UNIFORM BOUNDEDNESS PRINCIPLE AND CLOSED GRAPH THEOREM FOR BILINEAR MAPPINGS BETWEEN TOPOLOGICAL VECTOR SPACES

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We study the continuity and the boundedness of bilinear mappings between topological vector spaces. We investigate and characterize the compacity of these mappings. As an application, we prove bilinear versions of the Banach–Steinhaus and closed graph theorems in the framework of topological vector spaces.

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1. INTRODUCTION AND PRELIMINARIES

The theory of linear mappings on topological vector spaces emerged as a counterpart to that of linear operators on normed spaces and concerned mainly extension properties, continuity, boundedness, compact linear operators and spectral theory (see [6, 8, 9, 13–15, 17] and the references therein).

The aim of this paper is to study and investigate the continuity and the compacity of bilinear mappings defined on the cartesian product of two topological vector spaces with values in a topological vector space. We prove the bilinear Banach–Steinhaus theorem and the closed graph theorem for these mappings. As far as we know, that is a first attempt in this regard. In this direction, although within different frameworks, many papers have been devoted to establishing fundamental theorems for bilinear mappings (see [10] and [3] for bilinear mappings on asymmetric normed spaces, [2] for linear relations on asymmetric normed spaces, and [1] for bilinear relations on normed spaces).

The paper is divided into three sections. After the introductory one, in Section 2 we study the concept of continuity and boundedness of bilinear mappings in topological vector spaces giving the relationship between these two notions. Also, we introduce and characterize the compacity of these mappings. In Section 3 we establish some fundamental theorems: the bilinear

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Banach–Steinhaus theorem and the closed graph theorem for continuous bilinear mappings between topological vector spaces.

Throughout the paper X, Y and Z are topological vector spaces. A neighborhood U of the origin 0 in a topological vector space X is simply called a zero neighborhood.

Recall that a subset E of a topological vector space X is said to be bounded if it is absorbed by every zero neighborhood, i.e., if for every zero neighborhood U in X there exists $\lambda > 0$ such that $E \subset \lambda U$. Note that the notion of boundedness may not coincide with the metric notion of boundedness (see [13, p. 23]). Also, a locally bounded topological vector space is a topological vector space that possesses a bounded zero neighborhood. A subset E of X is said to be balanced if $\lambda E \subset E$ for every $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$.

The next theorem gives an important characterization of the boundedness by means of sequences in topological vector space.

THEOREM 1.1 ([13, p. 23]). *A subset E of a topological vector space X is bounded if and only if, for every sequence of scalars $(\alpha_n)_n \subset \mathbb{K}$ that converges to 0 and every sequence $(x_n)_n$ in E, the sequence $(\alpha_n x_n)_n$ converges to 0 in X.*

A linear map $T : X \longrightarrow Y$, between topological vector spaces, is bounded if it maps bounded sets into bounded sets. As consequence of the translational invariance of the topological vector spaces, if the linear mapping $T : X \longrightarrow Y$ is continuous at zero then it is continuous everywhere. For a linear functional $T : X \longrightarrow \mathbb{K}$ if $T(x) \neq 0$ for some $x \in X$, then T is continuous if and only if T maps some zero neighborhood in X into a bounded set in $\mathbb{K}$ (see [13, p. 15]). The space of all continuous linear functionals over X is denoted by X^* and is called dual of X.

A metric d on a vector space X is called invariant if

$$d(x+z, y+z) = d(x, y)$$

for all x, y, z in X. The topological vector space X is an F-space if its topology is induced by a complete translationally invariant metric. An F-space is a Banach space if in addition

$$d(\alpha x, 0) = |\alpha|\, d(x, 0)$$

for all x in X and all scalar α.

Further details on topological vector spaces can be found in [8,9,13,15,16].

2. BILINEAR MAPPINGS ON TOPOLOGICAL VECTOR SPACES

2.1. Continuity and Boundedness

Firstly, note that if X and Y are topological vector spaces over the same scalar field $\mathbb{K}$, then the Cartesian product $X \times Y$ is a topological vector space under the product topology (see [16, p. 19]). We are interested in the continuity and boundedness properties of a bilinear mapping $T : X \times Y \longrightarrow Z$.

Definition 2.1. A bilinear mapping $T : X \times Y \longrightarrow Z$ is continuous if it is continuous as a function between two topological spaces.

