ALGEBRAICITY OF NASH OR SMOOTH G-MANIFOLDS

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The Equivariant Nash conjecture states that any closed smooth G-manifold (G compact Lie) has an algebraic model. Now, let G be an abelian compact Lie group (respectively, an abelian compact affine Nash group) and M a closed smooth G-manifold (respectively, a closed smooth affine Nash G-manifold). With the aim to know whether the equivariant Nash conjecture is true for M, in this paper, we give some interesting necessary conditions. Moreover, if G is any compact affine Nash group, we find sufficient conditions so that a semialgebraic G-set or a C^k Nash G-manifold has an algebraic model.

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1. INTRODUCTION

The problem of the existence of an algebraic structure on differentiable objects is an important and a deep question in real algebraic geometry. We can say that the starting point of its study has been a paper by Siefert [23] from 1936, where he proved that, some particular, closed (i.e., C^∞ and compact) manifolds M are diffeomorphic to the union of connected components of a non-singular real algebraic variety. Later, Nash [15] proved that any closed manifold M is diffeomorphic to the union of connected component of a non singular real algebraic variety and posed the question whether M is diffeomorphic to a non-singular real algebraic variety; in other words, whether it has an algebraic model. Finally, in 1973, Tognoli [24] proved, in fact, that any closed smooth manifold has an algebraic model. It is important to remark that there exist non-compact manifolds which do not admit any algebraic structure. Nevertheless, we recall that by Shiota's result [21], any affine smooth (i.e., C^∞) Nash manifold, compact or not, is Nash diffeomorphic to a non-singular real algebraic variety. Tognoli's result opened the way to researches in several directions; in particular, in 1990, Dovermann, Masuda and Petrie [4] posed the question of the existence of algebraic structures in the equivariant category and

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formulated the following conjecture (the so-called "Equivariant Nash Conjecture"): "If G is a compact Lie group, then any closed smooth G-manifold is equivariantly diffeomorphic to a non-singular real algebraic G-variety". In this direction, some partial results are known, see, e.g., [2–7].

We previously said that an affine smooth Nash manifold always has an algebraic model; nevertheless, in general, very little is known about the equivariant algebraization problem in the Nash category. In previous papers [8, 9], we gave some results on this subject and in the present paper we are continuing that study. More precisely, we begin by giving sufficient conditions for the existence of algebraic structures on C^k Nash G-manifolds ($1 \leq k \leq \infty$) or on semialgebraic G-sets, where G is a compact affine Nash group.

Remark that a semialgebraic set in $\mathbb{R}^p$ does not necessarily admit a C^0 Nash manifold structure compatible with the semialgebraic structure, but a C^k Nash manifold in $\mathbb{R}^p$ ($0 \leq k \leq \infty$) is a semialgebraic set in $\mathbb{R}^p$. Since there exist non-affine Nash manifolds, in order to obtain algebraic structures, we must suppose the smooth Nash manifolds to be affine (if $k < \infty$, a C^k Nash manifold is always affine), and so the algebraization problem we consider is as follows: "Let M denote an affine closed C^k Nash manifold ($1 \leq k \leq \infty$) or a compact semialgebraic set in $\mathbb{R}^p$. Let G be a compact affine Nash group acting Nash on M (acting semialgebraically if M is semialgebraic). Do a Nash G-representation, with space $\mathbb{R}^m$, and a Nash G-diffeomorphism $M \to X$ (semialgebraic G-homeomorphism if M is semialgebraic), where X is a G-invariant non-singular real algebraic variety in $\mathbb{R}^m$, exist?" If the answer is positive, we say that X is an algebraic model of M, or that M has an algebraic G-variety structure. In this paper, we give some answers to this question.

In Section 3, we deal with semialgebraic G-sets in some $\mathbb{R}^p$, equipped with the usual topology in $\mathbb{R}^p$; all semialgebraic maps are continuous. So, if M is such a semialgebraic compact G-set, we found some results in order to assure the existence of an algebraic model of M (Theorems 3.3–3.5). For example, such a model exists if M is a homogeneous space.

Section 4 is devoted to affine C^k Nash manifolds ($1 \leq k \leq \infty$). Let M be such a closed manifold on which a compact affine Nash group acts. We prove that it is always Nash G-embeddable in a G-representation space, namely it is affine as a G-manifold; moreover, if the G-action is free or there is only one orbit type, it has an algebraic G-variety structure (Theorem 4.1). As a consequence of the affineness of M, we prove an approximation theorem of $C^k G$-maps by C^k Nash G-maps (Theorem 4.2).

