GRÖBNER FANS AND MINIMAL EMBEDDED TORIC RESOLUTIONS OF RATIONAL DOUBLE POINT SINGULARITIES

BÜŞRA KARADENİZ ŞEN

Communicated by Alexandru Dimca

Mourtada and Plénat (2018) give minimal embedded toric resolutions of ADE-singularities in $\mathbb{C}^3$ by constructing regular refinements of their dual Newton polyhedrons with the elements of their embedded valuation sets derived from the jet schemes constructed by the first author in 2014. On the other hand, in the works by Aroca et al., the authors represent the Gröbner fan of a Newton non-degenerate variety and prove that a regular refinement of the Gröbner fan of such a singularity yields an embedded toric resolution. In this paper, we reconstruct embedded toric resolutions of ADE-singularities. We give the explicit constructions of their Gröbner fans and refine them using the concept of profile. This provides an alternative and computationally effective framework to the one based on jet schemes.

AMS 2020 Subject Classification: 14B05, 14M25, 32S45.

Key words: Gröbner fan, embedded toric resolution, rational double point singularities.

1. INTRODUCTION

Let $X \subset \mathbb{C}^n$ be a variety and let $\operatorname{Sing}(X)$ be its singular locus. A resolution of singularities of X is a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is a non-singular variety and $\tilde{X} \setminus \pi^{-1}(\operatorname{Sing}(X)) \cong X \setminus \operatorname{Sing}(X)$. By [10], it is known that X admits a resolution.

J. Nash introduced in [20] the arc space $J_\infty(X) := \{\gamma : \operatorname{Spec} \mathbb{C}[[t]] \to X\}$ of X and he showed that the number of irreducible components of $J_\infty(X)$ passing through $\operatorname{Sing}(X)$ is at most the number of essential irreducible components of the exceptional fiber of a resolution. The conjecture the equality holds is called Nash Problem. In dimension 2, which means for surfaces, the Nash Problem was topologically solved with a positive answer in [8] by J. Fernandez Bobadilla and M. Pe Pereira. Later, it was algebraically solved in [7] by T. De Fernex and R. De Campo. In higher dimension, the Nash Problem had a positive answer for some special cases (see [5, 11, 22], for more detail). On the other hand, the m-th jet scheme $J_m(X) := \{\gamma_m : \operatorname{Spec} \frac{\mathbb{C}[[t]]}{\langle t^{m+1} \rangle} \to X\}$

doi: 10.59277/RRMPA.2025.59.83

of X is the set of truncated arcs. The arc space $J_\infty(X)$ of X may be seen as the limit of the jet scheme $J_m(X)$ of X [6]. After the studies over Nash Problem, in 2013, M. Lejeune-Jalabert, A. Reguera and H. Mourtada introduced the Inverse Nash Problem which is: "Given the jet scheme $J_m(X)$ of X, can we construct a resolution of singularities of X?". In [19], it was answered by C. Plénat and H. Mourtada for a special class of singularities which is called rational double point singularities (ADE-singularities). They constructed minimal embedded toric resolutions of them by using their jet schemes given in [18], where the authors gave the jet schemes of ADE-singularities and provided the embedded valuation sets which are the set of vectors coming from the some special irreducible components of the jet schemes. Moreover, in [18], one can find a relation between the number of irreducible components of jet schemes passing through the singular locus and the number of exceptional curves on the minimal resolution of the singularities. There is a one-to-one correspondence between them.

Furthermore, in [15], Y. Koreeda focused on the relation between the jet schemes of ADE-singularities and their minimal resolution graphs. He proved that a graph obtained by the jet schemes is isomorphic to the minimal resolution graph for A_n-singularities and D_4-singularity [15]. For the other members of ADE-singularities, showing this is still an open question. Despite this, it is a significant progress since it is obvious that there is a relation between the irreducible components of jet schemes and the minimal resolution graph of the singularities.

After all these studies, in this paper we give a different point of view of the resolution of singularities for ADE-singularities. Mainly, we focus on the resolutions by constructing regular refinements of the Gröbner fans of them.

In the 1970's, Newton non-degenerate hypersurface singularities were introduced by A. G. Khovanskii and A. G. Kouchnirenko. There is a guarantee that a Newton non-degenerate hypersurface singularity has a toric resolution [14, 16, 21, 23]. As we mentioned before, in [19], the authors constructed resolutions for ADE-singularities which are Newton non-degenerate hypersurface singularities by giving regular refinements of their dual Newton polyhedrons. On the other hand, in [1], the authors proved that the Gröbner fan is a polyhedral fan and a variety defined by Newton non-degenerate ideal on a toric variety with any characteristic of the base field admits an embedded toric resolution of singularities by constructing regular refinements of their Gröbner fans. The motivation of this article is the following theorem.

Theorem 1.1 ([1, 2]). *Let $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a Newton non-degenerate ideal and let $X = V(I) \subset X_C$ where X_C is the toric variety associated to the cone C. Let Σ be a regular refinement of C which is compatible with* *the Gröbner fan of X. Then the associated toric morphism $\pi_\Sigma : X_\Sigma \to X_C$ is a proper birational morphism and the irreducible components of the total transform π_Σ^{-1} are smooth and meet transversally.*

More precisely, here following [1] and [2], we give the explicit constructions of Gröbner fans of ADE-singularities. Then we provide minimal embedded toric resolutions of them by constructing regular refinements of their Gröbner fans. To do this, we look at the profile of each maximal dimensional Gröbner cones in their Gröbner fans and we use the vectors living inside the profile. The concept of profile was introduced in [3]. Then in [13], the authors extended the definition to non-simplicial cones and used it to show the minimality of the resolutions for a class of singularities which is called rational triple point singularities (RTP-singularities). Thus, the idea of using the profile to find suitable vectors for a regular refinement comes from the studies in [13].

This study is inspired by the works of C. Plénat and H. Mourtada. The reader finds the reconstructed version of the work from [19] using the Gröbner fan. What distinguishes this work is that we construct minimal embedded toric resolutions of ADE-singularities, a fundamental class of singularities, via their Gröbner fans, rather than the classical method of utilizing their dual Newton polyhedrons. Also, the reader finds a brief reminder of the works in [15, 18]. As all these come together, the present survey provides a broad perspective on the resolutions of ADE-singularities.

This article is organized as follows: we start with the profile of a cone in Section 2. The reader can also find some basic definitions in this section. Section 3 is devoted to explain Newton non-degeneracy and the notion of a Gröbner fan. In Section 4, we give minimal embedded toric resolutions of ADE-singularities. In each sub-section, one can find the detailed computations for each of the sub-classes.

2. PROFILE OF A CONE

Let $v_1, v_2, \dots, v_r$ be the vectors in $\mathbb{Z}^n$. A rational polyhedral cone in $\mathbb{R}^n$ generated by $v_1, v_2, \dots, v_r$ is the set

$$C := \langle v_1, v_2, \dots, v_r\rangle = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r \mid \lambda_i \in \mathbb{R}_{\geq 0}, i = 1, 2, \dots, r\}.$$

A rational polyhedral cone is called strongly convex if it does not contain any linear subspace different from $\{\mathbf{0}\}$. A vector is said to be primitive if all its coordinates are relatively prime. From now on, we assume that all cones C are strongly convex with primitive generators.

Let us associate with the set $C = \langle v_1, v_2, \dots, v_r\rangle \subset \mathbb{R}^n$, the following matrix, $M_C := [v_1 v_2 \dots v_r]$, where the columns of M_C are v_i's. Consider the

$r \times r$ minors M_r of M_C. The determinant of C is

$$\det(C) := gcd\big(\det(M_r)\big).$$

Definition 2.1. A cone $C \subset \mathbb{R}^n$ is called regular if $\det(C) = \pm 1$. Otherwise, C is called non-regular.

A collection of cones $\Sigma \subset \mathbb{R}^n$ is called a fan [4] if:

(i) Each face of a cone in Σ is a cone in Σ.

(ii) The intersection of any two cones $C_1, C_2 \in \Sigma$ is a face for both C_1 and C_2.

A fan $\Sigma \subset \mathbb{R}^n$ is called regular if each cone in Σ is regular.

