SPECIAL CURVES AND HAMILTONIAN SYSTEMS FROM POLYNOMIAL EIGENFUNCTIONS OF AN ELLIPTIC OPERATOR

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A family of polynomials linked to the set of the deltoid tangents and its associated algebraic hypersurfaces has been presented in recent years. In this paper, we study some related maximizing and free plane curves. We also analyse the bifurcations on polynomial Hamiltonian dynamical systems defined from such a family.

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1. INTRODUCTION

We study some plane curves and Hamiltonian systems related to a family of polynomials introduced in the last decade for the construction of algebraic hypersurfaces with many simple singularities ([8–10,13] and references within).

The one-parameter family of degree d polynomials $P_d(x, y, \mu)$ is discussed in Section 2, where we also consider the connection of $P_d(x, y, \mu)$ with eigenfunctions of a symmetric diffusion operator (see [13] for a generalisation to n variables) which corresponds to a Laplace–Beltrami operator for a metric with zero scalar curvature.

Special curves, called free and maximizing plane curves, are significant in the field of algebraic geometry. Maximizing curves can be used, for instance, in the construction of algebraic surfaces having the maximal Picard number. A link between maximizing curves and free curves has been found in [6]. In Section 3, we study some free and maximizing curves related to $P_d(x, y, \mu)$. In particular, we discuss maximizing curves of degrees eight and ten. A degree nine curve which is free but not maximizing is presented. We also show that a certain maximizing decic has a Miyaoka–Kobayashi number close to the maximum possible.

The critical points of $P_d(x, y, \mu)$ and some of their properties are presented in Section 4. We use them in Section 5 in order to analyse the bifurcation or

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catastrophe sets of the associated polynomial Hamiltonian dynamical systems. We find six bifurcation points if d is divisible by three and two bifurcation points otherwise.

2. A FAMILY OF DEGREE d POLYNOMIALS AS EIGENFUNCTIONS OF AN ELLIPTIC OPERATOR

We write $P_d(x, y, \mu)$ instead of the polynomials $\hat{J}_{d,\tau=\mu}(x, y)$ introduced in [10]. Up to a scaling factor, the polynomials $P_d(x, y, \mu)$ are connected to the union of the lines with d orientations linked with the affine Weyl group of the root lattice $\mathbf{A}_2$ and the set of the deltoid tangents [12]. The lines have equations

$$L_{d,\mu,s}(x, y) := y + (\cos 2\varphi - x)\tan\varphi + \sin 2\varphi = 0,$$
$$\phi = \frac{(6s-1)\pi}{6d} - \frac{\mu}{d},$$
$$s = -\left\lfloor \frac{d-2}{2} \right\rfloor, -\left\lfloor \frac{d-2}{2} \right\rfloor + 1, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor$$

with $(x, y) \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$. The basic degree-d polynomials are

$$P_d(x, y, \mu) := \lambda_{d,\mu} \prod_s L_{d,\mu,s}(x, y)$$

where

$$\lambda_{d,\mu} = (-1)^m 2d \text{ if } \ \mu = (6m - 3d - 1)\frac{\pi}{6}$$

(in this case, the line $L_{d,\mu,s}(x, y) = 0$ parallel to the y-axis is interpreted as the line $x + 1 = 0$) and

$$\lambda_{d,\mu} = 2\cos(\mu + \frac{d\pi}{2} + \frac{2\pi}{3}) \text{ if } \ \mu \neq (6m - 3d - 1)\frac{\pi}{6}, \ m \in \mathbb{Z}.$$

From [3], it follows that the symmetric diffusion operators defined on an open subset $\Omega \subset \mathbb{R}^n$ can be written

$$\mathcal{L} := \sum_{ij} g^{ij}(x)\partial^2_{x_i x_j} + \sum_i b^i(x)\partial_{x_i} \tag{1}$$

such that the symmetric matrix $(g^{ij}(x))$ is non-negative at every point $x = (x_1, x_2, \ldots, x_n) \in \Omega$. $\mathcal{L}$ is elliptic in the interior of Ω and an operator of this kind in $\mathbb{R}^2$, denoted by $\mathcal{L}_2$ in [10], has

$$g^{11}(x_1, x_2) = -\frac{1}{4}(3x_1^2 - x_2^2 - 6x_1 - 9), \tag{2}$$

$$g^{12}(x_1, x_2) = g^{21}(x_1, x_2) = -\frac{1}{2}(2x_1 x_2 + 3x_2), \tag{3}$$

$$g^{22}(x_1,x_2) = -\frac{1}{4}(3x_2^2 - x_1^2 + 6x_1 - 9), \tag{4}$$
$$b^1(x_1,x_2) = -x_1, b^2(x_1,x_2) = -x_2.$$

The operator $\mathcal{L}_2$ is elliptic in the interior of the region $-Q_\delta(x_1,x_2) \geq 0$, where

$$Q_\delta(x_1,x_2) = (x_1^2 + x_2^2)^2 - 8(x_1^3 - 3x_1x_2^2) + 18(x_1^2 + x_2^2) - 27 = 0$$

is the equation of a Steiner hypocycloid or deltoid δ.

For each μ, $J_d(x_1,x_2,\mu) := P_d(x_1,x_2,\mu) - 2\cos 3\mu$ is an eigenfunction of $\mathcal{L}_2$ [10]:

$$\mathcal{L}_2 J_d(x_1,x_2,\mu) = -d^2 J_d(x_1,x_2,\mu) \tag{5}$$

$\mathcal{L}_2$ has the form of the Laplace–Beltrami operator

$$\frac{1}{\sqrt{|g|}} \sum_{i,j} \partial_{x_i}(\sqrt{|g|} g^{ij} \partial_{x_j})$$

where g^{ij} is the (co)metric or inverse metric given in equations (2)–(4). The metric is $g_{ij} = (g^{ij})^{-1}$ and $|g| = \det g_{ij} = -\frac{3}{16}Q_\delta$. By computing the Christoffel symbols

$$\Gamma^m_{jk} := \frac{1}{2}\sum_i g^{im}(\partial_{x_k} g_{ij} + \partial_{x_j} g_{ki} - \partial_{x_i} g_{jk})$$

and the Riemann curvature tensor

$$R^m_{ijk} := \partial_{x_j}\Gamma^m_{ki} - \partial_{x_k}\Gamma^m_{ij} + \sum_s \Gamma^s_{ki}\Gamma^m_{js} - \sum_s \Gamma^s_{ij}\Gamma^m_{ks}$$

we obtain $R^m_{ijk} = 0$, hence also both the Ricci tensor $R_{ik} := \sum_j R^j_{ijk}$ and the scalar curvature $R := \sum_{i,k} g^{ik} R_{ik}$ are zero.