By $\Lambda(X, Y; Z)$ we denote the set of all continuous bilinear mappings between the topological vector spaces $X \times Y$ and Z. Note that a bilinear mapping, between topological vector spaces, is continuous if and only if it is continuous at the origin $(0, 0)$ (see [16, p. 87]).

The definition of bounded bilinear mappings between topological vector spaces is similar to the linear case, it is in terms of bounded sets.

Definition 2.2. A bilinear mapping $T : X \times Y \longrightarrow Z$ is called bounded if it maps bounded sets into bounded sets.

In the following, we prove the equivalence between the boundedness and continuity. We show that the result works for bilinear mappings with a metrizable domain. For the proof, we need the following preliminary results (see [13, p. 22]).

Lemma 2.3. *If X is a metrizable topological vector space and $(x_n)_n$ is a sequence in X such that $x_n \longrightarrow 0$, then there are positive scalars sequence $(\alpha_n)_n$ such that $\alpha_n \longrightarrow \infty$ and $\alpha_n x_n \longrightarrow 0$.*

Theorem 2.4. *Let X, Y, and Z be topological vector spaces. Every continuous bilinear mapping $T : X \times Y \longrightarrow Z$ is bounded. The converse is true if X and Y are metrizable.*

Proof. Assume that T is continuous. Let E be a bounded subset of $X \times Y$ and let W be a zero neighborhood in Z. Then there exists a zero neighborhood V in $X \times Y$ such that $T(V) \subset W$. On the other hand, by the boundedness of E we can choose $\lambda > 0$ such that $E \subset \lambda V$. Then

$$T(E) \subset T(\lambda V) = \lambda^2 T(V) \subset \lambda^2 W.$$

Therefore, T is bounded. Now, under the assumption that X and Y are metrizable we prove the reverse implication. Suppose T is bounded and take a sequence $((x_n, y_n))_n$ in $X \times Y$ such that $(x_n, y_n) \longrightarrow (0, 0)$. By Lemma 2.3, there are positive sequences $(\alpha_n)_n, (\beta_n)_n \subset \mathbb{K}$ such that $\alpha_n \longrightarrow \infty, \beta_n \longrightarrow \infty$ and $(\alpha_n x_n, \beta_n y_n) \longrightarrow (0,0)$. This implies that

$$\left\{T(\alpha_n x_n, \beta_n y_n) : n \in \mathbb{N}\right\}$$

is a bounded subset of Z. Therefore, by Theorem 1.1 we get

$$T(x_n, y_n) = \frac{1}{\alpha_n \beta_n} T(\alpha_n x_n, \beta_n y_n) \longrightarrow 0,$$

which means that T is continuous at $(0,0)$ and then T is continuous. $\square$

Remark 2.5. Note that in the linear case, we have an equivalence between the continuity and the boundedness of linear mappings with locally bounded domain into an arbitrary topological vector space (see [17]). Since every locally bounded Hausdorff topological vector space is metrizable (see [16, p. 30]), then from the above theorem, the equivalence remains true in the bilinear case.

2.2. **Compactness**

We introduce and characterize the compactness notions of bilinear mappings between topological vector spaces, according to the definition of compact bilinear operators on Banach spaces, we establish their fundamental properties, extending some results of Ramanujan and Schock [11], as well as those of Ruch [12], who provided improvements and corrections to the results in [11].

First note that every topological vector space is a Hausdorff space (see [13, Theorem 1.12]). A subset of a topological vector space is called relatively compact if its closure is compact.

Definition 2.6. Let X, Y and Z be topological vector spaces. A bilinear mapping $T : X \times Y \longrightarrow Z$ is said to be compact, in symbols $T \in \Lambda_{\mathcal{K}}(X, Y; Z)$, if it maps a bounded subset of $X \times Y$ into a relatively compact subset of Z.

As a consequence of this definition, we have a result that gives the characterization of the compact bilinear mapping between topological vector spaces.