What can we say if G is non-compact? It is known that if a G-manifold is equivariantly embeddable into a linear G-space, then the group G must be linear [11, 17]. So, let M be an affine closed smooth Nash manifold and G a linear Nash group acting on M. Then, we prove that M is affine as a Nash G-manifold if G is acting properly; moreover, it results that the existence of an affine G-structure is closely related with the existence of global slices (Theorem 4.3).

Finally, in Section 5, we deal with necessary conditions so that a closed smooth G-manifold M has an algebraic structure. We first suppose G to be an affine compact abelian Nash group and M a closed affine smooth Nash manifold. To obtain the conditions we deal with, the key point is considering the G-submanifolds $M(H_i)$ of M which are union of orbits of a given isotropy type: in fact, we find that if M has an algebraic G-variety structure, then each of such submanifolds must be G-diffeomorphic to a real algebraic G-variety (Corollary 5.2).

Now, conditions of the same type hold true in the smooth case, in order to equivariantly embed a closed smooth G-manifold into a G-representation space. Namely, if a closed smooth G-manifold (G compact abelian Lie) has an algebraic model, then each of the above submanifolds must be G-diffeomorphic to a real algebraic G-variety (Theorem 5.3).

Recall now that the submanifolds $M(H_i)$ in general are non-compact and that there exist non-compact manifolds which do not carry any algebraic structure. But Corollary 5.2 and Theorem 5.3 say that if M has an algebraic G-structure, then all these submanifolds $M(H_i)$ must have it too; thus, this is a non-trivial fact. In other words, the equivariant Nash conjecture is not true, both in Nash and in smooth setting, if at least one submanifold $M(H_i)$ is not G-diffeomorphic to a real algebraic G-variety, when G is abelian.

2. BASIC NOTIONS AND PRELIMINARY FACTS

We briefly recall some notions and facts (see [21, 22]). Let U, V be semialgebraic open sets in $\mathbb{R}^m$ and in $\mathbb{R}^n$, respectively. A C^r map $f : U \to V$ if $0 \leq r \leq \infty$ is said to be a C^r Nash map if its graph is semialgebraic. We know that if $r = \infty$, it is analytic. A C^r Nash manifold is a C^r manifold with a finite system of charts such that the coordinate changes are given by C^r Nash maps. A C^r Nash map $F : M \to N$ between C^r Nash manifolds is a C^r map which, in local coordinates, is given by C^r Nash maps. If there exists a C^r Nash embedding of M into some $\mathbb{R}^m$, we say that M is affine. If $r < \infty$, M is always affine, but there exist non-affine smooth (i.e., C^∞) Nash manifolds. If M is a C^r Nash submanifold of $\mathbb{R}^m$ then it is semialgebraic in $\mathbb{R}^m$, and if $r = \infty$ it is analytic. Conversely, for $r > 0$ a semialgebraic C^r submanifold of $\mathbb{R}^m$ admits a unique C^r Nash manifold structure compatible with the semialgebraic C^r structure.

A C^∞ Nash manifold G endowed with a group structure such that the group operations are C^∞ Nash maps is said to be a C^∞ Nash group. We consider only such groups and we call them simply Nash groups. They are affine if the Nash manifolds are affine. A Nash subgroup of G is a subgroup of G which is a C^∞ Nash regular submanifold of G. Of course, it is a Nash group and it is closed. One defines as usual the notion of a C^r Nash action of a Nash group on a C^r Nash manifold.

Definition 2.1. Let G be a Nash group and $0 \leq r \leq \infty$.

1. A C^r Nash G-manifold is a C^r Nash manifold with a given C^r Nash G-action on it (a semialgebraic action if $r = 0$);

2. A C^r Nash G-map between C^r Nash G-manifolds is an equivariant C^r Nash map;

3. A (linear) Nash representation of G is a smooth Nash homomorphism $G \to GL(n)$; this means a homomorphism of groups which is a smooth Nash map;

4. A C^r Nash G-manifold M (a semialgebraic G-set M) is said to have an affine Nash G-manifold structure if it is C^r Nash G-diffeomorphic (semialgebraically G-homeomorphic if $r = 0$ or M is a semialgebraic G-set) to a G-invariant C^r Nash submanifold L of some Nash G-representation space. If L is a non-singular real algebraic G-variety, M is said to have an algebraic G-variety structure and L is an algebraic model of M. We must recall that a non-singular real algebraic variety is a smooth manifold but there exist singular real algebraic varieties which are smooth manifolds [14, p. 12].