If $C \subset \mathbb{R}^n$ is a non-regular cone then one can obtain a regular sub-cone from C by adding vectors. This process is called regular refinement of C. Each non-regular cone has a regular refinement [9].

To construct a regular refinement of a non-regular cone C, we use the concept of profile introduced in [3]. The profile of a cone C is a specific bounded region. Here we count the vectors with integer entries living inside the profile of C. Specifically, we look at the vectors from the intersection of the profile of C and $\mathbb{Z}^n$.

Definition 2.2 ([3]). The primitive vector over 1-dimensional face of C is called extremal vector.

For $C = \langle v_1, v_2, \ldots, v_r\rangle \subset \mathbb{R}^n$, if $n = r$ then we say the cone C is simplicial. In [3], the authors give the definition of profile for a simplicial cone. In [13], the authors extend the definition of profile to non-simplicial cones. Here our purpose is to use vectors inside the profiles of the maximal dimensional Gröbner cones of X to construct an embedded toric resolution where X corresponds to each of the members of ADE-singularities.

Definition 2.3 ([13]). The profile p_C of a cone $C = \langle v_1, v_2, \ldots, v_r\rangle \subset \mathbb{R}^n$ is the convex hull such that its extremal vectors are exactly $v_1, v_2, \ldots, v_r$.

The extremal vectors of C are on the hyperplanes which are called the boundary of p_C. For the case when C is simplicial, we observe that all extremal vectors are on a unique hyperplane. For the case when C is non-simplicial, the boundary is the union of the hyperplanes. It may happen that though C is a non-simplicial cone, all extremal vectors are on a unique hyperplane as in the case of rational double point singularities.

3. NEWTON NON-DEGENERACY AND GRÖBNER FAN

Let X be a hypersurface defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n, c_{\boldsymbol{\alpha}} \in \mathbb{C}} c_{\boldsymbol{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The support set of f is

$$\mathrm{supp}(f) := \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n \mid c_{\boldsymbol{\alpha}} \neq 0\}.$$

The Newton polyhedron of f is the convex hull given as

$$NP(f) := \mathrm{conv}\{\boldsymbol{\alpha} + (\mathbb{R}_{\geq 0})^n \mid \boldsymbol{\alpha} \in \mathrm{supp}(f)\}.$$

Let F be a face of $NP(f)$. The restriction of f to the set $F \subset \mathbb{Z}^n$ is

$$f|_F := \sum_{\boldsymbol{\alpha} \in \mathrm{supp}(f) \cap F \subset \mathbb{Z}^n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Definition 3.1 ([2, 21]). A hypersurface singularity $X := V(f) \subset \mathbb{C}^n$ is called Newton non-degenerate if for each face F of $NP(f)$, the hypersurface $V(f|_F)$ has no singularities in $(\mathbb{C}^*)^n$.

Newton non-degenerate hypersurface singularities have been studied remarkably since we know that one may construct embedded toric resolutions of singularities from their dual Newton polyhedrons [14, 21, 23]. Here, we also study a special class of Newton non-degenerate hypersurface singularities but in a different way: we construct embedded toric resolutions of them from their Gröbner fans. Let us introduce the notion of Gröbner fan: let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{C}^n$. Take $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in (\mathbb{R}_{\geq 0})^n$. The $\boldsymbol{v}$-order of f is a number given as

$$o_{\boldsymbol{v}}(f) := \min\{v_1\alpha_1 + v_2\alpha_2 + \cdots + v_n\alpha_n \mid (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{supp}(f)\}.$$

The polynomial

$$In_{\boldsymbol{v}}(f) := \sum_{\{\boldsymbol{\alpha} \in \mathrm{supp}(f) \mid \sum_{i=1}^n v_i\alpha_i = o_{\boldsymbol{v}}(f)\}} c_{\boldsymbol{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

is called the $\boldsymbol{v}$-initial form of f. We define an equivalence relation as

$$\boldsymbol{v} \sim \boldsymbol{w} \iff In_{\boldsymbol{v}}(f) = In_{\boldsymbol{w}}(f)$$

for $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in (\mathbb{R}_{\geq 0})^n$. The closure of the set

$$C_{\boldsymbol{v}}(f) := \{\boldsymbol{w} \in (\mathbb{R}_{\geq 0})^n \mid In_{\boldsymbol{v}}(f) = In_{\boldsymbol{w}}(f)\}$$

is a polyhedral cone. The union of these cones forms a fan [17]. We use the fan formed by the intersection of this fan and the first orthant. It is called the Gröbner fan $G(X)$ of X [2]. Each cone in $G(X)$ is called a Gröbner cone. As a remark, one can compute the Gröbner fan by using a computer program which is called GFAN [12].

4. EMBEDDED TORIC RESOLUTION

The rational double point singularities are given by one of the following equations:

$$\begin{aligned}
A_n &: f(x,y,z) = xy - z^{n+1}, \quad n \in \mathbb{N} \\
D_n &: f(x,y,z) = z^2 - x(y^2 + x^{n-2}), \quad n \in \mathbb{N}, n \geq 4 \\
E_6 &: f(x,y,z) = z^2 + y^3 + x^4 \\
E_7 &: f(x,y,z) = x^2 + y^3 + yz^3 \\
E_8 &: f(x,y,z) = z^2 + y^3 + x^5.
\end{aligned}$$

Remark 4.1. Each member of these ADE-singularities is Newton non-degenerate.

Embedded toric resolutions of ADE-singularities were studied in [18, 19]. Differently here, we construct regular refinements of Gröbner fans of each member of them. This leads us to their embedded toric resolutions [1, 2]. Additionally, these resolutions are minimal which means that we have the next definition.

Definition 4.2. An embedded toric resolution is called minimal if:

(i) The vectors in the regular refinement are irreducible.

(ii) The resolution graph obtained from the refinement does not have any -1 curve. This means that the self-intersection of each vertex is different from -1 (this gives us that the resolution graph is minimal).

Note that the self-intersection of a vertex is computed as the number $-s \in \mathbb{Z}_{\leq 0}$ with $s.v = \sum_{i=1}^{m} v_i$. This is called the self-intersection of the vertex v in the graph where each v_i is adjacent to v in the regular refinement.

4.1. A_n-singularities

Consider the hypersurface $X = V(f) \subset \mathbb{C}^3$ where

$$f(x,y,z) = xy - z^{n+1}, \quad n \in \mathbb{N}.$$

Given $v_1 \in (\mathbb{R}_{\geq 0})^3$, let $In_{v_1}(f) = f$. The polyhedral cone associated with v_1 is

$$C_{v_1}(f) = \big\langle (n+1, 0, 1), (0, n+1, 1) \big\rangle.$$

Given $v_2 \in (\mathbb{R}_{\geq 0})^3$, let $In_{v_2}(f) = xy$. The polyhedral cone associated with v_2 is

$$C_{v_2}(f) = \big\langle (0,0,1), (n+1,0,1), (0, n+1, 1) \big\rangle.$$

Given $v_3 \in (\mathbb{R}_{\geq 0})^3$, let $In_{v_3}(f) = -z^{n+1}$. The polyhedral cone associated with v_3 is

$$C_{v_3}(f) = \big\langle (1,0,0), (0,1,0), (n+1,0,1), (0,n+1,1) \big\rangle.$$

The Gröbner fan $G(X)$ of X and its Gröbner cones can be seen in Figure 1.

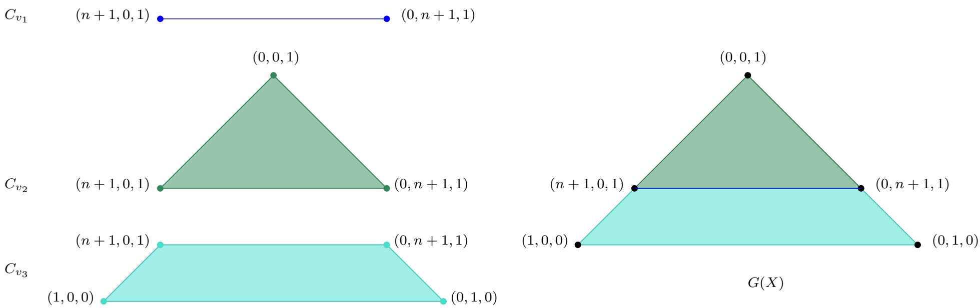

Figure 1 – The Gröbner fan and its Gröbner cones for A_n-singularities.