THEOREM 2.7. *Let $T : X \times Y \longrightarrow Z$ be a bilinear mapping. The following statements are equivalent.*

(i) *T is compact.*

(ii) *$T(U)$ is relatively compact in Z, for every bounded zero neighborhood U in $X \times Y$.*

(iii) *$T(V \times W)$ is relatively compact in Z, for every bounded zero neighborhood V in X and every bounded zero neighborhood W in Y.*

(iv) *For all bounded subsets $A \subset X, B \subset Y$, the subset $T(A \times B)$ is relatively compact in Z.*

Proof. (i)$\Longrightarrow$(ii) Is trivially true.

(ii)$\Longrightarrow$(iii) Follows from the fact that $V \times W$ is a bounded zero neighborhood in $X \times Y$.

(iii)$\Longrightarrow$(iv) The boundedness of A and B in X and Y, respectively, implies the existence of $\alpha > 0$ and $\beta > 0$ such that $\overline{T(A \times B)} \subset \alpha\beta\overline{T(V \times W)}$ for any bounded zero neighborhood V in X and any bounded zero neighborhood W in Y and the result follows.

(iv)$\Longrightarrow$(i) Let G be a bounded subset of $X \times Y$. Let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be the projections mappings defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. It is easy to see that $G \subset \pi_1(G) \times \pi_2(G)$. By the continuity of π_1 and π_2, the subsets $\pi_1(G)$ and $\pi_2(G)$ are bounded in X and Y, respectively. The result follows from the inclusion $\overline{T(G)} \subset \overline{T(\pi_1(G) \times \pi_2(G))}$. $\square$

The following results assert that the space $\Lambda_{\mathcal{K}}(X, Y; Z)$ of compact bilinear mappings is a linear subspace of the space $\Lambda(X, Y; Z)$ of all continuous bilinear mappings.

Proposition 2.8. *Let X, Y be metrizable topological vector spaces and Z a topological vector space. Every compact bilinear mappings $T : X \times Y \longrightarrow Z$, is continuous.*

Proof. Assume that $T \in \Lambda_{\mathcal{K}}(X, Y; Z)$. According to Theorem 2.4, it suffices to show that T is bounded. Let G a bounded subset of $X \times Y$. Then $\overline{T(G)}$ is compact in Z, from which it follows that $T(G)$ is bounded in Z (see [13, Theorem 1.15]) and the proof follows. $\square$

Proposition 2.9. *Let X, Y be two metrizable topological vector spaces and Z a topological vector space. If $T, S \in \Lambda_{\mathcal{K}}(X, Y; Z)$ and $\alpha \in \mathbb{K}$, then $\alpha T + S \in \Lambda_{\mathcal{K}}(X, Y; Z)$.*

Proof. The proof follows from the fact that the sum of two compact subsets is a compact subset (see [16, p. 26]). $\square$

Now we present the ideal properties of spaces of bilinear compact mappings.

Proposition 2.10. *Let X, Y, Z, W, E, F be topological vector spaces. If the mappings $u : E \longrightarrow X$, $v : W \longrightarrow Y$, $S : Z \longrightarrow F$ are linear continuous, and $T : X \times Y \longrightarrow Z$ is bilinear compact, then $S \circ T \circ (u, v) \in \Lambda_{\mathcal{K}}(E, W; F)$, where $(u, v)(z, w) := (u(z), v(w))$, $z \in E, w \in W$.*

Proof. Let G be a bounded subset of $E\times W$. As in the proof of Theorem 2.7 consider the projection mappings $\pi_1 : E \times W \longrightarrow E$ and $\pi_2 : E \times W \longrightarrow W$. Then we have $G \subset G_1 \times G_2$, where $G_1 = \pi_1(G)$ and $G_2 = \pi_2(G)$. The result follows immediately from the easy inclusion

$$S \circ T \circ (u, v)(G) \subset S\big(\overline{T(u(G_1) \times v(G_2))}\big). \qquad \square$$

3. FUNDAMENTAL THEOREMS

3.1. Bilinear open mapping theorem

Concerning the bilinear open mapping theorem, Rudin in [14, p. 67] asked the following question: If E, F and G are Banach spaces and $T : E \times F \longrightarrow G$ is a continuous bilinear map, does it follow that T is open at the origin? In general, the answer to this question is negative. A first counter example was found by Cohen [4] and, a bit later, Horowitz [7] gave a counter example by taking $E = \mathbb{R}^3$, $F = \mathbb{R}^3$, $G = \mathbb{R}^4$ and

$$T(x, y) = (x_1y_1, x_1y_2, x_1y_3 + x_3y_1 + x_2y_2, x_3y_2 + x_2y_1),$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

Since every Banach space is a special type of topological vector space, then the open mapping theorem for bilinear maps fails even in the topological vector space case.