3. ALGEBRAICITY OF SEMIALGEBRAIC G-SETS

First, recall that a C^0 Nash submanifold of $\mathbb{R}^p$ is a semialgebraic set, but a semialgebraic set in $\mathbb{R}^p$ may not be a Nash manifold. Second, we need some ingredients. We begin by collecting some facts in the following theorem.

Theorem 3.1. *Let G be a compact affine Nash group and K a closed subgroup of G. Let M, N be smooth Nash manifolds. Then we have:*

a) *G is Nash isomorphic to a closed subgroup of an orthogonal group and K is a Nash subgroup of G.*

b) *The homogeneous space G/K is a smooth Nash G-manifold which has an algebraic G-variety structure.*

c) *A smooth G-representation is of the class Nash.*

d) *The twisted product $G \times_K M$ is a smooth Nash manifold. We recall that the twisted product is the orbit space of the K action on $G \times M$ given by $k(g,m) = (gk^{-1}, km)$. The orbit of (g,m) is denoted by $[g,m]$. The canonical projection $\pi : G \times M \to G \times_K M$ is of class Nash and a map $f : G \times_K M \to N$ is of class Nash if and only if $f \circ \pi$ is.*

Proof. See [8, Corollaries 1.7, 1.5, Theorems 2.5, 2.6]. □

We use the averaging operator A. Here, we recall some basic facts about it. Let G be a compact Lie group and U, V orthogonal representations spaces of G. Denote by C^r (U, V) the set of all C^r maps $U \to V, 0 \leq r \leq \infty$. Let f be such a map and x a point of U. Denote the Haar measure on G by dg. Then

$$A(f)(x) = \int_G g^{-1} f(gx) dg.$$

LEMMA 3.2. 1. $A(f)$ is equivariant and $A(f) = f$ if f is equivariant.

2. The operator A induces a map $C^r(U,V) \to C^r(U,V), f \mapsto A(f)$, that is continuous with respect to the Whitney C^r topology.

3. If f is a polynomial, then so is $A(f)$.

Proof. See [4, Lemma 4.1]. □

We also use the notion of global slice. Let G be a Lie group, K a closed subgroup of G and M a $C^r G$-manifold ($0 \leq r \leq \infty$), or a semialgebraic G-subset of $\mathbb{R}^m$. A $C^r K$-submanifold S of M, or a semialgebraic K-subset of M, is said to be a global K-slice in M if the map $G \times_K S \to M$, [g, s] $\longmapsto g(s)$, is a $C^r G$-diffeomorphism (a semialgebraic G-homeomorphism if $r = 0$ or M is a semialgebraic G-set). It results that S is a closed subspace of M.

THEOREM 3.3. *Let M be a compact semialgebraic subset of $\mathbb{R}^m, G$ a compact affine Nash group acting semialgebraically on M and K a closed subgroup of G. Assume that there exists in M a global K-slice S which is a C^1 Nash manifold without boundary and with free K-action. Then M has an algebraic G-variety structure.*

Proof. First, remark that K is a Nash subgroup of G, by Theorem 3.1, a). Second, S is a closed manifold because it is compact and without boundary. Therefore, by [9, Theorem 5.3], it has an algebraic K-variety structure. Precisely this means that it exists a C^1 Nash K-diffeomorphism $q : S \to T$ between S and a non-singular real algebraic K-variety T of a Nash K-representation space. Therefore, since T is an affine smooth Nash K-manifold, $G \times_K T$ is a smooth Nash K-manifold (Theorem 3.1, d)) and by [9, Theorem 4.6] there exists a smooth Nash G-embedding h of G $\times_K T$ into a Nash G-representation space U. On the other hand, by [20, Corollary 1.4] there exists a smooth G-embedding $f : G \times {}_K T \to V$ into the smooth G-representation space V such that the image of f is a non-singular real algebraic G-variety L. Remark that this representation is of class Nash by Theorem 3.1, c).

Consider now the smooth G-diffeomorphism $f \circ h^{-1} : h(G \times_K T) \to L, C^1$ approximate this map, in the strong topology of the space of smooth maps $h(G \times_K T) \to V$, by a polynomial map $p : U \to V$ and then use the averaging operator A (Lemma 3.2). We obtain a G-polynomial map $A(p) : U \to V$ which approximates $A(f \circ h^{-1}) = f \circ h^{-1}$. If the approximation is close enough, using the retraction of a Nash G-tubular neighbourhood of L in V [12, Proposition 2.3], we find a smooth Nash G-map $\lambda : U \to L$ which is a G-diffeomorphism on $h(G \times_K T)$ and hence, we have the smooth Nash G-diffeomorphism $\lambda \circ h : G \times {}_K T \to L$. So we have the following semialgebraic G-homeomorphisms:

$$M \to G \times_K S \to G \times_K T \to L.$$

The first homeomorphism is given by hypothesis; the second follows from the C^1 Nash K-diffeomorphism $q : S \to T$ and by Theorem 3.1, d); the third is $\lambda \circ h$. Since L lies in the Nash G-representation space V, L is an algebraic model of M. □

The next theorem characterizes the semialgebraic G-sets (G compact affine Nash), with only one orbit type, which have an algebraic model.