The operator $\mathcal{L}_2$ can be interpreted as a projection of the Euclidean Laplacian $\Delta_{\mathbb{R}^2}$. We use the operator

$$\mathcal{G}_{\mathcal{L}}(f,g) := \frac{1}{2}\big(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\big) \tag{6}$$

and we see that $g^{jk} = \mathcal{G}_{\mathcal{L}}(x_j,x_k), b^j = \mathcal{L}x_j$.

In terms of $z = x_1 + \sqrt{-1}x_2$ and $\bar{z} = x_1 - \sqrt{-1}x_2$, the operator $\mathcal{L}_2$ has a simpler form, denoted by $\mathcal{L}_{2C}$, with [10]

$$\mathcal{G}_{2C}(z,z) = -z^2 + 3\bar{z}, \mathcal{G}_{2C}(z,\bar{z}) = \mathcal{G}_{2C}(\bar{z},z) = 9 - z\bar{z}, \mathcal{G}_{2C}(\bar{z},\bar{z}) = -\bar{z}^2 + 3z,$$
$$\mathcal{L}_{2C} z = -z, \mathcal{L}_{2C}\bar{z} = -\bar{z}. \tag{7}$$

Under a change of variables $y = (y_k(x))_{k=1,2,\ldots}$, we have in general

$$\mathcal{L}\phi(y) = \sum_{ij} \mathcal{G}_{\mathcal{L}}(y_i,y_j)\partial^2_{y_i y_j}\phi(y) + \sum_i (\mathcal{L}y_i)\partial_{y_i}\phi(y). \tag{8}$$

The so-called generalised cosine (see [9] and references therein)

$$h(u,v) := e^{-2\pi u\sqrt{-1}} + e^{-2\pi v\sqrt{-1}} + e^{2\pi(u+v)\sqrt{-1}} \tag{9}$$

can be used to make a change of variables which shows that $\mathcal{L}_2$ is a projection of $\Delta_{\mathbb{R}^2}$. We identify the sets $\mathbb{R}^2$ and $\mathbb{C}$ and we make the substitutions $2\pi u = -\omega\cdot z$ and $2\pi v = -\bar{\omega}\cdot z$ in equation (9), where $z = x_1+\sqrt{-1}x_2 \in \mathbb{C}$, $z_1\cdot z_2 = \mathrm{Re}(z_1\bar{z}_2)$ and $1,\omega,\bar{\omega}$ are the cubic roots of unity. The variable change $Z : \mathbb{R}^2 \to \mathrm{Int}\delta$, where $\mathrm{Int}\delta$ denotes the interior of the region whose border is the deltoid δ, is

$$Z(z) = e^{(1\cdot z)\sqrt{-1}} + e^{(\omega\cdot z)\sqrt{-1}} + e^{(\bar{\omega}\cdot z)\sqrt{-1}}. \tag{10}$$

It can be shown that $\Delta_{\mathbb{R}^2}Z = -Z, \Delta_{\mathbb{R}^2}\bar{Z} = -\bar{Z}$ and $\mathcal{G}_{\Delta_{\mathbb{R}^2}} = \mathcal{G}_{2C}$ where $\mathcal{G}_{2C}$ is given by equation (7) for $z = Z, \bar{z} = \bar{Z}$, hence

$$\mathcal{L}_{2C} f(Z,\bar{Z}) = \Delta_{\mathbb{R}^2} f(Z,\bar{Z}). \tag{11}$$

The Jacobian

$$\left|\frac{\partial(x_1',x_2')}{\partial(x_1,x_2)}\right| = 4\sqrt{3}\big(\cos(3x_1) - \cos(\sqrt{3}x_2)\big)\sin(\sqrt{3}x_2),$$

where $x_1' = \mathrm{Re}Z(z), x_2' = \mathrm{Im}Z(z)$, is zero in three families of lines, with equations $3x_1 = \pm\sqrt{3}x_2 + 2k\pi, \sqrt{3}x_2 = l\pi, k,l \in \mathbb{Z}$, forming a periodic triangular tiling in $\mathbb{R}^2$. Z is a one-to-one map from the border of a triangle, which is a fundamental region of the affine Weyl group $\mathbf{A}_2$, to δ. Also, Z is a one-to-one map from the interior of the triangle onto $\mathrm{Int}\delta$ and maps $\mathbb{R}^2$ onto $\mathrm{Int}\delta$.

3. SOME FREE AND MAXIMIZING CURVES

In this section, we first recall some basic facts on free and maximizing curves in $\mathbb{P}^2$ and then we present some examples based on the polynomials $P_d(x,y,\mu)$. We also comment on specific curves with high Miyaoka–Kobayashi number.

Let $\mathrm{Der}(S) = \{\partial := a\cdot\partial_x + b\cdot\partial_y + c\cdot\partial_z, a,b,c \in S\}$ be the free S-module of $\mathbb{C}$-linear derivations of the polynomial ring S. The graded S-module of derivations for a reduced curve $C : F = 0$ preserving the ideal $\langle F\rangle$ is denoted by $\mathrm{D}(F) = \{\partial \in \mathrm{Der}(S) : \partial F \in \langle F\rangle\}$. If $\delta_E = x\partial_x + y\partial_y + z\partial_z$ is the Euler derivation and $\mathrm{D}_0(F) = \{\partial \in \mathrm{Der}(S) : \partial F = 0\}$ is the graded S-module of Jacobian syzygies of F, then we have $\mathrm{D}(F) = \mathrm{D}_0(F) \oplus S\cdot\delta_E$.

Definition 3.1 ([5]). We say that a reduced curve $C : F = 0$ is free if $\mathrm{D}(F)$, or equivalently $\mathrm{D}_0(F)$, is a free graded S-module. The exponents (d_1,d_2) of a free curve C are the degrees of a basis for the graded S-module $\mathrm{D}_0(F)$ with rank 2.

If C is a curve of even degree d with only simple singularities, then the "number" of simple singularities (by counting each simple singularity $X_k, X = A, D, E$ by its index k) of C is less or equal $\frac{3}{4}d^2 - \frac{3}{2}d + 1$ [16]. Now we use these results to construct algebraic curves with many simple singularities of types A, D and E [8, 10]. The singularities have local equations [2]

$$\begin{aligned} &A_k : x^{k+1} + y^2 = 0; \\ &D_k : x(x^{k-2} + y^2) = 0; \\ &E_6 : x^3 + y^4 = 0; \\ &E_7 : x(x^2 + y^3) = 0. \end{aligned}$$

In [6] it is shown that reduced plane curves C of odd degree $d = 2m + 1$ with only ADE singularities have a total Tjurina number at most $\tau(C) = 3m^2 + 1$. Curves with maximum total Tjurina number were studied in the 80s (see [16] and [20]). Later studies have motivated the following definition.

Definition 3.2 ([5]). A curve $C : F = 0$ of degree d having only ADE-singularities is maximizing if either $d = 2m$ and $\tau(C) = 3m(m - 1) + 1$ or $d = 2m + 1$ and $\tau(C) = 3m^2 + 1$.