Notation 4.3. We draw the cones/fans by taking the intersection with $x + y + z = 1$ to visualize easily.

Remark 4.4. The Gröbner fan $G(X)$ of X is the same as its dual Newton polyhedron which is given in [19]. As previously mentioned, the authors construct an embedded toric resolution by means of regular refinement of its dual Newton polyhedron with the vectors coming from its jet schemes. Here we look at the profile to obtain special vectors to do regular refinement of $G(X)$.

For each maximal dimensional cones (C_{v_2} and C_{v_3}) of $G(X)$, the profiles p_{C_2} and $p_{C_{v_3}}$ are given by the region shown with the pink color in Figure 2.

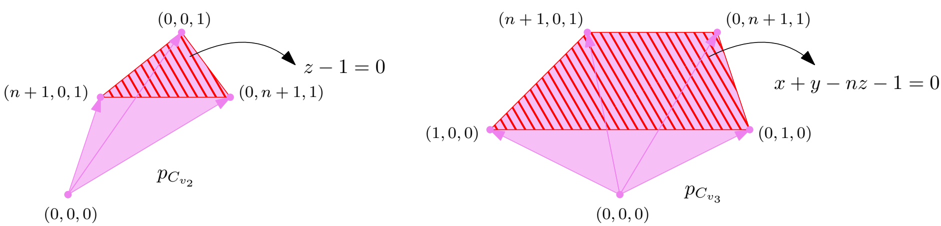

Figure 2 – The profiles of maximal dimensional Gröbner cones for A_n-singularities.

The boundaries of $p_{C_{v_2}}$ and $p_{C_{v_3}}$ are

$$H_1 : z - 1 = 0 \text{ and } H_2 : x + y - nz - 1 = 0,$$

respectively. Note that although C_{v_3} is a non-simplicial cone, all its extremal vectors are on a unique hyperplane.

THEOREM 4.5. i) *For A_n-singularities, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible which means they are free over $\mathbb{Z}$.*

ii) *For A_n-singularities, the elements on the skeleton of $G(X)$ give the minimal resolution graph of the singularities.*

Proof. i) The cones C_{v_2} and C_{v_3} are the maximal dimensional Gröbner cones for $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_2}}$ are

- $(n+1, 0, 1), (n, 1, 1), \ldots, (0, n+1, 1)$
- $(n, 0, 1), (n-1, 1, 1), \ldots, (0, n, 1)$

$\vdots$

- $(1, 0, 1), (0, 1, 1)$
- $(0, 0, 1)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_3}}$ are

- $(n+1, 0, 1), (n, 1, 1), \ldots, (0, n+1, 1)$
- $(1, 0, 0), (0, 1, 0)$.

We construct a regular refinement of $G(X)$ with these elements. This refinement can be seen in Figure 3.

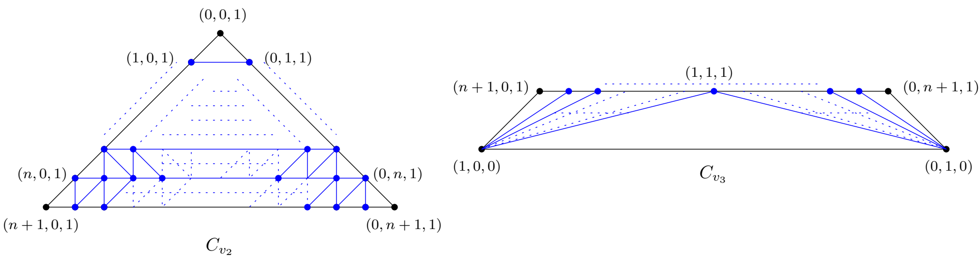

Figure 3 – A regular refinement for A_n-singularities.

Each element in the refinement is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ is given in Figure 4.

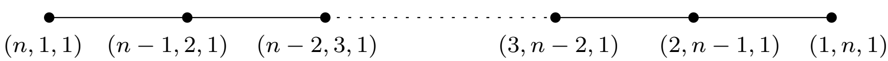

Figure 4 – The resolution graph of A_n-singularities.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence it is a minimal resolution graph. □

COROLLARY 4.6. *For A_n-singularities, the set of vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$ and $\mathbb{Z}^3$ are exactly the same as the set of embedded valuations of X.*

This fact can be seen combinatorially by looking at [18, 19].

COROLLARY 4.7. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.5(i), the elements give an embedded toric resolution and these elements are irreducible. By Theorem 4.5(ii), the resolution graph is minimal. Hence, the embedded toric resolution we construct is minimal. □

As we mentioned in the introduction, there are some significant studies about the resolutions of ADE-singularities. Here, one can find the next results.

THEOREM 4.8 ([18]). *For A_n-singularities, when $n = m$, in m-jet scheme over the singular locus there are n irreducible components which is equal to the number of vertices of its minimal resolution graph.*

These irreducible components $J^i_m(X)$, $1 \leq i \leq n$, are given by the following ideals in $\mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_m, z_0, \ldots, z_m]$, respectively:

$$\begin{aligned} I^1_m &:= \langle x_0, y_0, z_0, x_1, \ldots, x_{(m-1)} \rangle \\ I^2_m &:= \langle x_0, y_0, z_0, x_1, \ldots, x_{(m-2)}, y_1 \rangle \\ &\vdots \\ I^{n-1}_m &:= \langle x_0, y_0, z_0, x_1, y_1, y_2 \ldots, y_{(m-2)} \rangle \\ I^n_m &:= \langle x_0, y_0, z_0, y_1, y_2 \ldots, y_{(m-1)} \rangle. \end{aligned}$$

In [15], Y. Koreeda gives the following construction. When $n = m$, for A_n-singularities consider the set of intersection of the irreducible components:

$$K := \left\{ J^i_m(X) \cap J^j_m(X) \mid i \neq j, i, j \in \{1, 2, \ldots, n\} \right\}.$$

LEMMA 4.9 ([15]). *We have the inclusion relation*

$$I^k_m \cap I^\ell_m \subset I^i_m \cap I^j_m$$

where $i \leq k \leq \ell \leq j$ which gives

$$J^i_m(X) \cap J^j_m(X) \subset J^k_m(X) \cap J^\ell_m(X).$$

All maximal elements of the set K with respect to inclusion relation form a subset E.

Conjecture 4.10 ([15]). *For some fix m, the graph G is constructed with (V, E) as*

(i) *The irreducible components $J_m^i(X)$ are the vertices of G.*

(ii) *The elements of the set E are the edges of G.*

For A_n-singularities, when $n = m$, the vertex set

$$\{J_m^1(X), J_m^2(X), \ldots, J_m^n(X)\}$$

and the edges set is

$$\{J_m^i(X) \cap J_m^j(X) \mid j = i + 1\}.$$

The graph obtained by the construction is given in Figure 5.

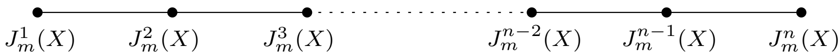

Figure 5 – The graph of A_n-singularities obtaining from the construction.

Note that each $J_m^i(X)$ corresponds to the vector $(n - i + 1, i, 1)$.

Corollary 4.11 ([15]). *For A_n-singularities, the graph obtained by the construction is isomorphic to its minimal resolution graph.*

4.2. D_n-singularities

Consider the hypersurface $X = V(f) \subset \mathbb{C}^3$ where

$$f(x, y, z) = z^2 - xy^2 - x^{n-1}, \quad n \in \mathbb{N}, n \geq 4.$$

The Gröbner cones are:

$$\begin{aligned}
C_{v_1}(f) &= \langle (2, n-2, n-1) \rangle \\
C_{v_2}(f) &= \langle (2, 0, 1), (2, n-2, n-1) \rangle \\
C_{v_3}(f) &= \langle (0, 0, 1), (2, n-2, n-1) \rangle \\
C_{v_4}(f) &= \langle (0, 1, 0), (2, n-2, n-1) \rangle \\
C_{v_5}(f) &= \langle (2, 0, 1), (0, 0, 1), (2, n-2, n-1) \rangle \\
C_{v_6}(f) &= \langle (0, 1, 0), (0, 0, 1), (2, n-2, n-1) \rangle \\
C_{v_7}(f) &= \langle (1, 0, 0), (0, 1, 0), (2, 0, 1), (2, n-2, n-1) \rangle
\end{aligned}$$

such that $In_{v_1}(f) = f$, $In_{v_2}(f) = z^2 - xy^2$, $In_{v_3}(f) = -xy^2 - x^{n-1}$, $In_{v_4}(f) = z^2 - x^{n-1}$, $In_{v_5}(f) = -xy^2$, $In_{v_6}(f) = -x^{n-1}$ and $In_{v_7}(f) = z^2$.

