

A CHIEF FACTOR CHARACTERIZATION OF THE SOLVABLE RADICAL OF A FINITE GROUP

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Let G be a finite group and let $S(G)$ denote the solvable radical of G (the unique maximal subnormal solvable subgroup of G). In this article we present a chief factor characterization of $S(G)$ that was suggested by the classic chief factor characterization of the Fitting subgroup of G (the unique maximal subnormal nilpotent subgroup of G). We close by extending these two results to a proper normal subgroup of G .

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Our notation and terminology are standard and tend to follow Gorenstein [1] and Huppert [3]. As in these references, let G be a finite group. By definition, if $x, y \in G$, then $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$. Let X be a subset of G so that

$$C_G(X) = \{g \in G \mid xg = gx \text{ for all } x \in X\}$$

and

$$N_G(X) = \{g \in G \mid x^g \in X \text{ for all } x \in X\}.$$

Also, let Y be a subset of G , then $\langle X, Y \rangle$ is the subgroup of G generated by all elements $[x, y]$ for all $x \in X$ and $y \in Y$. If X and Y are subgroups of G , then $\langle X, Y \rangle = \langle Y, X \rangle$ and the subgroup of G generated by X and Y , $\langle X, Y \rangle$, is contained in $N_G(\langle X, Y \rangle)$. Next, $X \leq N_G(Y)$ if and only if $\langle X, Y \rangle \leq Y$. Note that $G' = [G, G]$ and G is a simple group if $\{1_G, G'\}$ are the only normal subgroups of G . Therefore, G is abelian if and only if $G' = 1$.

Thus, if X is a normal subgroup of G , then $gX = Xg$ for all $g \in G$ and the set $\{gX \mid g \in G\}$ (of cosets of X in G) forms a group by multiplication of order $|G|/|X|$ called the quotient group of G over X on which G acts by conjugation and is denoted by G/X . Let $1 \neq G$ be a finite group and let

$$(1) \quad G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where $n \geq 1$ be a chief series of G .

Here, n is the length of the chief series, $G_i \trianglelefteq G$ for all $0 \leq i \leq n$ and G_i is a maximal normal subgroup of G contained in G_{i+1} and G_{i+1}/G_i called a chief factor of G for all $0 \leq i \leq n-1$. Thus G acts by conjugation on G_{i+1}/G_i and G_{i+1}/G_i is G -simple for all $0 \leq i \leq n-1$. Thus G_{i+1}/G_i is a minimal normal subgroup of G/G_i and either G_{i+1}/G_i is an elementary abelian p -group for some prime p or G_{i+1}/G_i is a direct sum of non-abelian simple groups which are G -conjugate and hence is perfect for all $0 \leq i \leq n-1$, see Kurzweil and Stellmacher [3, Theorems 1.7.4 and 1.7.5].

Let $\Sigma = \{G_{i+1}/G_i \mid 0 \leq i \leq n-1\}$.

By the Jordan–Hölder Theorem ([3, Theorem 1.8.1]), any two chief series, A and B of G have length n and the chief factors of A and B can be paired such that paired chief factors are isomorphic G -groups.

LEMMA 1. *Let X and Y be normal subgroups of G with $X \leq Y$ so that G acts by conjugation on Y/X . Let U be a subgroup of G so that U acts by conjugation on Y/X . Thus, $[Y/X, U] = \langle (y^{-1}X)(y^uX) \mid y \in Y \text{ and } u \in U \rangle$. Then*

$$(a) \quad [Y/X, U] = [Y, U]X, \text{ and}$$

$$(b) \quad U \leq C_G(Y/X) \text{ if and only if } [Y, U] \leq X.$$

Proof. Let $u \in U$ and $y \in Y$. Then

$$(y^{-1}X)(u^{-1}X)(yX)(uX) = [y, u]X$$

and (a) is proved. Then (b) is immediate and we are done. \square

Let $F(G)$ denote the Fitting subgroup of G (the unique maximal subnormal nilpotent subgroup of G) and let $S(G)$ denote the solvable radical of G (the unique maximal subnormal solvable subgroup of G) ([3, Theorem 6.3.1]).

As is classic, we have from [2, III, Theorem 4.3]

$$(2) \quad F(G) = \bigcap_{G_{i+1}/G_i \in \Sigma} C_G(G_{i+1}/G_i).$$

Let $\Gamma = \{G_{i+1}/G_i \in \Sigma \mid (G_{i+1}/G_i)' = G_{i+1}/G_i\}$.

Our main result is the next theorem.

THEOREM 2.

$$S(G) = \bigcap_{G_{i+1}/G_i \in \Gamma} C_G(G_{i+1}/G_i).$$

Proof. Let $H = \bigcap_{G_{i+1}/G_i \in \Gamma} C_G(G_{i+1}/G_i)$ so that $H \text{ char } G$. Next, let $G_{i+1}/G_i \in \Gamma$ so that $[G_{i+1}/G_i, S(G)] = ([G_{i+1}, S(G)]G_i)/G_i$. Here

$$[G_{i+1}, S(G)] \leq G_{i+1} \cap S(G)$$

and is normal in G . Thus $[G_{i+1}/G_i, S(G)]$ is a normal solvable subgroup of the perfect group G_{i+1}/G_i . We conclude that $S(G) \leq H$.

Assume that $S(G) \not\leq H$. Then there is a normal subgroup L of G such that $S(G) \not\leq L \trianglelefteq H$ and $L/S(G)$ is a minimal normal subgroup of $G/S(G)$. If $L/S(G)$ is abelian, then $L' \leq S(G)$ and $L \leq S(G)$ by [3, Theorem 6.1.2] which is a contradiction. Thus

$$(L/S(G))' = L'/S(G) = L/S(G) = (L'S(G))/S(G).$$

Here, $L/S(G)$ is G -isomorphic to an element of Γ . Thus $H \leq C_G(L/S(G))$ so that $L' = [L, L] \leq S(G)$ by Lemma 1 (b). This contradiction establishes Theorem 2. \square

Let $1 \neq G^* \trianglelefteq G$. Then we obtain the next corollary.

COROLLARY 3.

- (a) $F(G^*) = G^* \cap F(G) = \bigcap_{G_{i+1}/G_i \in \Gamma} C_{G^*}(G_{i+1}/G_i)$, and
 (b) $S(G^*) = G^* \cap S(G) = \bigcap_{G_{i+1}/G_i \in \Gamma} C_{G^*}(G_{i+1}/G_i)$.

Proof. Since $F(G^*) = G \cap F(G)$ and $S(G^*) = G^* \cap S(G)$, we are done. \square

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