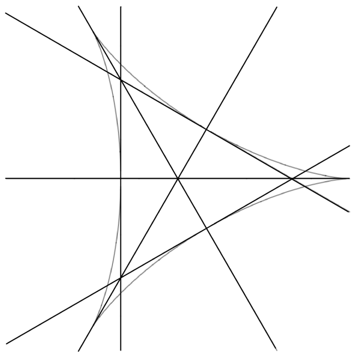

Figure 1 – Maximizing decic M_{10}.

Related to the polynomials discussed in Section 2, is the following decic curve (see Figure 1 and also Figure 6 when $\mu = k\pi/36, k = 30$)

$$M_{10} := P_6(x, y, 5\pi/6) \cdot \delta$$

which has $3E_7 \oplus 3D_6 \oplus 4D_4 \oplus 6A_1$, so it is maximizing with $\tau(M_{10}) = 61$ ([8]).

A maximizing octic M_8 can be constructed if we subtract from the decic M_{10} two of the three lines tangent to the deltoid ordinary points. The curve

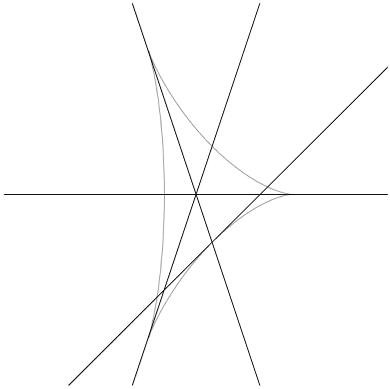

Figure 2 – Maximizing octic over the rationals $\widetilde{M}_8$.

is not defined over the rationals, but we can get one defined over the rationals which is given by $\widetilde{M}_8(x,y) := M_8(x, \frac{y}{\sqrt{3}})$ (Figure 2) and has equation

$$\widetilde{M}_8(x,y) = (-18x^2y + 9x^3y - 9x^2y^2 + 2y^3 - xy^3 + y^4)(-27 + 18x^2 - 8x^3 + x^4 + 6y^2 + 8xy^2 + (2x^2y^2)/3 + (y^4)/9).$$

We can check that $\tau(\widetilde{M}_8) = 37$ with the Singular [15] code (short input is used, e.g., $x^4 + 3x^2$ is denoted $x4 + 3x2$).

```
LIB "sing.lib";
ring R = 0, (x,y,z), dp;
poly f = (-18x2yz+9x3y-9x2y2+2y3z-xy3+y4)
       *(-27z4+18x2z2-8x3z+x4+6y2z2+8xy2z+(2x2y2)/3+(y4)/9);
ideal sl = jacob(f); // the singular locus of f
ideal newsl = groebner(sl); //a groebner basis
mult(newsl); //total tjurina number
```

Another maximizing octic T_8 formed by the union of a deltoid, one bitangent line and three cuspidal tangents was presented in [6]. It has $3E_7 \oplus D_4 \oplus 2A_3 \oplus 6A_1$ whereas, $\widetilde{M}_8$ has $3E_7 \oplus D_6 \oplus D_4 \oplus 6A_1$. Other maximizing octics were constructed in [5] but they also have different types of singularities distribution.

The relation between maximizing curves and free curves is characterised by the following theorem.

Theorem 3.3 ([6]). *A curve $C : F = 0$ of degree d having only ADE-singularities is maximizing if and only if either $d = 2m$ and C is a free curve with the exponents $(m-1, m)$ or $d = 2m+1$ and C is a free curve with exponents $(m-1, m+1)$.*

As a result of Theorem 3.3, we see that the exponents of the free curves $\widetilde{M}_8$ and M_{10} are $(3,4)$ and $(4,5)$, respectively. A maximizing nonic (therefore, a nonic free curve with exponents (3,5)) has been obtained in [5], but maximizing curves of odd degree are exceptional [17]. In order to study the cases of curves which are free but not maximizing, other methods must be used [4]. Let $R = \mathbb{C}[x_0, x_1, \ldots, x_n]$ be the graded ring of polynomials in $x_0, x_1, \ldots, x_n$ with complex coefficients. The vector space of degree k homogeneous polynomials in R is denoted by R_k. For any $f \in R_k$, the graded Milnor or Jacobian algebra is denoted by $M(f)$. For a degree d hypersurface $\mathcal{S} : f = 0$ with isolated singularities in $\mathbb{P}^n(\mathbb{C})$ the following integers are defined:

1) Coincidence threshold:

$$ct(\mathcal{S}) := \max\left\{q : \dim M(f)_k = \dim M(f_s)_k, \forall k \leq q\right\},$$

with f_s a polynomial in R_d, such that $\mathcal{S}_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n(\mathbb{C})$.

2) Stability threshold:

$$st(\mathcal{S}) := \min\left\{q : \dim M(f)_k = \tau(\mathcal{S}), \forall k \geq q\right\}.$$

Coincidence and stability thresholds can be used to check if a given curve is free.

Theorem 3.4 ([4]). (i) *For a reduced free plane curve $C : F = 0$ of degree d one has $ct(F) + st(F) = T$ where $T = 3(d-2)$.*

(ii) *Conversely, suppose that the reduced plane curve $C : F = 0$ of degree d satisfies $ct(F) + st(F) \leq T + 1$. Then C is free.*

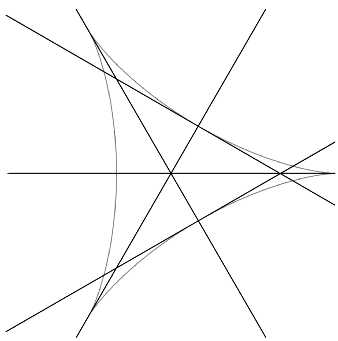

Figure 3 – A free nonic curve C_9 which is not maximizing.

If we analyse the singularities of the nonic curve

$$C_9 := (-48x^2y + 48x^3y - 12x^4y + 16y^3 - 16xy^3 + 40x^2y^3 - 12y^5)$$
$$(-27 + 18x^2 - 8x^3 + x^4 + 18y^2 + 24xy^2 + 2x^2y^2 + y^4),$$

obtained by subtracting from M_{10} the line $x+1=0$, which is tangent to one of the deltoid ordinary points (Figure 3) we see that $\tau(C_9) = 48 = 3m^2$ therefore, it is not maximizing. However C_9 is free because the computation with Singular shows that $ct(C_9) = 11, st(C_9) = 10$ and $T = 3(d-2) = 21 = ct(C_9) + st(C_9)$.

A family of curves which are significant in the construction of complex ball-quotient surfaces are the Miyaoka–Kobayashi (MK for short) curves which are certain complex projective plane curves of even degree higher than four, defined and studied by Hirzebruch and Ivinskis. If Sing(C) denotes the set of singularities of the curve C then the MK number is

$$m(C) := \sum_{p \in \mathrm{Sing}(C)} m(p),$$

where $m(p)$ is the MK number of the singularity p (see Definition 2.1 in [7]). In the case of ADE singularities, there are explicit expressions for $m(p)$ given in Lemma 3.2 in [7]. A MK curve C of even degree $d > 5$ is characterised by $m(C) = \frac{d}{2}(5d-6)$ (Definition 2.4 in [7]).