The Gröbner fan $G(X)$ of X is the union $\cup_{i=1}^{7} C_{v_i}$ which can be seen in Figure 6.

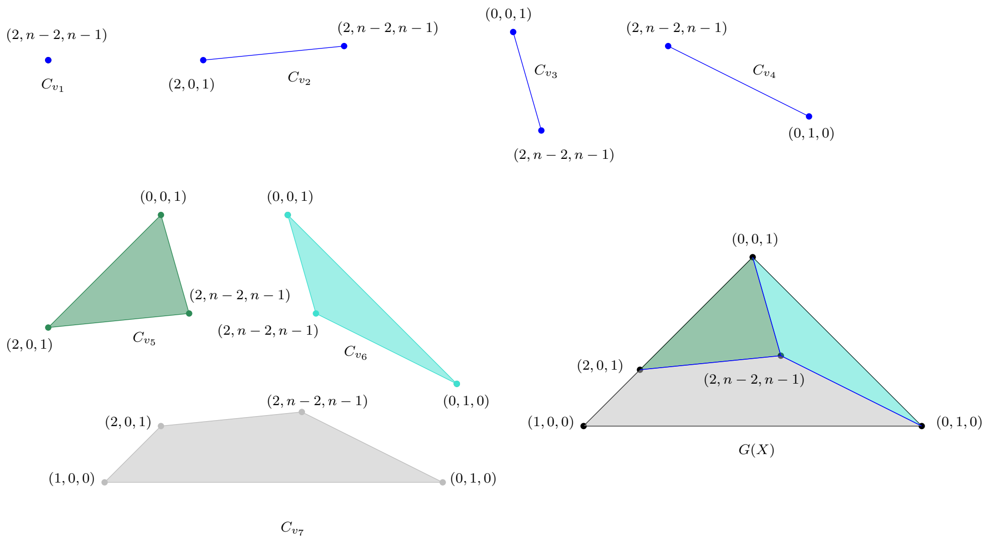

Figure 6 – The Gröbner fan and its Gröbner cones for D_n-singularities.

Remark 4.12. The Gröbner fan $G(X)$ of X is same as its dual Newton polyhedron [19].

For each maximal dimensional cones (C_{v_5}, C_{v_6} and C_{v_7}) of $G(X)$, the profiles $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are given in Figure 7.

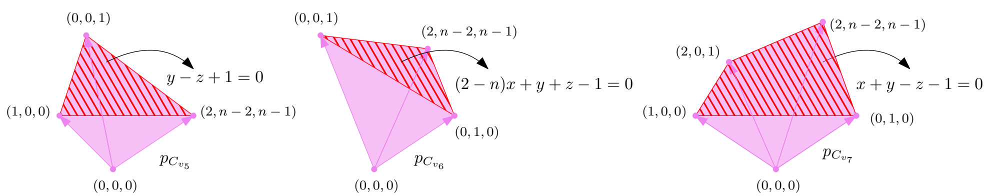

Figure 7 – The profiles of maximal dimensional Gröbner cones for D_n-singularities.

The boundaries of $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are:

$$H_1 : y - z + 1 = 0,$$
$$H_2 : (2 - n)x + y + z - 1 = 0 \text{ and}$$
$$H_3 : x + y - z - 1 = 0,$$

respectively. Note that although C_{v_7} is a non-simplicial cone, all its extremal vectors are on a unique hyperplane.

We have two different sub-cases depending if n is even or odd.

Case 1. n is even.

THEOREM 4.13. i) *For D_n-singularities, n is even, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible which means they are free over $\mathbb{Z}$.*

ii) *For D_n-singularities, n is even, the elements on the skeleton of $G(X)$ give the minimal resolution graph of the singularities.*

Proof. i) The cones C_{v_5}, C_{v_6} and C_{v_7} are the maximal dimensional cones for $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_5}}$ are:

- $(1,0,1),(1,1,2),\dots,(1,\frac{n-2}{2},\frac{n}{2})$
- $(2,0,1),(2,1,2),\dots,(2,n-2,n-1)$
- $(0,0,1)$

The elements for $\mathbb{Z}^3 \cap p_{C_{v_6}}$ are

- $(0,0,1),(0,1,0),(2,n-2,n-1),(1,\frac{n-2}{2},\frac{n}{2})$
- $(1,0,0),(0,1,0)$

The elements for $\mathbb{Z}^3 \cap p_{C_{v_7}}$ are

- $(2,0,1),(2,1,2),\dots,(2,n-2,n-1)$
- $(1,1,1),(1,2,2),\dots,(1,\frac{n}{2}-\frac{n}{2}-1)$
- $(1,0,0),(0,1,0)$.

A regular refinement of the Gröbner fan $G(X)$ of X is given in Figure 8.

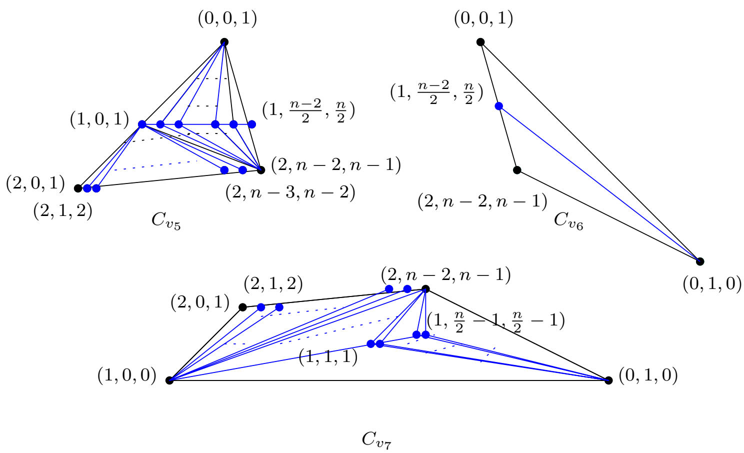

Figure 8 – A regular refinement for D_n-singularities, n is even.

Each element in the refinement is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ can be seen in Figure 9.

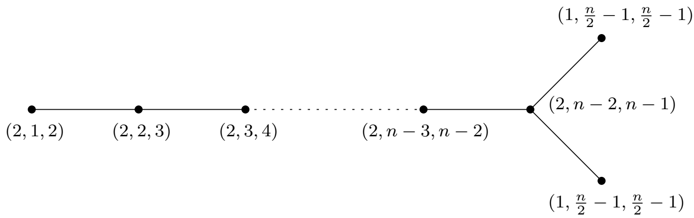

Figure 9 – The resolution graph of D_n-singularities, n is even.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence, it is a minimal resolution graph. □

COROLLARY 4.14. *For D_n-singularities, n is even, the set of the vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$ and $\mathbb{Z}^3$ are exactly the same as the set of embedded valuations set of X given in* [18].

COROLLARY 4.15. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.13(i), the elements give an embedded toric resolution and these elements are irreducible. By Theorem 4.13(ii), the resolution graph is minimal. Hence, the embedded toric resolution we construct is minimal. □

Case 2. n is odd.