THEOREM 3.4. *Let G be a compact affine Nash group acting semialgebraically on the compact semialgebraic set M with only one orbit type G/H. Then M has an algebraic G-variety structure if and only if the H-fixed point set $M^H = \{x \in M; h(x) = x, \forall h \in H\}$ is a closed C^1 Nash manifold.*

Proof. First, we suppose M^H to be a closed C^1 Nash manifold. Let $K = N/H$, where N is the normalizer of H in G, and recall that there is a semialgebraic G-homeomorphism $\beta : M \to G \times_N M^H$ [19, Proposition 2.9]. We want to prove that this last Nash G-manifold has an algebraic G-variety structure. In order to do this, remark that K is a compact affine Nash group, by Theorem 3.1, a), b), acting freely on M^H; moreover, its action is of class C^1 Nash, as follows from the next commutative diagram and by Theorem 3.1, d)

$$\begin{array}{ccc} N \times M^H & & \\ \downarrow & \searrow & \\ N \times_H M^H = (N/H) \times M^H & \longrightarrow & M^H \end{array} \qquad \begin{array}{ccc} (n, x) & & \\ \downarrow & \searrow & \\ (nH, x) & \longrightarrow & nx \end{array}$$

Therefore, the closed C^1 Nash K-manifold M^H has an affine Nash K-manifold structure by [9, Theorem 4.1] and hence, by [9, Theorem 5.2], it has an algebraic K-variety structure. This means that there exist a Nash K-representation $\lambda : K \to GL(t)$, with space $\mathbb{R}^t$, and a C^1 Nash K-diffeomorphism $f : M^H \to T \subset \mathbb{R}^t$, where T is a non-singular real algebraic K-variety. Remark now that λ induces the homomorphism $\varrho = \lambda \circ \pi : N \to GL(t)$, where $\pi : N \to N/H$ is the canonical Nash projection. Therefore, we have the Nash N-representation ϱ and hence a Nash N-action on $\mathbb{R}^t$. Moreover, the K-map f is a N-map: if $x \in M^H, n \in N$, we have

$$f(nx) = f(nhx) = nhf(x), \forall h \in H$$

and it follows that $f(nx) = nf(x)$. So M^H has an algebraic N-variety structure and therefore, by [9, Theorem 4.6], $G \times_N M^H$ has an affine C^1 Nash G-manifold structure, that is, there exists a C^1 Nash G-embedding h: $G \times_N M^H \to \mathbb{R}^p$ into a Nash G-representation space. On the other hand, by [20, Corollary 1.4] there is a C^1 G-embedding d: $G \times_N M^H \to \mathbb{R}^q$ into a smooth (and hence Nash) G-representation space and such that the image of d is a non-singular real algebraic G-variety L.

Therefore, using the averaging operator A and a Nash G-tubular neighbourhood of L in $\mathbb{R}^q$ and proceeding as in the proof of Theorem 3.3, we can find a C^1 Nash G-diffeomorphism $\psi : h(G \times_N M^H) \to L$ and then the semialgebraic G-homeomorphism $\psi \circ h \circ \beta : M \to L$.

Next, let us suppose that there exists a semialgebraic G-homeomorphism $\vartheta : M \to L$ between M and a non-singular real algebraic G-variety L in a Nash G-representation space. We now consider the H fixed point set L^H. It is a closed smooth Nash manifold [13, Proposition 2.25]. Therefore, endow $M^H = \vartheta^{-1}(L^H)$ with the smooth Nash structure of L^H by means of ϑ. □

THEOREM 3.5. *Let G be a compact affine Nash group, and K a closed subgroup of G. Assume that G acts semialgebraically on a homogeneous compact semialgebraic set M. Then*

a) *The set M has an algebraic G-variety structure.*

b) *The twisted product $G \times_K M$ is semialgebraically G-homeomorphic to an algebraic G-variety.*