For degrees higher than seven, no example of MK curve is known [7] and the curves presented in this paper are not MK curves. We then consider the problem of finding curves with the maximum m. According to [7, Lemma 3.2], $m(w) = 3(\tau(w) + \epsilon(w))$ for any isolated hypersurface singularity w, where

$$\epsilon(A_k) = \frac{k}{k+1}(k \geq 1), \epsilon(D_k) = \frac{4k-9}{4k-8}(k \geq 4),$$
$$\epsilon(E_6) = \frac{23}{24}, \epsilon(E_7) = \frac{47}{48}, \epsilon(E_8) = \frac{119}{120}$$

and $\tau(w)$ is the Tjurina number of the singularity. Both maximizing octics $\widetilde{M}_8$ and T_8 are not MK curves but T_8 has an $m(C)/3$ closer to the value $\tau(C) + \epsilon(C) = \frac{d}{6}(5d-6)$. In fact,

$$m(\widetilde{M}_8) < m(T_8) = 135.9375 < \frac{d}{2}(5d-6) = 136.$$

The maximizing decic M_{10} is not a MK curve but

$$m(M_{10}) = 3(\tau(C) + \epsilon(C)) = 219.75$$

is close to $\frac{d}{2}(5d-6) = 220$ and higher than other maximizing decics.

4. THE CRITICAL POINTS OF $P_d(x, y, \mu)$

In Section 5, we study Hamiltonian systems related to the polynomials $P_d(x, y, \mu)$. It is then necessary to use certain characteristics of the critical points of $P_d(x, y, \mu)$.

THEOREM 4.1 ([10]). *The critical points and critical values of $P_d(x, y, \mu)$ have the following properties:*

(p1) *The critical values are*

$$\zeta = 0, \zeta_M(\mu) = 6\cos\mu + 2\cos 3\mu, \zeta_{m_1}(\mu) = \zeta_M\Big(\mu - \frac{2\pi}{3}\Big), \zeta_{m_2}(\mu) = \zeta_M\Big(\mu + \frac{2\pi}{3}\Big).$$

(p2) *The critical points with critical values $\zeta_M(\mu), \zeta_{m_1}(\mu)$ and $\zeta_{m_2}(\mu)$ have the same coordinates $\forall \mu \in \mathbb{R}$.*

(p3) *At $\mu \in \Lambda_\Sigma := \{k\frac{\pi}{3}, k \in \mathbb{Z}, 0 \leq k \leq 5\}$, either $P_d(x, y, \mu)$ or $-P_d(x, y, \mu)$ have all the maxima with critical value 8 and all the minima with value -1.*

(p4) *All the maxima have values $3\sqrt{3}$ and all the minima have values $-3\sqrt{3}$ at $\mu \in \Lambda_S := \{(2k+1)\frac{\pi}{6}, k \in \mathbb{Z}, 0 \leq k \leq 5\}$.*

(p5) *For each $\mu \in \Lambda_3 := [0, 2\pi) \setminus \{k\frac{\pi}{6}, k \in \mathbb{Z}, 0 \leq k \leq 11\}$, $P_d(x, y, \mu)$ has critical values $\zeta = 0$ and $\zeta_m, m = 1, 2, 3$, with $0 < |\zeta_3| < 1 < |\zeta_2| < 3\sqrt{3} < |\zeta_1| < 8$ and $\operatorname{sgn}(\zeta_1) \neq \operatorname{sgn}(\zeta_2) = \operatorname{sgn}(\zeta_3)$.*

The critical points of $P_d(x, y, \mu)$ are

$$(12) \qquad (x_c, y_c) = \big(\cos(2\pi(u_c + v_c)) + \cos(2\pi u_c) + \cos(2\pi v_c), \\ \sin(2\pi(u_c + v_c)) - \sin(2\pi u_c) - \sin(2\pi v_c)\big)$$

and a direct computation of the critical values leads to the following expressions for u_c, v_c ($k, l \in \mathbb{Z}$) [10]:

(a) $\zeta_M(\mu) = 6\cos\mu + 2\cos 3\mu$; $u_c = \frac{3k+1}{3d}, v_c = \frac{3l+1}{3d}$;
(b) $\zeta_{m_1}(\mu) = 6\cos(\mu - \frac{2\pi}{3}) + 2\cos 3\mu$; $u_c = \frac{3k+2}{3d}, v_c = \frac{3l+2}{3d}$;
(c) $\zeta_{m_2}(\mu) = 6\cos(\mu + \frac{2\pi}{3}) + 2\cos 3\mu$; $u_c = \frac{k}{d}, v_c = \frac{l}{d}$;
(d) $\zeta = 0$;
(d1) $u_c = \frac{6k-1}{6d} - \frac{\mu}{\pi d}, v_c = \frac{6l-1}{6d} - \frac{\mu}{\pi d}$;
(d2) $u_c = \frac{6k-1}{6d} - \frac{\mu}{\pi d}, v_c = \frac{3l+1}{3d} + \frac{2\mu}{\pi d}$;
(d3) $u_c = \frac{3k+1}{3d} + \frac{2\mu}{\pi d}, v_c = \frac{6l-1}{6d} - \frac{\mu}{\pi d}$.

The points (u_c, v_c) in the (u, v)-plane are situated in the fundamental region, denoted by W, of the affine Weyl group of $\mathbf{A}_2$. They satisfy

$\max\{-2v, v\} < u < \frac{1-v}{2}$ [9] when $\mu \notin \Lambda_S$ whereas, when $\mu \in \Lambda_S$ (case (p4) in Theorem 4.1) some points are in the border of W. This allows to compute the possible values of k, l and hence the number of critical points of $P_d(x, y, \mu)$, which are situated inside Intδ in the (x, y)-plane, except when $\mu \in \Lambda_S$ where some critical points with $\zeta = 0$ lie on δ ([10], Figures 2, 3).

5. BIFURCATIONS ON THE POLYNOMIAL HAMILTONIAN SYSTEMS ASSOCIATED WITH $P_d(x, y, \mu)$

We now study one-parameter families of autonomous ordinary differential equations $\dot{\mathbf{r}} = f(\mathbf{r}, \mu)$, which generate flows, where $\mathbf{r}(t) = (x(t), y(t)) \in \mathbb{R}^2$ are the dependent variables, $t \in \mathbb{R}$ is the independent variable and $\mu \in \mathbb{R}$ is the parameter. The simplest solutions are constants corresponding to $f(\mathbf{r}, \mu) = 0$, and are called stationary points or critical points. Bifurcations or catastrophes deal with the change in the number of solutions of $f(\mathbf{r}, \mu) = 0$ as the parameter μ varies, and then can be treated as part of the singularity theory [1,2]. They were motivated to some extent by some questions in biology [21] and have been applied to a variety of problems in science and engineering [14].

