2 i\nu_{k,\ell,i}$, which takes the shape

$$H^*_{K_3,k,\ell}(s) := \prod_j L(\mathrm{sym}^{n_j} f, s)^{\omega_j},$$

for some suitable constants $n_j \geqslant 2, \omega_j \geqslant 1, j \geqslant 1$, and the constants $\nu_{k,\ell,i}, i = 1,2$ are defined as (35). For the L-function $H_{K_3,k,\ell}(s)$ twisted by the non-principal character $\chi(\mathrm{mod}\ 4)$, we also set

$$H^*_{K_3,k,\ell}(s,\chi) := \prod_j L(\mathrm{sym}^{n_j} f\otimes\chi, s)^{\omega_j}.$$

By appealing to Lemma 2.9, and (49), we have

$$Q_1(T_1) := \int_{T_1}^{2T_1}\Big|L\Big(\frac{5}{7}+it,\chi\Big)\Big|^{12}dt \ll T_1^{1+\varepsilon}, \tag{62}$$

$$Q_2(T_1) := \int_{T_1}^{2T_1}\Big|L\Big(f\otimes\chi, \frac{5}{7}+it\Big)\Big|^{3}dt$$

$$\ll \sup_{T_1\leqslant t\leqslant 2T_1}\left|L\left(f\otimes\chi,\frac{5}{7}+it\right)\right|\int_{T_1}^{2T_1}\left|L\left(f\otimes\chi,\frac{5}{7}+it\right)\right|^2dt$$

$$\ll T_1^{\frac{2}{3}\cdot\frac{2}{7}+2\cdot\frac{2}{7}+\varepsilon}\ll T_1^{\frac{16}{21}+\varepsilon}, \tag{63}$$

and

$$Q_3(T_1):=\int_{T_1}^{2T_1}\left|H^*_{K_3,k,\ell}\left(\frac{5}{7}+it,\chi\right)\right|^2dt$$

$$\ll T_1^{\frac{2}{7}\nu^*_{k,\ell}+\varepsilon}\ll T_1^{\frac{2}{7}(3k)^\ell-\frac{2}{7}\nu_{k,\ell,1}-\frac{4}{7}\nu_{k,\ell,2}+\varepsilon}. \tag{64}$$

From Lemma 2.9, Remark 2.10, and (62)–(64), along with Hölder's inequality, we get

$$\begin{aligned}
I_1&=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\mathcal{L}_{K_3,k,\ell}(s)\frac{x^s}{s}dt\\
&\ll x^{\frac{19}{7}+\varepsilon}\int_1^T\left|L\left(\frac{5}{7}+it,\chi\right)^{\nu_{k,\ell,1}}L\left(f\otimes\chi,\frac{5}{7}+it\right)^{\nu_{k,\ell,2}}\right.\\
&\qquad\left.\cdot H^*_{K_3,k,\ell}\left(\frac{5}{7}+it,\chi\right)t^{-1}\right|dt+x^{\frac{19}{7}+\varepsilon}\\
&\ll x^{\frac{19}{7}+\varepsilon}\sup_{1\leqslant T_1\leqslant T/2}\sup_{T_1\leqslant t\leqslant 2T_1}T_1^{-1}\left|\zeta\left(\frac{5}{7}+it\right)\right|^{\nu_{k,\ell,1}-2}\left|L\left(f,\frac{5}{7}+it\right)\right|^{\nu_{k,\ell,2}-1}\\
&\qquad\cdot Q_1(T_1)^{\frac{1}{6}}Q_2(T_1)^{\frac{1}{3}}Q_3(T_1)^{\frac{1}{2}}\\
&\ll x^{\frac{19}{7}+\varepsilon}T^{\frac{2}{7}(2\mu(\nu_{k,\ell,1}-2)+\frac{2}{3}(\nu_{k,\ell,2}-1))+\frac{1}{6}+\frac{1}{3}\cdot\frac{16}{21}+\frac{1}{2}\cdot\frac{2}{7}\cdot\nu^*_{k,\ell}-1+\varepsilon}\\
&\ll x^{\frac{19}{7}+\varepsilon}T^{\frac{1}{7}(3k)^\ell+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu-\frac{97}{126}+\varepsilon}.
\end{aligned} \tag{65}$$

Here, we apply the bound $\widetilde{U}_{k,\ell}(s)\ll 1$ for $\Re(s)\geqslant\frac{5}{2}+\varepsilon$.

Let $\widetilde{H}_{K_3,k,\ell}(s,\chi)$ be the L-function defined as for which $\widetilde{H}_{K_3,k,\ell}(s)$ is twisted by non-principal character $\chi(\mathrm{mod}\ 4)$, where $\widetilde{H}_{K_3,k,\ell}(s)$ is defined as equation (43). For the integrals over the horizontal segments I_2 and I_3, by appealing to Remarks 2.7, 2.10, Lemma 2.9, and (50), we have