Proof. a) Let G_x be the isotropy group of G at $x \in M$. By [19, Proposition 2.6], M is semialgebraically G-homeomorphic to the semialgebraic G-set G/G_x. On the other hand, G/G_x is a smooth Nash G-manifold which is Nash G-diffeomorphic to a non-singular real algebraic G-variety L (Theorem 3.1, b)). It follows that M and L are semialgebraically G-homeomorphic.

b) Let $f: M \to L$ be the semialgebraic G-homeomorphism found in a). We claim that $G \times_K M$ and $G \times_K L$ are semialgebraically G-homeomorphic. This follows from the following commutative diagram:

$$\begin{array}{ccc} G \times M & \longrightarrow & G \times L \\ \downarrow & & \downarrow \\ G \times_K M & \longrightarrow & G \times_K L \end{array} \quad , \quad \begin{array}{ccc} (g, x) & \longrightarrow & (g, f(x)) \\ \downarrow & & \downarrow \\ [g, x] & \longrightarrow & [g, f(x)] \end{array}$$

where the arrow below is a semialgebraic G-homeomorphism because the other arrows are semialgebraic maps [19, Propositions 2.1, 2.5].

Consider now the G-map

$$\gamma : G \times_K L \to (G/K) \times L, [g, x] \to (gK, gx)$$

between the Nash G-manifold $G \times {}_K L$ and the algebraic G-variety $(G/K) \times L$: it is a Nash G-diffeomorphism. In fact, if $\pi : G \times L \to G \times {}_K L$ is the canonical projection, the map $\gamma \circ \pi : G \times L \to (G/K) \times L, \gamma \circ \pi(g, x) = (gK, gx)$ is of class Nash and hence γ is of class Nash by Theorem 3.1, d).

The inverse map γ^{-1} is given by

$$\gamma^{-1} : (G/K) \times L \to G \times_K L, (gK, x) \mapsto [g, g^{-1}x]$$

and it is of class Nash as follows from the commutative diagram

$$\begin{array}{ccc} G \times L & \longrightarrow & G \times L \\ \downarrow & & \downarrow \\ G/K \times L & \longrightarrow & G \times_K L \end{array} \quad , \quad \begin{array}{ccc} (g, x) & \longrightarrow & (g, g^{-1}x) \\ \downarrow & & \downarrow \\ (gK, x) & \longrightarrow & [g, g^{-1}x] \end{array}$$

where the arrows different from γ^{-1} are of class Nash. Hence, $G \times_K M$ is semialgebraically G-homeomorphic to an algebraic G-variety of a G-representation space. □

4. ALGEBRAICITY OF C^k NASH G-MANIFOLDS ($1 \leq k \leq \infty$)

We begin by giving a generalisation of [9, Theorem 4.1].

THEOREM 4.1. *Let G be a compact affine Nash group acting on a closed C^k Nash submanifold M of $\mathbb{R}^m (1 \leq k \leq \infty)$. Then:*

a) *M has an affine Nash G-manifold structure and is G-diffeomorphic to a smooth Nash G-manifold.*

b) *If the G-action is free, M has an algebraic G-variety structure.*

c) *If there is only one orbit type, M has an algebraic G-variety structure.*

Proof. a) By [18, Theorem B] there exists a C^k G-embedding $f : M \to \mathbb{R}^p$ into a G-representation space and $f(M)$ is a smooth G-manifold. By [12, Theorem 1] there is a smooth G-diffeomorphism $\varphi : f(M) \to V$, where V is a smooth Nash G-submanifold of the Nash G-representation space $\mathbb{R}^n$ (Theorem 3.1, c)). So we have the equivariant C^k diffeomorphism $F = \varphi \circ f : M \to V$. Now C^1 approximate F by a polynomial map $q : \mathbb{R}^m \to \mathbb{R}^n$ and then, using the averaging operator A, consider the polynomial map $A(q)(x) = \int_G g^{-1}q(x)dg : \mathbb{R}^m \to \mathbb{R}^n$ and its restriction to M. If q is close enough to F, then $A(q)$ is close to $A(F) = F$, and therefore $A(q)(M)$ lies to a Nash G-tubular neighbourhood of V in $\mathbb{R}^n$. Therefore, we can get a C^k Nash G-diffeomorphism $\lambda : M \to V$.

b) It follows from a) and [9, Theorem 5.2].