The defining differential equations we consider can be derived from a polynomial Hamiltonian $H(x, y)$ for which we have the next result ([19], p. 298).

THEOREM 5.1. *Any nondegenerate critical point of an analytic Hamiltonian system*

$$\dot{x} = \partial_y H(x, y), \dot{y} = -\partial_x H(x, y) \tag{13}$$

is either a saddle or a centre; again (x_0, y_0) is a saddle for equation (13) if and only if it is a saddle of the Hamiltonian function $H(x, y)$ and a strict local maximum or minimum of the function $H(x, y)$ is a centre for equation (13).

We analyse codimension-one bifurcations [18] for a type of dynamical systems connected with $P_d(x, y, \mu)$. We treat the Hamiltonian system

$$\dot{x} = \partial_y P_d(x, y, \mu), \dot{y} = -\partial_x P_d(x, y, \mu). \tag{14}$$

Although the results we obtain are valid for the system given by equation (14) one can also consider a related gradient system $\dot{\mathbf{x}} = -\mathrm{grad}F(\mathbf{x})$, where $\mathrm{grad}F = (\partial_x F, \partial_y F)^T$ ([19], p. 302).

THEOREM 5.2. *The planar system given by $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a Hamiltonian system, if and only if the system orthogonal to it, given by $\dot{x} = g(x, y), \dot{y} = -f(x, y)$ is a gradient system.*

The critical or stationary points of the orthogonal gradient system

$$\dot{x} = -\partial_x P_d(x, y, \mu), \dot{y} = -\partial_y P_d(x, y, \mu) \tag{15}$$

are the same as those of the system given by equation (14). The centres of a planar system correspond to the nodes of its orthogonal system and the saddles (foci) of the planar system are the saddles (foci) of its orthogonal system [19]. Also, at the regular points the trajectories of these systems are orthogonal.

The bifurcation or catastrophe sets of the dynamical system defined by equation (14) are analysed in this section by taking into account Theorem 4.1.

THEOREM 5.3. *The polynomial Hamiltonian system*

$$\dot{x} = \partial_y P_d(x, y, \mu)$$
$$\dot{y} = -\partial_x P_d(x, y, \mu)$$

has

(a) *two bifurcation points at* $\mu = \frac{\pi}{6}, \frac{\pi}{2}$ *for* $d \neq 3n$*;*

(b) *six bifurcation points at* $\mu = (2l+1)\frac{\pi}{6}$, $l = 0, 1, \dots, 5$ *for* $d = 3n$.

Proof. We consider both the degenerate and the nondegenerate critical points of $P_d(x, y, \mu)$. The bifurcations occur when degenerate critical points appear. In the proof of Theorem 4.1 (Theorem 2.2 in [10]) there is a description of the type of line arrangements associated with the level curve $P_d(x, y, \mu) = 0$ in terms of the behaviour of $\zeta_M(\mu), \zeta_{m_1}(\mu), \zeta_{m_2}(\mu)$ for $\mu \in [0, 2\pi) = \Lambda_\Sigma \cup \Lambda_S \cup \Lambda_3$ ([10], Figure 1):

(1) $\mu \in \Lambda_\Sigma$: one of the functions ζ has a maximum with critical value 8 and the other two have value -1, or a minimum with critical value -8 and the other two 1. The critical points of $P_d(x, y, \mu)$ are nondegenerate and the curves $P_d(x, y, \mu) = 0$ are simple arrangements of lines denoted by Σ_D^d ($\forall d$) and Σ_C^d (only for $d = 3n$).

(2) $\mu \in \Lambda_S$: one ζ has an inflection point with critical value 0 and the others have values $-3\sqrt{3}$ and $3\sqrt{3}$, the curves $P_d(x, y, \mu) = 0$ being simplicial arrangements denoted by S_D^d ($\forall d$) or S_C^d (only for $d = 3n$) hence, we find degenerate critical points in those cases.

(3) $\mu \in \Lambda_3$: the curves $P_d(x, y, \mu) = 0$ are simple arrangements not of type Σ_C^d and Σ_D^d but they have the same number of critical points with critical value 0, namely $\binom{d}{2}$, and the other critical values transform as described in what follows.

There are two types of critical or stationary points of the associated Hamiltonian system: saddles in the vertices of the line arrangements, and centres at the interior of the triangles or the closed non-triangular cells (quadrilateral, pentagonal or hexagonal cells). We now analyse the evolution of the critical points of $P_d(x,y,\mu)$ as μ varies for the cases (a) $d \neq 3n$ and (b) $d = 3n$, which are qualitatively different:

(a) $d \neq 3n$

(a1) $0 \leq \mu \leq \frac{\pi}{6}$. The critical values of $P_d(x,y,0)$, associated with the simple line arrangements Σ_D^d with dihedral symmetry, have been analysed in (Lemma 1, [9]). There are $\frac{(d-1)(d-2)}{6}$ local maxima with $\zeta_M = 8$ inside the non-triangular zones. The $\frac{(d-1)(d-2)}{6}$ local minima with $\zeta_{m_1} = -1$ and the $\frac{(d-1)(d-2)}{6}$ local minima with $\zeta_{m_2} = -1$ are located inside the triangular ones. As μ increases its value within the interval $[0, \frac{\pi}{6}]$, the curves $P_d(x,y,\mu) = 0$ are still simple arrangements ([10], Figure 2, for $d = 5$), the triangular zones containing the critical points with ζ_{m_1} decrease in size and, for each triangle (half of the total), its three vertices, which are A_1 singularities of the curve $P_d(x,y,\mu) = 0$ corresponding to $\zeta = 0$, collapse into one D_4 singularity when $\mu = \frac{\pi}{6}$: $\zeta_{m_1}(\frac{\pi}{6}) = 0$ and the values (u_c, v_c) of the critical points with $\zeta(\frac{\pi}{6}) = 0$ are

$$\text{(d1)} \left(\frac{3k-1}{3d}, \frac{3l-1}{3d}\right), \text{ (d2)} \left(\frac{3k-1}{3d}, \frac{3l+2}{3d}\right), \text{ (d3)} \left(\frac{3k+2}{3d}, \frac{3l-1}{3d}\right),$$

which, by appropriate relabelling, coincide with those linked to ζ_{m_1} which are always in Intδ. The remaining critical points with $\zeta(\frac{\pi}{6}) = 0$ are on δ ([10], Figure 2(d), for $d = 5$) and if we consider the union of δ and the simplicial arrangement S_D^d then they correspond to $d-1$ D_6 singularities [11].