THEOREM 4.16. i) *For D_n-singularities, n is odd, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible, which means they are free over $\mathbb{Z}$.*

ii) *For D_n-singularities, n is odd, the elements on the skeleton of $G(X)$ give the minimal resolution graph of the singularities.*

Proof. i) The cones C_{v_5}, C_{v_6} and C_{v_7} are the maximal dimensional cones in $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_5}}$ are:

- $(1, 1, 2), (1, 2, 3), \ldots, (1, \frac{n-3}{2}, \frac{n-1}{2})$
- $(2, 0, 1), (2, 1, 2), \ldots, (2, n-2, n-1)$
- $(0, 0, 1)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_6}}$ are:

- $(0,0,1),(0,1,0),(2,n-2,n-1),(1,\frac{n-1}{2},\frac{n-1}{2})$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_7}}$ are:

- $(2,0,1),(2,1,2),\ldots,(2,n-2,n-1)$
- $(1,1,1),(1,2,2),\ldots,(1,\frac{n-1}{2},\frac{n-1}{2})$
- $(1,0,0),(0,1,0)$.

A regular refinement of the Gröbner fan $G(X)$ of X is given in Figure 10.

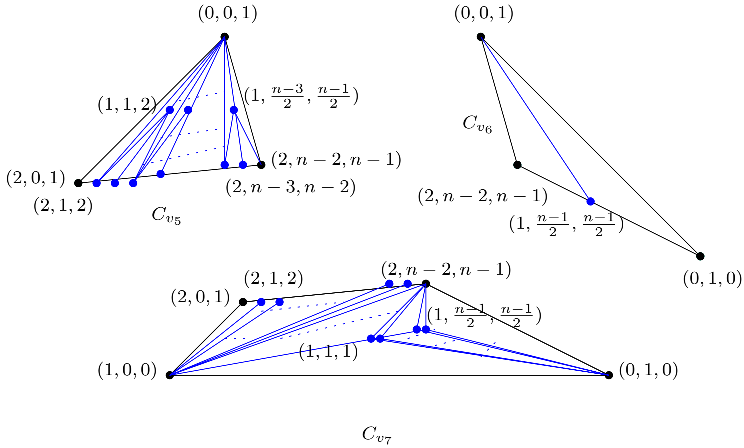

Figure 10 – A regular refinement for D_n-singularities, n is odd.

Each element in the refinement is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ can be seen in Figure 11.

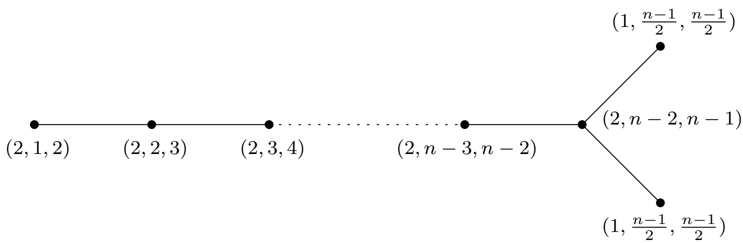

Figure 11 – The resolution graph of D_n-singularities, n is odd.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence, it is a minimal resolution graph. $\square$

Corollary 4.17. *For D_n-singularities, n is odd, the set of the vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$*

and $\mathbb{Z}^3$ *are exactly the same as the set of embedded valuations of* X *given in* [18].

COROLLARY 4.18. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.16(i), the elements give an embedded toric resolution and these elements are irreducible. By Theorem 4.16(ii), the resolution graph is minimal. Hence, the embedded toric resolution we construct is minimal. □

THEOREM 4.19 ([18]). *For* D_n*-singularities, when* $m \geq 2n-3$*, in the* m*-jet scheme over the singular locus there are* n *irreducible components which is equal to the number of vertices of the minimal resolution graph.*

In [15], Y. Koreeda gave the construction for D_4-singularity. For $n > 4$, showing the construction is still an open question.

4.3. E_6-singularity

Consider the hypersurface $X = V(f) \subset \mathbb{C}^3$ where

$$f(x,y,z) = z^2 + y^3 + x^4.$$

The Gröbner cones are:

$$\begin{aligned}
C_{v_1}(f) &= \langle (3,4,6) \rangle \\
C_{v_2}(f) &= \langle (1,0,0),(3,4,6) \rangle \\
C_{v_3}(f) &= \langle (0,0,1),(3,4,6) \rangle \\
C_{v_4}(f) &= \langle (0,1,0),(3,4,6) \rangle \\
C_{v_5}(f) &= \langle (1,0,0),(0,0,1),(3,4,6) \rangle \\
C_{v_6}(f) &= \langle (0,1,0),(0,0,1),(3,4,6) \rangle \\
C_{v_7}(f) &= \langle (1,0,0),(0,1,0),(3,4,6) \rangle
\end{aligned}$$

such that

$$In_{v_1}(f) = f,\ In_{v_2}(f) = z^2 + y^3,$$
$$In_{v_3}(f) = y^3 + x^4,\ In_{v_4}(f) = z^2 + x^4,$$
$$In_{v_5}(f) = y^3,\ In_{v_6}(f) = x^4 \text{ and } In_{v_7}(f) = z^2.$$

The Gröbner fan $G(X)$ of X is the union $\cup_{i=1}^{7} C_{v_i}$ which can be seen in Figure 12.

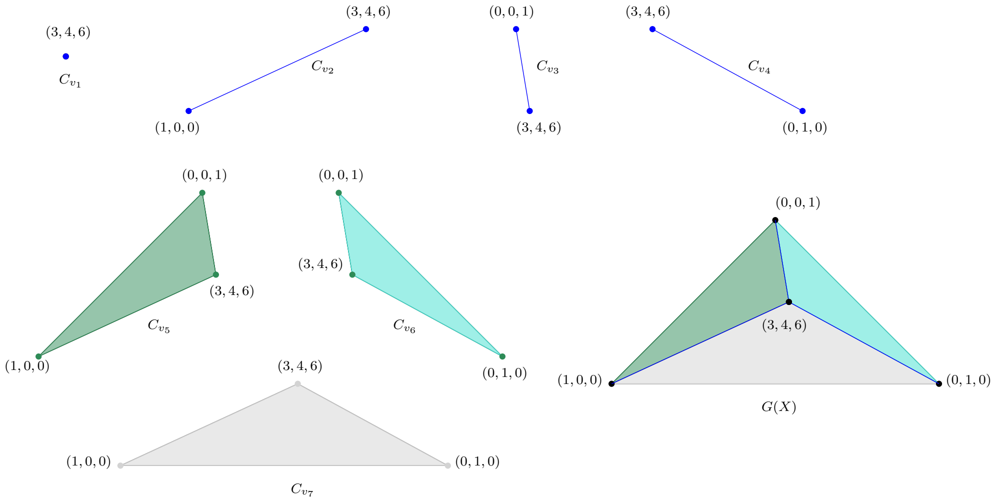

Figure 12 – The Gröbner fan and its Gröbner cones for E_6-singularity.

Remark 4.20. The Gröbner fan $G(X)$ of X is the same as its dual Newton polyhedron [19].

For each maximal dimensional cones (C_{v_4}, C_{v_5} and C_{v_6}) of $G(X)$, the profiles $p_{C_{v_4}}$, $p_{C_{v_5}}$ and $p_{C_{v_6}}$ are given in Figure 13.

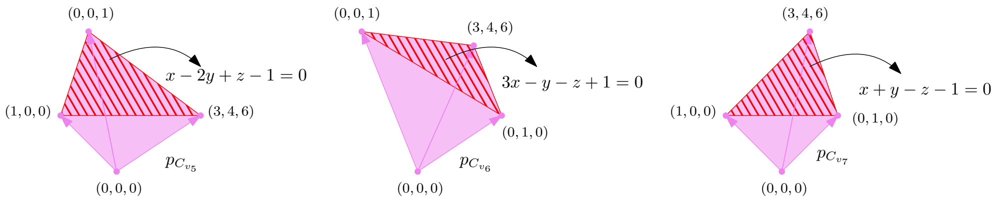

Figure 13 – The profiles of maximal dimensional Gröbner cones for E_6-singularity.

The boundaries of $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are $H_1 : x - 2y + z - 1 = 0$, $H_2 : 3x - y - z + 1 = 0$ and $H_3 : x + y - z - 1 = 0$, respectively.