c) Let $\lambda : M \to V$ be the C^k Nash G-diffeomorphism found in a). The H-fixed point set V^H is a closed smooth Nash submanifold of V. Now, recall from [19, Proposition 2.9] that there exists a semialgebraic G-homeomorphism $\psi : G \times_N V^H \to V$, where N is the normalizer of H in G. This homeomorphism is smooth, as it follows from the next commutative diagram:

$$\begin{array}{ccc} G \times V^H & & \\ \downarrow & \searrow & \\ G \times_N V^H & \longrightarrow & V \end{array} \qquad\qquad \begin{array}{ccc} (g,t) & & \\ \downarrow & \searrow & \\ [g,t] & \longrightarrow & gt \end{array}$$

where the arrow below is ψ and the other arrows are smooth maps. So ψ, and hence ψ^{-1}, are of class Nash, and therefore there is a C^k Nash G-diffeomorphism $M \to G \times_N V^H$. Now, repeat the proof of Theorem 3.4 about $G \times_N M^H$: one obtains that $G \times_N V^H$ has an algebraic G-variety structure, and therefore, the same happens for M. □

The embedding Theorem 4.1 allows us to obtain an approximation theorem of G-maps by Nash G-maps.

THEOREM 4.2. *Let G be an affine compact Nash group acting on the affine closed C^r Nash manifolds M and L $(1 \leq r \leq \infty)$. Then, the C^r Nash G-maps are dense in the space of the C^t G-maps $(1 \leq t \leq r)$ endowed with the strong topology.*

Proof. The proof is standard. Let $f : M \to L$ be a $C^t G$-map. By Theorem 4.1 embed M and L into G-representation spaces. Then, since M is compact, approximate f by a polynomial map, use the averaging operator A and a Nash G-tubular neighbourhood of L: one gets a C^r Nash G-map C^t-close to f. □

If G is a Lie group acting on a smooth manifold M, in order to embed M into a G-representation space, G must be linear [11,17] (remark that a compact affine Nash group is linear (Theorem 3.1, a)).

If K is a compact subgroup of a linear Nash group $G \subset GL(n)$, it is a Nash subgroup of G. In fact, K is an algebraic group in the vector space $L(n)$ of all $n \times n$ matrices [16, p. 133], and therefore, it is a Nash subgroup of $L(n)$; so it is a Nash subgroup of G.

We need to recall the notion of a proper G-space. A locally compact space X on which a Lie group G acts is said to be a proper G-space, and G is said to act properly on X, if the set $G_A = \{g \in G; gA \cap A \neq \varnothing\}$ is a compact subset of G for every compact subset A of X. This is equivalent to the fact that the map $G \times X \to X \times X, (g, x) \mapsto (gx, x)$ is proper.

THEOREM 4.3. *Let M be a closed affine smooth Nash manifold and let $G \subset GL(n)$ be a linear Nash group acting properly on M. Let K be a maximal compact subgroup of G. Then*

a) *There exists in M a smooth Nash global K-slice S which has an affine Nash K-manifold structure.*

b) *If S is a closed manifold and the K-action is free, S has an algebraic K-variety structure.*

c) *M has an affine Nash G-manifold structure.*

Proof. a) First remark that, because G is a semialgebraic manifold, it has a finite number of connected components; therefore, by [1] there exists in M a smooth global K-slice S', which is compact since it is a closed subspace of M. The subgroup K is Nash, compact and affine: therefore, by [12, Theorem 1] there is a smooth K-embedding $f : S' \to \Omega$ of S' into a K-representation space such that $f(S')$ is a smooth Nash K-submanifold of Ω. By Theorem 4.1, we can suppose M to be a Nash K-submanifold of a K-representation space. In particular, it results that S has an affine Nash K-manifold structure.

Let $i : S' \to M$ be the smooth canonical inclusion and consider the smooth K-map $i \circ f^{-1} : f(S') \to M$. By the proof of Theorem 4.2, we can C^1 approximate this map by a smooth Nash K-map, say h, and therefore, the map $j = h \circ f : S' \to M$ approximates i. Now endow S' with the Nash structure of $f(S')$ and denote S', with this structure, by ${S'}_n$. So $f : {S'}_n \to f(S')$ becomes a Nash K-diffeomorphism and hence $j : S'_n \to M$ a Nash K-map. If j is close enough to i, it is a Nash K-embedding and, by [10, Lemma 6.1], $S = j(S'_n)$ is a smooth global K-slice in M. We want to prove that S is a global Nash K slice, that is that the smooth K-diffeomorphism $\alpha : G \times_K S \to M, [g, s] \mapsto gs$, and its inverse, are of class Nash. To do this, let $q : G \times S \to G \times {}_K S$ be the canonical projection. The map $\alpha \circ q, (g, s) \mapsto$ gs, is of class Nash and hence also α. The inverse map is of class Nash by the inverse function theorem in a Nash setting.