(a2) $\frac{\pi}{6} < \mu \leq \frac{\pi}{3}$. Each non-hyperbolic point of $P_d(x,y,\frac{\pi}{6})$, which is a D_4 singularity of the curve $P_d(x,y,\frac{\pi}{6}) = 0$, splits into 3 saddles and 1 centre of the Hamiltonian system. We have local minima inside the $\frac{(d-1)(d-2)}{6}$ non-triangular zones containing the critical value ζ_{m_2} which decreases from $\zeta_{m_2}(\frac{\pi}{6}) = -3\sqrt{3}$ until $\zeta_{m_2}(\frac{\pi}{3}) = -8$. The local maxima are inside the $\frac{(d-1)(d-2)}{6}$ triangular zones containing ζ_M decreasing from $\zeta_M(\frac{\pi}{6}) = 3\sqrt{3}$ until $\zeta_M(\frac{\pi}{3}) = 1$, and inside the $\frac{(d-1)(d-2)}{6}$ triangular zones containing ζ_{m_1} increasing from $\zeta_{m_1}(\frac{\pi}{6}) = 0$ until $\zeta_{m_1}(\frac{\pi}{3}) = 1$. The curve at $\mu = \frac{\pi}{3}$ is, as for $\mu = 0$, a simple arrangement Σ_D^d. The first bifurcation therefore occurs when $\mu = \frac{\pi}{6}$.

(a3) $\frac{\pi}{3} < \mu \leq \frac{\pi}{2}$. The vertices of the $\frac{(d-1)(d-2)}{6}$ triangles containing local maxima with a critical value ζ_M in the interior collapse into D_4 singularities when $\mu = \frac{\pi}{2}$ analogously to the cases for ζ_{m_1} in (a1): $\zeta_M(\frac{\pi}{2}) = 0$ and the values (u_c, v_c) of the critical points with $\zeta(\frac{\pi}{2}) = 0$ are

$$\text{(d1)} \left(\frac{3k-2}{3d}, \frac{3l-2}{3d}\right), \text{ (d2)} \left(\frac{3k-2}{3d}, \frac{3l+4}{3d}\right), \text{ (d3)} \left(\frac{3k+4}{3d}, \frac{3l-2}{3d}\right),$$

which can be rewritten as $(\frac{3k+1}{3d}, \frac{3l+1}{3d})$. The other $d-1$ critical points with $\zeta(\frac{\pi}{2}) = 0$ are on δ and remain A_1 singularities of the planar curve.

(a4) $\frac{\pi}{2} < \mu \leq \frac{2\pi}{3}$. The second bifurcation takes place when $\mu = \frac{\pi}{2}$. The curves and trajectories for $\mu = \frac{2\pi}{3}$ are, up to a rotation, the same as those for $\mu = 0$ and there are no more distinct bifurcations for $\mu > \frac{\pi}{2}$.

(b) $d = 3n$

(b1) $0 \leq \mu \leq \frac{\pi}{6}$. The curves $P_d(x, y, 0) = 0$ are formed by the simple line arrangements Σ_C^d with cyclic symmetry. There are $\frac{d(d-3)}{6}$ local maxima with $\zeta_M = 8$ inside the non-triangular zones. Located at the triangular zones there are $\frac{d(d-3)}{6}$ local minima with $\zeta_{m_1} = -1$ and $1 + \frac{d(d-3)}{6}$ local minima with $\zeta_{m_2} = -1$ ([9, Lemma 1]). In the interval $[0, \frac{\pi}{6})$, the curves are always simple arrangements ([10], Figure 3, for $d = 6$). The triangular zones containing the critical points with critical value ζ_{m_1} shrink as μ increases its value and, for each triangle, the three A_1 singularities of the curve situated at its vertices, which correspond to $\zeta = 0$, are transformed into one D_4 singularity when $\mu = \frac{\pi}{6}$, because $\zeta_{m_1}(\frac{\pi}{6}) = 0$. The critical points with $\zeta(\pi/6) = 0$ have the following (u_c, v_c):

$$\text{(d1)} \left(\frac{3k-1}{9n}, \frac{3l-1}{9n}\right), \text{ (d2)} \left(\frac{3k-1}{9n}, \frac{3l+2}{9n}\right), \text{ (d3)} \left(\frac{3k+2}{9n}, \frac{3l-1}{9n}\right),$$

which can be relabelled in such a way that they coincide with those corresponding to ζ_{m_1}. The remaining critical points with $\zeta(\frac{\pi}{6}) = 0$ are on δ and correspond to d D_6 singularities of $\delta \cup S_C^d$ [11].

(b2) $\frac{\pi}{6} < \mu \leq \frac{\pi}{3}$. Each one of the d non-hyperbolic points of $P_d(x, y, \frac{\pi}{6})$ splits into 3 saddles and 1 centre of equation (14). The local minima are inside the $1 + \frac{d(d-3)}{6}$ non-triangular zones with critical value ζ_{m_2} which decreases from $\zeta_{m_2}(\frac{\pi}{6}) = -3\sqrt{3}$ until $\zeta_{m_2}(\frac{\pi}{3}) = -8$. There are local maxima inside the $\frac{d(d-3)}{6}$ triangular zones with critical value ζ_M, which decreases from $\zeta_M(\frac{\pi}{6}) = 3\sqrt{3}$ until $\zeta_M(\frac{\pi}{3}) = 1$, and there are $\frac{d(d-3)}{6}$ triangular zones containing local maxima with critical value ζ_{m_1} which increases from $\zeta_{m_1}(\frac{\pi}{6}) = 0$ until $\zeta_{m_1}(\frac{\pi}{3}) = 1$. The curve at $\mu = \frac{\pi}{3}$ is, in contrast to $\mu = 0$, a simple arrangement Σ_D^d. As in (a), the first bifurcation is found when $\mu = \frac{\pi}{6}$.

(b3) $\frac{\pi}{3} < \mu \leq \frac{\pi}{2}$. When $\mu = \frac{\pi}{2}$ the vertices of the $\frac{d(d-3)}{6}$ triangles containing in the interior local maxima with a critical value ζ_M produce D_4 singularities: $\zeta_M(\frac{\pi}{2}) = 0$ and the pairs (u_c, v_c) of the critical points with $\zeta(\pi/2) = 0$ are

$$\text{(d1)} \left(\frac{3k-2}{9n}, \frac{3l-2}{9n}\right), \text{ (d2)} \left(\frac{3k-2}{9n}, \frac{3l+4}{9n}\right), \text{ (d3)} \left(\frac{3k+4}{9n}, \frac{3l-2}{9n}\right),$$

which can also be expressed as $(\frac{(3k+1)}{9n}, \frac{(3l+1)}{9n})$. In the deltoid curve there are d critical points with critical value $\zeta(\frac{\pi}{2}) = 0$ that are A_1 singularities of $P_d(x, y, \frac{\pi}{2}) = 0$ which is a simplicial arrangement S_C^d as in the case $\mu = \frac{\pi}{6}$.