Theorem 4.21. i) *For E_6-singularity, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible, which means they are free over $\mathbb{Z}$.*

ii) *For E_6-singularity, the elements on the skeleton of $G(X)$ give the minimal resolution graph of the singularity.*

Proof. i) The cones C_{v_5}, C_{v_6} and C_{v_7} are the maximal dimensional cones in $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_5}}$ are:

- $(1,0,0),(0,0,1),(3,4,6),(1,1,2),(2,2,3)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_6}}$ are:

- $(0,1,0),(0,0,1),(3,4,6),(1,1,2),(2,3,4)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_7}}$ are:

- $(1,0,0),(0,1,0),(3,4,6),(1,1,1),(1,2,2),(2,2,3),(2,3,4)$.

A regular refinement of the Gröbner fan $G(X)$ of X is in Figure 14.

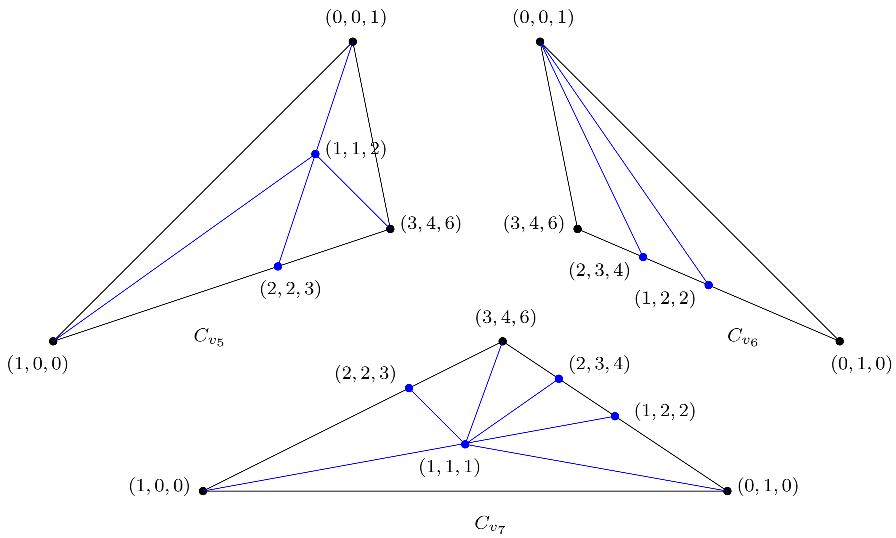

Figure 14 – A regular refinement for E_6-singularity.

Each element in the refinement is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ given in Figure 15.

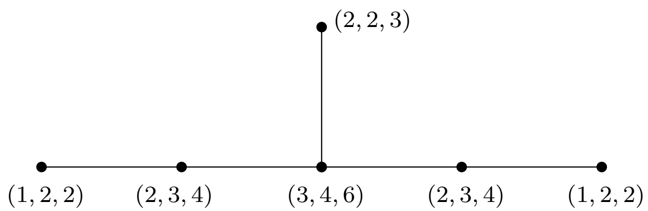

Figure 15 – The resolution graph of E_6-singularity.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence, it is a minimal resolution graph. $\square$

Corollary 4.22. *For E_6-singularity, the set of the vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$ and $\mathbb{Z}^3$ are exactly the same as the set of embedded valuations of X given in* [18].

Corollary 4.23. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.21(i), the elements give an embedded toric resolution and these elements are irreducible. By Theorem 4.21(ii), the resolution graph is minimal. Hence, the embedded toric resolution is minimal. □

Theorem 4.24 ([18]). *For E_6-singularity, when $m \geq 11$, in the m-jet scheme over the singular locus there are six irreducible components which is equal to the number of vertices of its minimal resolution graph.*

4.4. E_7-singularity

Consider the hypersurface $X = V(f) \subset \mathbb{C}^3$ where

$$f(x,y,z) = x^2 + y^3 + yz^3.$$

The Gröbner cones are:

$$\begin{aligned}
C_{v_1}(f) &= \langle (9,6,4) \rangle \\
C_{v_2}(f) &= \langle (1,0,0), (9,6,4) \rangle \\
C_{v_3}(f) &= \langle (0,0,1), (9,6,4) \rangle \\
C_{v_4}(f) &= \langle (1,2,0), (9,6,4) \rangle \\
C_{v_5}(f) &= \langle (1,0,0), (0,0,1), (9,6,4) \rangle \\
C_{v_6}(f) &= \langle (0,1,0), (0,0,1), (1,2,0), (9,6,4) \rangle \\
C_{v_7}(f) &= \langle (1,0,0), (1,2,0), (9,6,4) \rangle
\end{aligned}$$

such that

$$In_{v_1}(f) = f,\ In_{v_2}(f) = y^3 + yz^3,$$
$$In_{v_3}(f) = x^2 + y^3, In_{v_4}(f) = x^2 + yz^3,$$
$$In_{v_5}(f) = y^3,\ In_{v_6}(f) = x^2,\ \text{and } In_{v_7}(f) = yz^3.$$

The Gröbner fan $G(X)$ of X is the union $\cup_{i=1}^{7} C_{v_i}$ which can be seen in Figure 16.

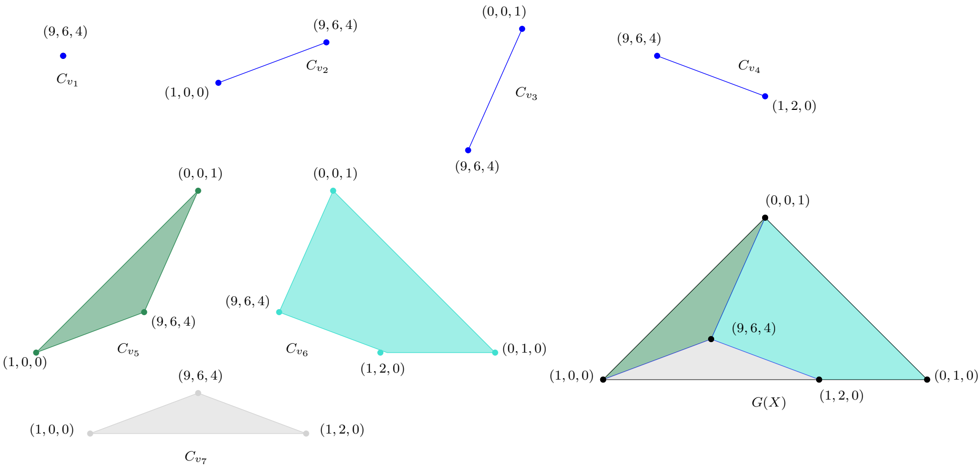

Figure 16 – The Gröbner fan and its Gröbner cones for E_7-singularity.

Remark 4.25. The Gröbner fan $G(X)$ of X is the same as its dual Newton polyhedron [19].

For each maximal dimensional cones (C_{v_5}, C_{v_6} and C_{v_7}) of $G(X)$, the profiles $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are given in Figure 17.

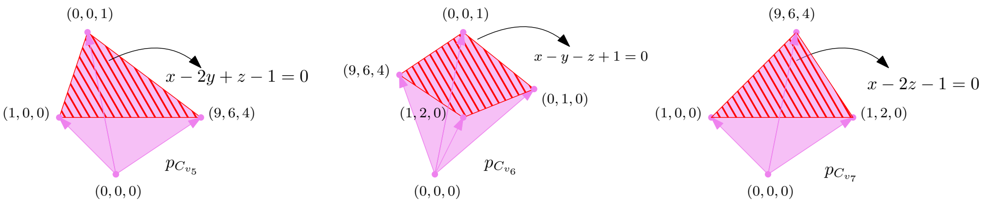

Figure 17 – The profiles of maximal dimensional Gröbner cones for E_7-singularity.

The boundaries of $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are $H_1 : x - 2y + z - 1 = 0$, $H_2 : x - y - z + 1 = 0$ and $H_3 : x - 2z - 1 = 0$, respectively. Note that although C_{v_6} is a non-simplicial cone, all its extremal vectors are on a unique hyperplane.