b) It follows from Theorem 4.1.

c) It follows from a) and [9, Theorem 6.6]. □

5. ON THE EQUIVARIANT NASH CONJECTURE

In this section, we look for necessary conditions in order that the Equivariant Nash Conjecture holds true, as to any closed smooth G-manifold (G compact) has an algebraic G-variety structure. We begin by working in the Nash category. So, let G be a compact affine Nash group and V a non-singular real algebraic G-variety in a Nash G-representation space. Assume that there exists a finite number of orbit types, with isotropy types $(H_1), \ldots, (H_n)$, and let $V(H_i) = \{x \in V; G_x = gH_ig^{-1}$ for some $g \epsilon G\}$ be the union of the orbits with isotropy type $(H_i)\,(i = 1, \ldots, n)$. $V(H_i)$ is a smooth G-submanifold of V and it is a semialgebraic set of V [19, Proposition 2.8]; hence it is a smooth Nash G-submanifold of V. Suppose now G is abelian. Therefore, we have $(H_i) = H_i$, and $V(H_i) = \{x \in V; G_x = .H_i\}\,(i = 1, \ldots, n)$.

The key result is the following theorem.

Theorem 5.1. *Let G be an abelian compact affine Nash group and V a non-singular real algebraic G-variety in a Nash G-representation space $\mathbb{R}^m$. Assume that there exists a finite number of orbit types with isotropy types $(H_1), \ldots, (H_n)$. Then the Nash G-manifolds $V(H_1), \ldots, V(H_n)$ are Nash G-diffeomorphic to real algebraic G-varieties of Nash G-representation spaces.*

Proof. Let us consider the decomposition of V by the disjoint union of $V(H_i) : V = \sqcup V(H_i)$. If $n = 1$, there is nothing to prove. Now, we proceed as follows. First, give a partial ordering to the isotropy groups: $H_i \leq H_j \Leftrightarrow H_i$ is a subgroup of H_j. It is known that there exists a maximal element among the isotropy groups.

Suppose that $n = 2$ and hence $V = V(H_1) \sqcup V(H_2)$. We can also suppose that H_1 is maximal. Consider the H_1 fixed point set V^{H_1} of V. Because H_1 is maximal we have: $V(H_1) = V^{H_1} = V \cap (\mathbb{R}^m)^{H_1}$. Since $(\mathbb{R}^m)^{H_1}$ is a linear space, it follows that $V(H_1)$ is an algebraic G-variety in $\mathbb{R}^m$. Now let

$p_1 : \mathbb{R}^m \to \mathbb{R}$ be a polynomial such that $V = \{x \in \mathbb{R}^m; p_1(x) = 0\}$ and let $q_1 : \mathbb{R}^m \to \mathbb{R}$ be an invariant polynomial such that we obtain the next equality $V(H_1) = \{x \in \mathbb{R}^m; q_1(x) = 0\}$. Such a polynomial q_1 can be constructed as in [4, Lemma 4.7]. From the decomposition of the space V, we have that $V(H_2) = V - V(H_1) = \{x \in \mathbb{R}^m; p_1(x) = 0, q_1(x) \neq 0\}$. Now consider the algebraic G-variety

$$A_1 = \big\{(x, y) \in \mathbb{R}^m \oplus \mathbb{R}; p_1(x) = 0, yq_1(x) = 1\big\},$$

where the G-action on $\mathbb{R}$ is trivial. So, we obtain the Nash G-diffeomorphism $V(H_2) \to A_1, x \mapsto (x, 1/q_1(x))$.

Suppose that $n = 3$ and hence $V = V(H_1) \sqcup V(H_2) \sqcup V(H_3)$. We can suppose H_1 maximal. From what we saw before, $V(H_1)$ is an algebraic G-variety in $\mathbb{R}^m$ and $V - V(H_1) = V(H_2) \sqcup V(H_3)$ is Nash G-diffeomorphic to the algebraic G-variety $A_1 = \{x \in \mathbb{R}^{m+1}; p_2(x) = 0\}$.