$$\begin{aligned}
I_2+I_3&\ll x^{2+\varepsilon}\int_{\frac{5}{7}}^{1+\varepsilon}|H_{K_3,k,\ell}(\sigma+iT)x^\sigma T^{-1}|d\sigma\\
&\ll x^{2+\varepsilon}\sup_{\frac{5}{7}\leqslant\sigma\leqslant 1+\varepsilon}x^\sigma|L(\sigma+iT,\chi)|^{\nu_{k,\ell,1}}|L(f\otimes\chi,\sigma+iT)|^{\nu_{k,\ell,2}}\\
&\qquad\cdot|L(\mathrm{sym}^2f\otimes\chi,\sigma+iT)|^{\nu_{k,\ell,3}}|L(\mathrm{sym}^3f,\sigma+iT)|^{\nu_{k,\ell,4}}\\
&\qquad\cdot|\widetilde{H}_{K_3,k,\ell}(\sigma+iT,\chi)|T^{-1}\\
&\ll x^{2+\varepsilon}\max_{\frac{5}{7}\leqslant\sigma\leqslant 1+\varepsilon}x^\sigma T^{(\nu_{k,\ell,1}\varepsilon+\frac{2}{3}\nu_{k,\ell,2}+\frac{6}{5}\nu_{k,\ell,3}+\frac{1}{2}((3k)^\ell-\sum_{j=1}^3 j\nu_{k,\ell,j}))(1-\sigma)+\varepsilon}
\end{aligned}$$

$$\cdot T^{-1}$$

(66) $$\ll \frac{x^{3+\varepsilon}}{T} + x^{\frac{19}{7}+\varepsilon} T^{\frac{1}{7}(3k)^{\ell}-\frac{1}{7}\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{3}{35}\nu_{k,\ell,3}-1+\varepsilon}.$$

Therefore, by inserting the estimates (65) and (66) into (61), we get

(67) $$S_{K_3,k,\ell,2}(x) \ll x^{\frac{19}{7}+\varepsilon} T^{\frac{1}{7}(3k)^{\ell}+\frac{1}{7}(4\mu-1)\nu_{k,\ell,1}-\frac{2}{21}\nu_{k,\ell,2}-\frac{8}{7}\mu-\frac{97}{126}+\varepsilon} + \frac{x^{3+\varepsilon}}{T}.$$

We choose $T = x^{\frac{2}{7\theta_{k,\ell}}}$ in (67), then

$$S_{K_3,k,\ell,2}(x) = O(x^{3-\frac{2}{7\theta_{k,\ell}}+\varepsilon}),$$

where $\theta_{k,\ell}$ is defined the same as in (51). This completes the proof of Proposition 3.2. $\square$

4. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we first deal with the proof of Theorem 1.1, then we complete the proof of Theorem 1.2.

By Propositions 3.1 and 3.2, along with (28), it follows that

$$S_{K_3,k,\ell}(x) = x^3 P_{K_3,k,\ell}(\log x) + O(x^{\delta_{k,\ell}+\varepsilon}),$$

where $\delta_{k,\ell} = 3 - \frac{2}{7\theta_{k,\ell}}$, and $P_{K_3,k,\ell}(t)$ denotes a polynomial in t with degree $\nu_{k,\ell,1} + \nu_{k,\ell,4} - 1$, and the constant $\theta_{k,\ell}$ is given by (51), and the constants $\nu_{k,\ell,i}, 1 \leqslant i \leqslant 4$ are defined as (35).

Now, we turn to the demonstration of the proof of Theorem 1.2. From the classical monograph of Iwaniec and Kowalski [26, (1.76)], we know that

(68) $$R_6(x) := \sum_{n\leqslant x} r_6(n) = \frac{(\pi x)^3}{\Gamma(4)} + O(x^2 \log x).$$

From the variance formula [5, Theorem 3], for random variable X defined on a countable sample space $\mathbb{V}$, we have

(69) $$\mathrm{Var}(X) = E(X^2) - E^2(X).$$

With the help of Theorem 1.1 and (68), we have

$$\begin{aligned} E\big((\tau_k^{K_3}(n))^{\ell}\big)_{\mathcal{D}} &= \frac{S_{K_3,k,\ell}(x)}{R_6(x)} \\ &= \pi^{-3}\Gamma(4) P_{K_3,k,\ell}(\log x) + O(x^{\delta_{k,\ell}-3+\varepsilon})\big(1 + O(x^{-1}\log x)\big) \\ &= \pi^{-3}\Gamma(4) P_{K_3,k,\ell}(\log x) + O(x^{\delta_{k,\ell}-3+\varepsilon}), \end{aligned} \quad (70)$$

where $\widetilde{P}_{K_3,k,\ell}(t)$ is a polynomial in t with degree $\nu_{k,\ell,1}+\nu_{k,\ell,4}-1$. Then, by (69), (70), we deduce that

$$\begin{aligned}\operatorname{Var}\big((\tau_k^{K_3}(n))^\ell\big)_{\mathcal{D}} &= E\big((\tau_k^{K_3}(n))^{2\ell}\big)_{\mathcal{D}} - E^2\big((\tau_k^{K_3}(n))^\ell\big)_{\mathcal{D}}\\ &= \pi^{-3}\Gamma(4)P_{K_3,k,2\ell}(\log x) - \big(\pi^{-3}\Gamma(4)\big)^2 P^2_{K_3,k,\ell}(\log x)\\ &\qquad + O(x^{\delta_{k,2\ell}-3+\varepsilon})\\ &= \widetilde{P}_{K_3,k,2\ell}(\log x) + O(x^{\delta_{k,2\ell}-3+\varepsilon}),\end{aligned}$$

where $\widetilde{P}_{K_3,k,2\ell}(t)$ is a polynomial in t with degree $\nu_{k,2\ell,1}+\nu_{k,2\ell,4}-1$. This completes the proof of Theorem 1.2.

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