(b4) $\frac{\pi}{2} < \mu \leq \frac{2\pi}{3}$. Also, as in (a) the second bifurcation takes place when $\mu = \frac{\pi}{2}$. Each D_4 singularity of $P_d(x, y, \frac{\pi}{2}) = 0$ splits into 3 saddles and 1 centre of equation (14). We have local maxima inside the $\frac{d(d-3)}{6}$ non-triangular zones with the critical value ζ_{m_1} increasing from $\zeta_{m_1}(\frac{\pi}{2}) = 3\sqrt{3}$ until $\zeta_{m_1}(\frac{2\pi}{3}) = 8$, local minima inside the $\frac{d(d-3)}{6}$ triangular zones with ζ_M decreasing from $\zeta_M(\frac{\pi}{2}) = 0$ until $\zeta_M(\frac{2\pi}{3}) = -1$, and local minima inside the $1 + \frac{d(d-3)}{6}$ triangular zones with ζ_{m_2} increasing from $\zeta_{m_2}(\frac{\pi}{2}) = -3\sqrt{3}$ until $\zeta_{m_2}(\frac{2\pi}{3}) = -1$. The level curves for $\mu = \frac{2\pi}{3}$ are, up to a rotation and reflection, the same as those for $\mu = 0$ but the trajectories have opposite orientations.

(b5) $\frac{2\pi}{3} < \mu \leq \frac{5\pi}{6}$. As μ increases its value within the interval $(\frac{2\pi}{3}, \frac{5\pi}{6}]$ the curves are still simple arrangements, the $1 + \frac{d(d-3)}{6}$ triangular zones containing the critical points with ζ_{m_2} decrease in size and give one D_4 singularity when $\mu = \frac{5\pi}{6}$: $\zeta_{m_2}(\frac{5\pi}{6}) = 0$ and the values (u_c, v_c) of the critical points with $\zeta(\frac{5\pi}{6}) = 0$ are

$$\text{(d1)}\ \left(\frac{k-1}{3n}, \frac{l-1}{3n}\right),\ \text{(d2)}\ \left(\frac{k-1}{3n}, \frac{l+2}{3n}\right),\ \text{(d3)}\ \left(\frac{k+2}{3n}, \frac{l-1}{3n}\right),$$

which, by appropriate relabelling, coincide with those associated with ζ_{m_2}. The remaining critical points with $\zeta(\frac{5\pi}{6}) = 0$ are on δ and correspond to the d D_6 singularities of $\delta \cup S_D^d$ [11].

(b6) $\frac{5\pi}{6} < \mu \leq \pi$. The third bifurcation occurs when $\mu = \frac{5\pi}{6}$. Each non-hyperbolic point of $P_d(x, y, \frac{5\pi}{6})$ splits into 3 saddles and 1 centre of the Hamiltonian system. We have local minima inside the $\frac{d(d-3)}{6}$ non-triangular zones with ζ_M decreasing from $\zeta_M(\frac{5\pi}{6}) = -3\sqrt{3}$ until $\zeta_M(\pi) = -8$, local maxima inside the $1 + \frac{d(d-3)}{6}$ triangular zones with ζ_{m_2} increasing from $\zeta_{m_2}(\frac{5\pi}{6}) = 0$ until $\zeta_{m_2}(\pi) = 1$, and the $\frac{d(d-3)}{6}$ triangular zones contain local maxima with ζ_{m_1} decreasing from $\zeta_{m_1}(\frac{5\pi}{6}) = 3\sqrt{3}$ until $\zeta_{m_1}(\pi) = 1$. The level curves for $\mu = \pi$ are the same as those for $\mu = 0$ but the trajectories have opposite orientations.

(b7) $\pi < \mu \leq 2\pi$. The level curves are the same as $0 < \mu \leq \pi$ but the trajectories have also opposite orientations. We then find three additional bifurcation points at $\mu = \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$. $\square$

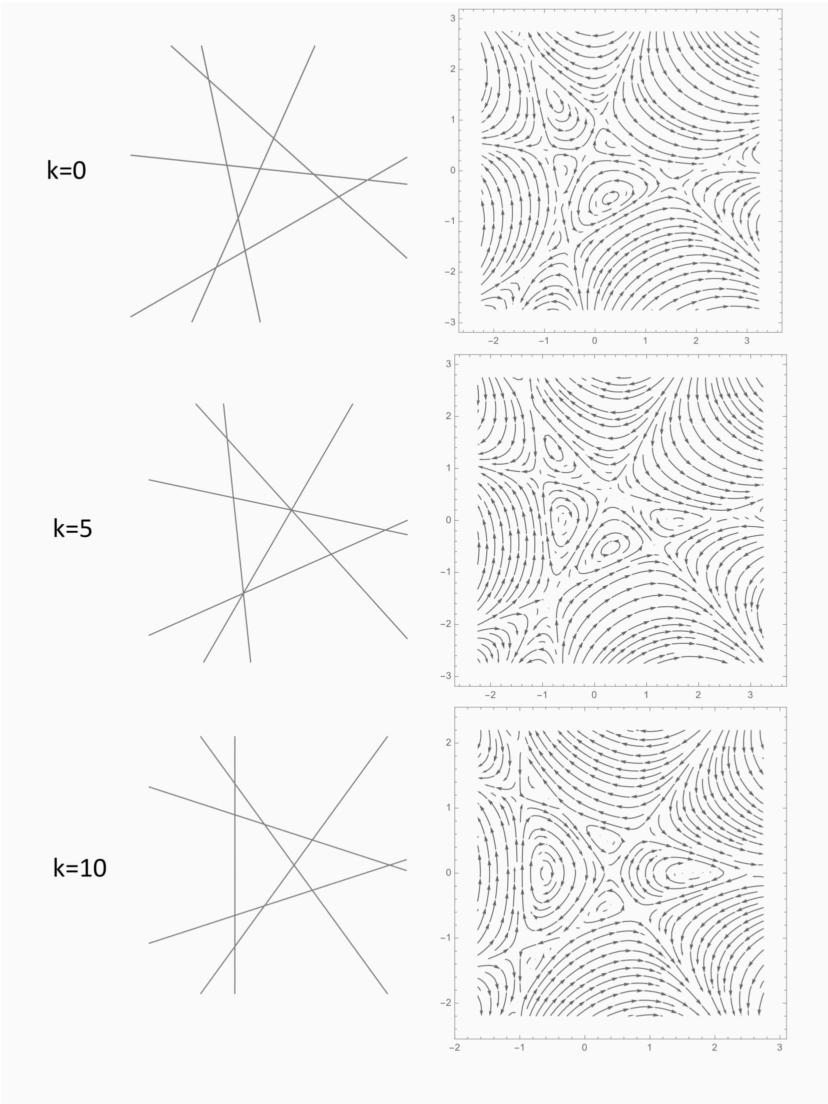

Figure 4 – Level curves $P_5(x, y, \mu) = 0$ (left) and trajectories (right) for $\mu = k\pi/30$, $k = 0, 5, 10$. Local minima (maxima) of $P_5(x, y, \mu)$ correspond to centres with trajectories having clockwise (anticlockwise) orientation.