Theorem 4.26. i) *For E_7-singularity, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible, which means they are free over $\mathbb{Z}$.*

ii) *For E_7-singularity, the elements on the skeleton of $G(X)$ give the minimal resolution graph of the singularity.*

Proof. i) The cones C_{v_5}, C_{v_6} and C_{v_7} are the maximal dimensional cones in $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_5}}$ are:

• $(1,0,0), (0,0,1), (9,6,4), (6,4,3), (5,3,2), (3,2,2), (2,1,1)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_6}}$ are:

• $(0,1,0), (0,0,1), (9,6,4), (1,2,0), (7,5,3), (5,4,2), (3,3,1), (6,4,3), (3,2,2)$

• $(4,3,2), (2,2,1), (1,1,1)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_7}}$ are:

• $(1,0,0), (1,2,0), (9,6,4), (1,1,1), (3,2,1), (5,3,2), (7,5,3), (5,4,2), (3,3,1)$.

A regular refinement of the Gröbner fan $G(X)$ of X is given in Figure 18.

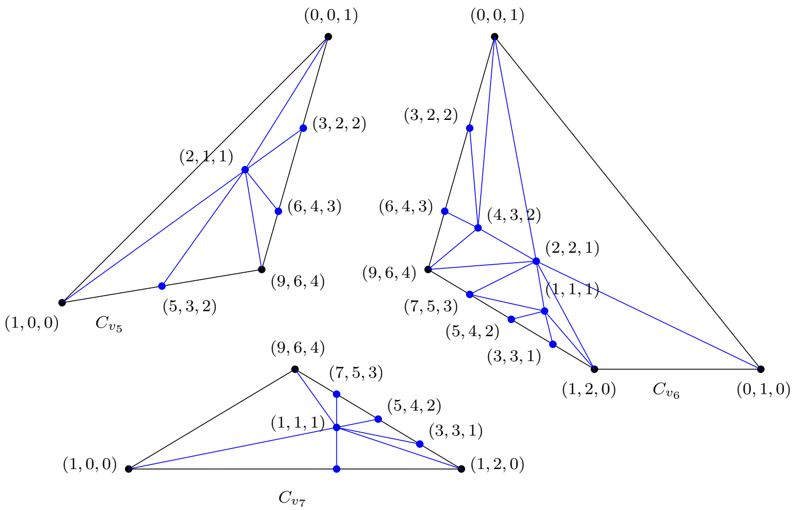

Figure 18 – A regular refinement for E_7-singularity.

Each element is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ is given in Figure 19.

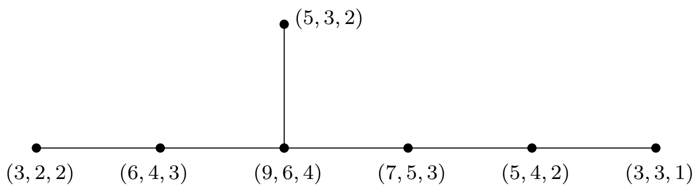

Figure 19 – The resolution graph of E_7-singularity.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence, it is a minimal resolution graph. □

COROLLARY 4.27. *For E_7-singularity, the set of the vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$ and $\mathbb{Z}^3$ are exactly the same as the set of embedded valuations of X given in* [18].

COROLLARY 4.28. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.26(i), the elements give an embedded toric resolution and these elements are irreducible. By Theorem 4.26(ii), the resolution graph is minimal. Hence, the embedded toric resolution is minimal. □

THEOREM 4.29 ([18]). *For E_7-singularity, when $m \geq 17$, in the m-jet scheme over the singular locus there are seven irreducible components which is equal to the number of vertices of its minimal resolution graph.*

4.5. E_8-singularity

Consider the hypersurface $X = V(f) \subset \mathbb{C}^3$ where

$$f(x, y, z) = z^2 + y^3 + x^5.$$

The Gröbner cones are:

$$\begin{aligned}
C_{v_1}(f) &= \langle (6, 10, 15) \rangle \\
C_{v_2}(f) &= \langle (1, 0, 0), (6, 10, 15) \rangle \\
C_{v_3}(f) &= \langle (0, 0, 1), (6, 10, 15) \rangle \\
C_{v_4}(f) &= \langle (0, 1, 0), (6, 10, 15) \rangle \\
C_{v_5}(f) &= \langle (1, 0, 0), (0, 0, 1), (6, 10, 15) \rangle \\
C_{v_6}(f) &= \langle (0, 1, 0), (0, 0, 1), (6, 10, 15) \rangle \\
C_{v_7}(f) &= \langle (1, 0, 0), (0, 1, 0), (6, 10, 15) \rangle
\end{aligned}$$

such that

$$In_{v_1}(f) = f, \; In_{v_2}(f) = z^2 + y^3,$$
$$In_{v_3}(f) = y^3 + x^5, \; In_{v_4}(f) = z^2 + x^5,$$
$$In_{v_5}(f) = y^3, \; In_{v_6}(f) = x^5 \text{ and } In_{v_7}(f) = z^2.$$

The Gröbner fan $G(X)$ of X is the union $\cup_{i=1}^{7} C_{v_i}$ which can be seen in Figure 20.

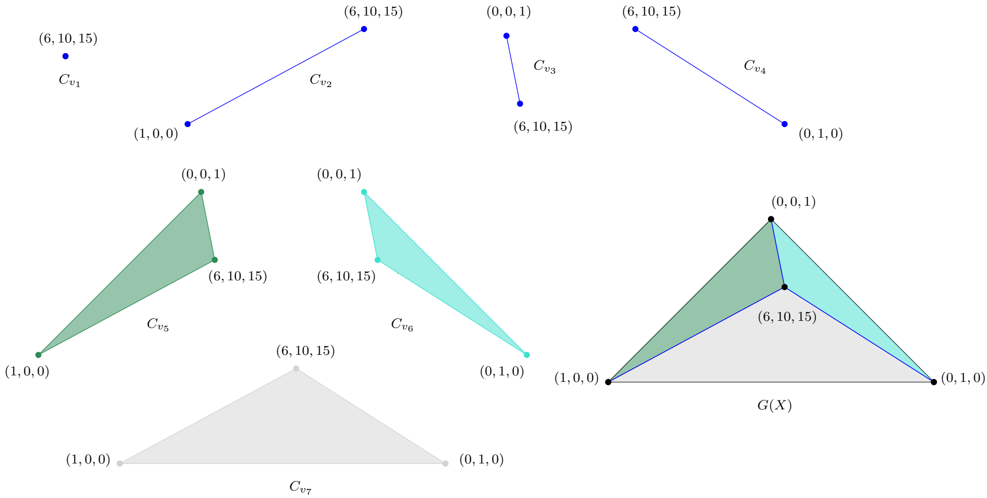

Figure 20 – The Gröbner fan and its Gröbner cones for E_8-singularity.

Remark 4.30. The Gröbner fan $G(X)$ of X is the same as its dual Newton polyhedron [19].

For each maximal dimensional cones (C_{v_5}, C_{v_6} and C_{v_7}) of $G(X)$, the profiles $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are given in Figure 21.

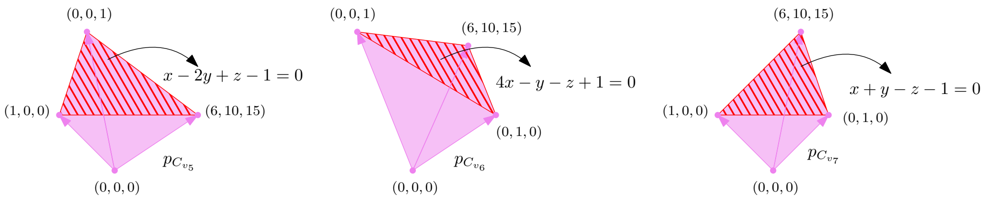

Figure 21 – The profiles of maximal dimensional Gröbner cones for E_8-singularity.

The boundaries of $p_{C_{v_5}}$, $p_{C_{v_6}}$ and $p_{C_{v_7}}$ are $H_1 : x - 2y + z - 1 = 0$, $H_2 : 4x - y - z + 1 = 0$ and $H_3 : x + y - z - 1 = 0$, respectively.