3.2. Bilinear closed graph theorem

Let $T : X \times Y \longrightarrow Z$ be a bilinear mapping between topological vector spaces. The graph of T, in symbols $Gr(T)$, is the set of elements

$$\big((x, y), z\big) \in (X \times Y) \times Z$$

such that $z = T(x, y)$. Consider the space $(X \times Y) \times Z$ endowed the product topology. Associate to each $x \in X$ and $y \in Y$ the linear mappings $T_x : Y \longrightarrow Z$ and $T_y : X \longrightarrow Z$ by defining

$$T_x(y) = T(x, y) = T_y(x).$$

The bilinear mapping T is separately continuous if T_x and T_y are continuous for every fixed $x \in X$ and $y \in Y$, respectively. Obviously, the continuity implies the separately continuity, but the reverse implication is true if X is an F-space and Y is metrizable (see [13, Theorem 2.17]).

The closed graph theorem for the continuous bilinear mappings between topological vector spaces can be derived from [13, Proposition 2.15 and Theorem 2.17]. The proof of this result is an adaptation of the proof of main result in [5].

THEOREM 3.1. *Assume that X is F-space, Y is metrizable and the graph of T is closed in $(X \times Y) \times Z$. Then T is continuous.*

Proof. For fixed $x_0 \in X$ and $y_0 \in Y$, consider the subsets K_1 and K_2,

$$K_1 = Gr(T) \cap \big((X \times \{y_0\}) \times Z\big),$$

$$K_2 = Gr(T) \cap \big((\{x_0\} \times Y) \times Z\big).$$

It is easy to check that subsets K_1 and K_2 are closed in $(X \times Y) \times Z$, and $Gr(T_{x_0}) = \phi(K_1)$ and $Gr(T_{y_0}) = \psi(K_2)$, for every $x \in X$ and $y \in Y$, where ϕ is the homeomorphism

$$\phi : \big(X \times \{y_0\}\big) \times Z \longrightarrow X \times Z, \quad \phi\big((x, y_0), z\big) = (x, z)$$

and ψ is the homeomorphism

$$\psi : \big(\{x_0\} \times Y\big) \times Z \longrightarrow Y \times Z, \quad \psi\big((x_0, y), z\big) = (y, z).$$

Then $Gr(T_{x_0})$ and $Gr(T_{y_0})$ are closed in $X \times Z$ and $Y \times Z$, respectively. The closed graph theorem (see [13, Proposition 2.15]) asserts that T_{x_0} and T_{y_0} are continuous. It follows that T is separately continuous. Therefore, by [13, Theorem 2.17], T is continuous. $\square$

3.3. **Banach–Steinhaus theorem**

We now present the uniform boundedness principle (or Banach–Steinhaus theorem) for the bilinear mappings between topological vector spaces. As far as we know, that is a first attempt in this regard.

Recall that a subset E of topological vector space X is called nowhere dense if its closure has an empty interior. Also, E is called of the first category if it is a countable union of nowhere dense sets. E that is not of the first category is of the second category.

The next result is the Banach–Steinhaus theorem for linear mappings in the framework topological vector spaces, it can be found in [13, p. 44] and it is used to prove the bilinear Banach–Steinhaus theorem.

THEOREM 3.2. *Let X and Y be topological vector spaces. Let $\mathcal{F}$ be a collection of continuous linear mappings from X to Y. Consider the set B that consists of elements $x \in X$ such that $\{T(x) : T \in \mathcal{F}\}$ is bounded in Y. If B is of the second category, then $B = X$.*

Recall that a family $\mathcal{S}$ of bilinear mappings from $X \times Y$ to Z is said to be equicontinuous if for every zero neighborhood W in Z there exists some zero neighborhood V in $X \times Y$ such that $T(V) \subset W$ for all $T \in \mathcal{S}$.

We now present the main result of this section. We establish the Banach–Steinhaus theorem in the special case where the set $B \subset X \times Y$ is of the form $B = F \times G$. Whether the theorem extends to the entire product space $X \times Y$ remains an open question.