The isotropy groups occurring in the smooth manifold $V - V(H_1)$ are H_2 and H_3 and we can suppose that H_2 is maximal among them. Therefore,

$$V(H_2) = \big(V - V(H_1)\big)(H_2) = \big(V - V(H_1)\big)^{H_2} = \big(V - V(H_1)\big) \cap (\mathbb{R}^{m+1})^{H_2},$$

where in the last equality we have identified $V - V(H_1)$ with its image A_1 in $\mathbb{R}^{m+1}$. So $V(H_2)$ is Nash G-diffeomorphic to an algebraic G-variety in $\mathbb{R}^{m+1}$ and hence, by identifying $V(H_2)$ with this G-variety, we can write the following: $V(H_2) = \{x \in \mathbb{R}^{m+1}; q_2(x) = 0\}$, where $q_2 : \mathbb{R}^{m+1} \to \mathbb{R}$ is an invariant polynomial. Consider now the algebraic G-variety

$$A_2 = \big\{(x, y) \in \mathbb{R}^{m+1} \oplus \mathbb{R}; p_2(x) = 0, yq_2(x) = 1\big\}.$$

With the previous identifications, we get the map

$$V(H_3) = V - V(H_1) - V(H_2) \to A_2, x \mapsto \big(x, 1/q_2(x)\big),$$

which is a Nash G-diffeomorphism.

Next, we can proceed by using the decomposition $V = V(H_1) \sqcup \cdots \sqcup V(H_n)$ (where $n \geq 3$). We obtain that: $V(H_1), \ldots, V(H_{n-2})$ are Nash G-diffeomorphic to algebraic G-varieties and $\mathrm{W} = V - V(H_1) - \cdots - V(H_{n-2})$ is, up to Nash G-diffeomorphisms, an algebraic G-variety in $\mathbb{R}^{m+n-2}$, with polynomial equation $p_{n-1}(x) = 0, x \in W$. The isotropy groups in W are H_{n-1} and H_n and we can suppose H_{n-1} maximal. Therefore, up to Nash G-diffeomorphisms, we can write

$$V(H_{n-1}) = W(H_{n-1}) = W^{H_{n-1}} = W \cap (\mathbb{R}^{m+n-2})^{H_{n-1}}$$

and hence $V(H_{n-1})$ is Nash G-diffeomorphic to an algebraic G-variety in $\mathbb{R}^{m+n-2}$, with equation $q_{n-1}(x) = 0$, where q_{n-1} is a G-invariant polynomial. Finally, there is the Nash G-diffeomorphism

$$V(H_n) = W - V(H_{n-1}) \to A_{n-1}$$

$$= \{x \in \mathbb{R}^{m+n-2} \oplus \mathbb{R}; p_{n-1}(x) = 0, yq_{n-1}(x) = 1\}, x \mapsto (x, 1/q_{n-1}(x)),$$

into a Nash G representation space. □

Now, let G be an abelian compact affine Nash group and M an affine closed smooth Nash manifold on which G acts. It is well known that there exists a finite number of orbit types in M, with isotropy types $H_1, \ldots, H_n$. Therefore, consider $M(H_1), \ldots, M(H_n)$, that is the G-manifolds union of the orbits with isotropy type $H_1, \ldots, H_n$; they are Nash submanifolds of M. If M has an algebraic G-variety structure, there exists an equivariant Nash diffeomorphism $f : M \to V$, where V is a non-singular real algebraic G-variety in the Nash G-representation space $\mathbb{R}^m$. The restriction of f to $M(H_i)$, for each $i = 1, \ldots, n$, gives equivariant Nash diffeomorphisms

$$M(H_1) \to V(H_1), \ldots, M(H_n) \to V(H_n).$$

Therefore, by Theorem 5.1, each $M(H_i)$ must be Nash G-diffeomorphic to a real algebraic G-variety.

Recall now that if G and M are compact, then there exists a finite number of orbit types. Thus, we have just proved the following corollary.

Corollary 5.2. *Let G be an abelian affine compact Nash group acting on a closed affine smooth Nash manifold M. If M has an algebraic G-variety structure, then all G-manifolds $M(H_1), \ldots, M(H_n)$ must be Nash G-diffeomorphic to real algebraic G-varieties.*

Remark now that Theorem 5.1 and Corollary 5.2 hold true with the same proofs if we consider only smooth manifolds and maps. Therefore, again under the hypothesis of compactness of G and M, we have the next theorem.

Theorem 5.3. *Let G be an abelian compact Lie group and M a closed smooth manifold on which G acts. If M has an algebraic G-variety structure then all G-manifolds $M(H_i)\,(i = 1, \ldots, n)$ must be G-diffeomorphic to real algebraic G-varieties.*

As the G-manifolds $M(H_i)$ are in general non-compact and there exist non-compact manifolds which do not carry any algebraic structure, Corollary 5.2 and Theorem 5.3 lead to the following natural question, in order to give an answer to the equivariant Nash conjecture: if M is any closed G-manifold (G abelian), do all G-manifolds $M(H_i)$ have an algebraic structure?

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