THEOREM 4.31. i) *For E_8-singularity, the elements of the set consisting of $\mathbb{Z}^3 \cap p_{C_{v_i}}$ give an embedded toric resolution of X where C_{v_i} is a maximal dimensional Gröbner cone in $G(X)$. Moreover, these elements are irreducible, which means they are free over $\mathbb{Z}$.*

ii) *For E_8-singularity, the elements on the skeleton of $G(X)$ give minimal resolution graph of the singularity.*

Proof. i) The cones C_{v_5}, C_{v_6} and C_{v_7} are the maximal dimensional cones in $G(X)$. The elements for $\mathbb{Z}^3 \cap p_{C_{v_5}}$ are:

- $(0,1,0), (0,0,1), (6,10,15), (2,2,3), (3,4,6), (4,6,9), (5,8,12), (1,1,2)$
- $(2,3,5), (3,5,8)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_6}}$ are:

- $(0,1,0), (0,0,1), (6,10,15), (2,4,5), (4,7,10), (3,5,8), (1,2,3)$.

The elements for $\mathbb{Z}^3 \cap p_{C_{v_7}}$ are:

- $(1,0,0), (0,1,0), (6,10,15), (2,2,3), (3,4,6), (4,6,9), (5,8,12), (2,4,5), (4,7,10)$
- $(1,2,2), (2,3,4), (3,4,6), (3,5,7), (1,2,2), (2,3,4)$.

A regular refinement of the Gröbner fan $G(X)$ of X is given in Figure 22.

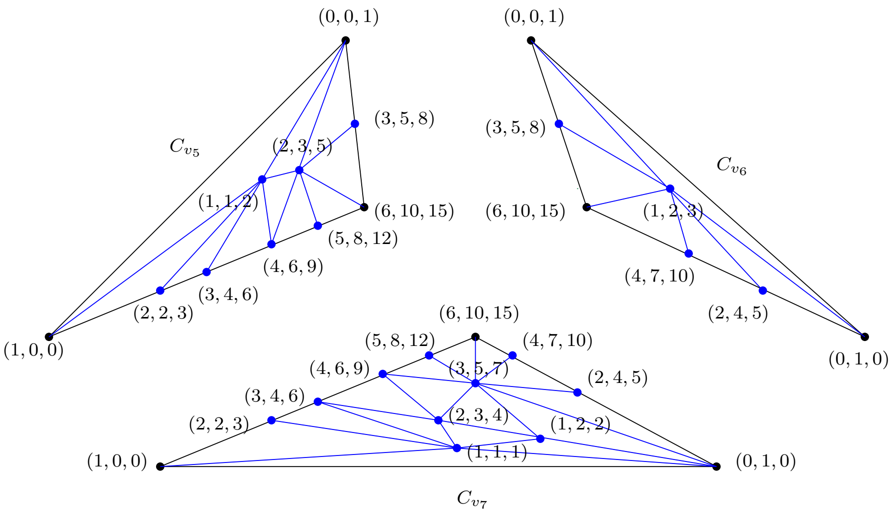

Figure 22 – A regular refinement for E_8-singularity.

Each element in the refinement is irreducible since it cannot be written as a sum of two other elements.

ii) The elements on the skeleton of $G(X)$ is given in Figure 23.

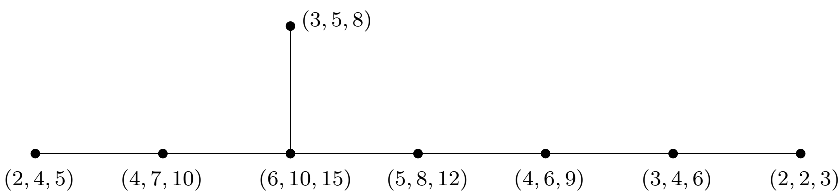

Figure 23 – The resolution graph of E_8-singularity.

The self-intersection of each vertex is -2 so it has no -1 curves. Hence, it is a minimal resolution graph. □

Corollary 4.32. *For E_8-singularity, the set of the vectors in the intersection of profile of the maximal dimensional Gröbner cones of $G(X)$ and $\mathbb{Z}^3$ are exactly the same as the set of embedded valuations set of X given in* [18].

Corollary 4.33. *The embedded toric resolution that we construct is minimal.*

Proof. By Theorem 4.31(i), the elements give a resolution and these elements are irreducible. By Theorem 4.31(ii), the resolution graph is minimal. Hence, the embedded toric resolution is minimal. □

Acknowledgments. The author would like to express sincere gratitude to the anonymous referee for the prompt and careful review.

REFERENCES

[1] F. Aroca, M. Gomes-Morales and H. Mourtada, *Gröbner fan and embedded resolutions of ideals on toric varieties.* Beitr. Algebra Geom. **65** (2024), *1*, 217–228.

[2] F. Aroca, M. Gomez-Morales, and K. Shabbir, *Torical modification of Newton non-degenerate ideals.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **107** (2013), *1*, 221–239.

[3] C. Bouvier and G. Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations de variétés toriques.* Tohoku Math. J. (2) **47** (1995), *1*, 125–149.

[4] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric Varieties.* Grad. Stud. Math. 124, American Mathematical Society, Providence, RI, 2011.

[5] T. de Fernex, *Three-dimensional counter-examples to the Nash problem.* Compos. Math. **149** (2013), *9*, 1519–1534.

[6] T. de Fernex, *The space of arcs of an algebraic variety.* In: *Algebraic Geometry.* Proc. Sympos. Pure Math. (Salt Lake City, 2015) 97, pp. 169–197. Amer. Math. Soc., Providence, RI, 2018.

[7] T. de Fernex and R. Docampo, *Terminal valuations and the Nash problem.* Invent. Math. **203** (2016), *1*, 303–331.

[8] J. Fernandez de Bobadilla and M. Pe Pereira, *The Nash problem for surfaces.* Ann. of Math. (2) **176** (2012), *3*, 2003–2029.

[9] W. Fulton, *Introduction to Toric Varieties.* Ann. of Math. Stud. 131, Princeton Univ. Press, Princeton, NJ, 1993.

[10] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero* I, II. Ann. of Math. (2) **79** (1964), 109–326, 205–326.

[11] S. Ishii and J. Kollar, *The Nash problem on arc families of singularities.* Duke Math. J. **120** (2003), *3*, 601–620.

[12] A.N. Jensen, *Computing Gröbner fans and tropical varieties in Gfan*. In: *Software for Algebraic Geometry*. IMA Vol. Math. Appl. 148, pp. 33–46. Springer, New York, 2008.

[13] B. Karadeniz Şen, C. Plénat, and M. Tosun, *Minimality of a toric embedded resolution of rational triple points after Bouvier–Gonzalez–Sprinberg*. Kodai Math. J. **47** (2024), *3*, 395–427.

[14] A.G. Khovanskii, *Newton polyhedra (resolution of singularities)*. J. Soviet Math. **27** (1984), 2811–2830.

[15] Y. Koreeda, *On the configuration of the singular fibers of jet schemes of rational double points*. Comm. Algebra **50** (2022), *4*, 1802–1820.

[16] A.G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*. Invent. Math. **32** (1976), *1*, 1–31.

[17] T. Mora and L. Robbiano, *The Gröbner fan of an ideal*. J. Symbolic Comput. **6** (1988), *2-3*, 183–208.

[18] H. Mourtada, *Jet schemes of rational double point singularities*. In: *Valuation Theory in Interaction*. EMS Ser. Congr. Rep., pp. 373–388. Eur. Math. Soc., Zürich, 2014.

[19] H. Mourtada and C. Plénat, *Jet schemes and minimal toric embedded resolutions of rational double point singularities*. Comm. Algebra **46** (2018), *3*, 1314–1332.

[20] J.F. Nash, *Arc structure of singularities*. Duke Math. J. **81** (1995), *1*, 31–38.

[21] M. Oka, *On the resolution of the hypersurface singularities*. In: *Complex Analytic Singularities*. Adv. Stud. Pure Math. 8, pp. 405–436. North-Holland, Amsterdam, 1987.

[22] C. Plénat and P. Popescu-Pampu, *A class of non-rational surface singularities with bijective Nash map*. Bull. Soc. Math. France **134** (2006), *3*, 383–394.

[23] A.N. Varchenko, *Zeta-function of monodromy and Newton's diagram*. Invent. Math. **37** (1976), *3*, 253–262.

Received 21 April 2025

Gebze Technical University,
Department of Mathematics,
41400, Kocaeli, Turkey
busrakaradeniz@gtu.edu.tr