THEOREM 3.3. *Let X, Y, and Z be topological vector spaces. Let $\mathcal{S}$ be a collection of continuous bilinear mappings from $X \times Y$ to Z and B the set of all $(x, y) \in X \times Y$ such that $\mathcal{S}(x, y) = \{T(x, y) : T \in \mathcal{S}\}$ is bounded in Z. If B has the form $B = F \times G$, where $F \subset X$ and $G \subset Y$, and if B is of the second category, then $B = X \times Y$ and $\mathcal{S}$ is equicontinuous.*

Proof. Firstly, we prove that F and G are both of the second category. Indeed, suppose that F is of the first category, then there exists a countable family of nowhere dense sets $(A_i)_{i \in I}$ such that $F \times G = \cup_{i \in I}(A_i \times G)$. Since

$$\overset{\circ}{\overline{A_i \times G}} = \overset{\circ}{\overline{A_i}} \times \overset{\circ}{\overline{G}} = \emptyset$$

for every $i \in I$, it follows that B is of the first category, and this contradicts the assumptions. For a fixed $y \in G$, consider $\mathcal{S}_y$ the collection of all continuous linear mappings $T_y : X \longrightarrow Z$ defined by $T_y(x) = T(x, y)$. It is clear that F coincides with the set of $x \in X$ such that $\{T_y(x) : T_y \in \mathcal{S}_y\}$ are bounded in Z. Then by the previous theorem, we get $F = X$. We apply this argument, with T_y replaced by $T_x : X \longrightarrow Z$ defined by $T_x(y) = T(x, y)$, to obtain $G = Y$. Hence $B = X \times Y$.

To see the equicontinuity of $\mathcal{S}$, take W a zero neighborhood in Z. We have proved that $T(X \times Y)$ is bounded. Thus, there exists $\lambda > 0$ such that $T(X \times Y) \subset \lambda^2 W$, and then $T(\frac{1}{\lambda} X \times Y) \subset W$ for all $T \in \mathcal{S}$. The proof is finished. $\square$

One of the most important consequences of the bilinear Banach–Steinhaus theorem, in the framework of topological vector spaces, is that the limit of sequence of continuous bilinear mappings between topological vector spaces is always continuous.

A sequence $(x_n)_n$ in a topological vector space X is called a Cauchy sequence if for each zero neighborhood V in X, there exists some $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $x_m - x_n \in V$.

PROPOSITION 3.4. *Let X, Y, Z be topological vector spaces and $(T_n)_n$ a sequence of bilinear continuous mappings from $X \times Y$ to Z. Let $\mathcal{C}$ be the set of all $(x, y) \in X \times Y$ for which $(T_n(x, y))_n$ is a Cauchy sequence in Z. If $\mathcal{C}$ has the form $B = F \times G$, where $F \subset X$ and $G \subset Y$, and if $\mathcal{C}$ is of the second category, then $\mathcal{C} = X \times Y$.*

Proof. We can assume that $\mathcal{C}$ has the form $\mathcal{C} = F \times G$ with $F \subset X$ and $G \subset Y$. As in a similar way that is done in the proof of the above theorem, we see that F and G are both of the second category. For a fixed $y \in G$, consider the continuous linear mappings $T_{n,y} : X \longrightarrow Z$ defined by $T_{n,y}(x) = T_n(x, y)$. It is clear that F coincides with the set of $x \in X$ such that $(T_{n,y}(x))_n$ is a Cauchy sequence in Z. Then by (a) in [13, Theorem 2.7], we get $F = X$. By a similar argument, we get $G = Y$. $\square$

As a consequence of the above proposition and (b) in [13, Theorem 2.7], we obtain the following.

COROLLARY 3.5. *Let X, Y, Z be topological vector spaces and $(T_n)_n$ a sequence of bilinear continuous mappings from $X \times Y$ to Z. Let $\mathcal{D}$ be the set of all $(x, y) \in X \times Y$ for which $T(x, y) = \lim_{n\longrightarrow+\infty} T_n(x, y)$ exists. If $\mathcal{D}$ has the form $B = F \times G$, where $F \subset X$ and $G \subset Y$, and if $\mathcal{D}$ is of the second category, and if Z is an F-space, then $\mathcal{D} = X \times Y$ and the bilinear mapping $T : X \times Y \longrightarrow Z$ is separately continuous.*

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