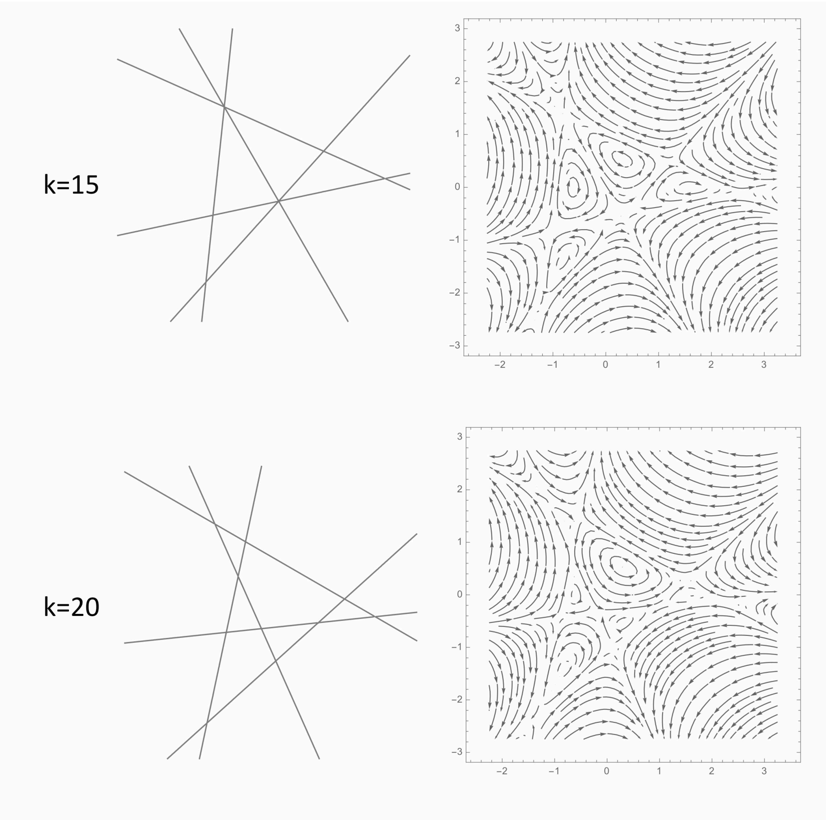

Figure 5 – Level curves $P_5(x, y, \mu) = 0$ (left) and trajectories (right) for $\mu = k\pi/30$, $k = 15, 20$.

In Figures 4 and 5, we have represented the trajectories of the Hamiltonian system corresponding to $P_5(x, y, \mu)$. The level curve $P_5(x, y, \mu) = 0$ is drawn on the left and the trajectories of the Hamiltonian flow on the right. There are two bifurcation points at $\mu = \frac{k\pi}{30}$, $k = 5, 15$. The case $k = 20$ is topologically equivalent to $k = 0$: they are related by a rotation. The non-hyperbolic points are the result of the collapsing of 3 saddles and 1 centre. Both bifurcations consist in the splitting of the non-hyperbolic points (D_4-singularities of the level curves) into 3 saddles (A_1-singularities of the level curves) and 1 centre in the interior of the triangle whose vertices are the 3 saddles.

The trajectories for $d = 6$ can be seen in Figures 6 and 7 where we have represented both the level curves and the orbits in the same coordinate frames. We can see some qualitative differences in relation to the case $d = 5$, among them the appearance of two types of both simple and simplicial line arrangements with dihedral and cyclic symmetries. The parameter values of the bifurcation points are $\mu = \frac{k\pi}{36}$, $k = 6, 18, 30, 42, 54, 66$, which correspond to all $\mu \in \Lambda_S$.

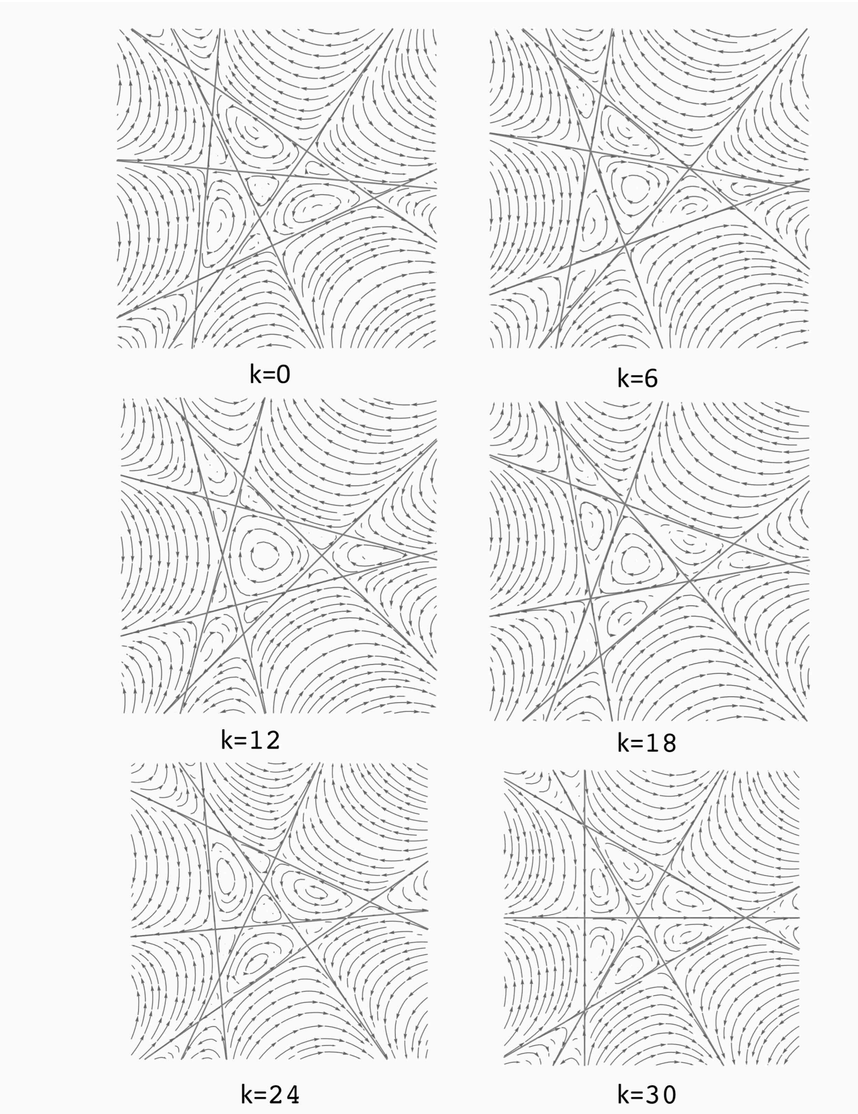

Figure 6 – Level curves $P_6(x, y, \mu) = 0$ and trajectories for $\mu = k\pi/36$, $k = 0, 6, 12, 18, 24, 30$.

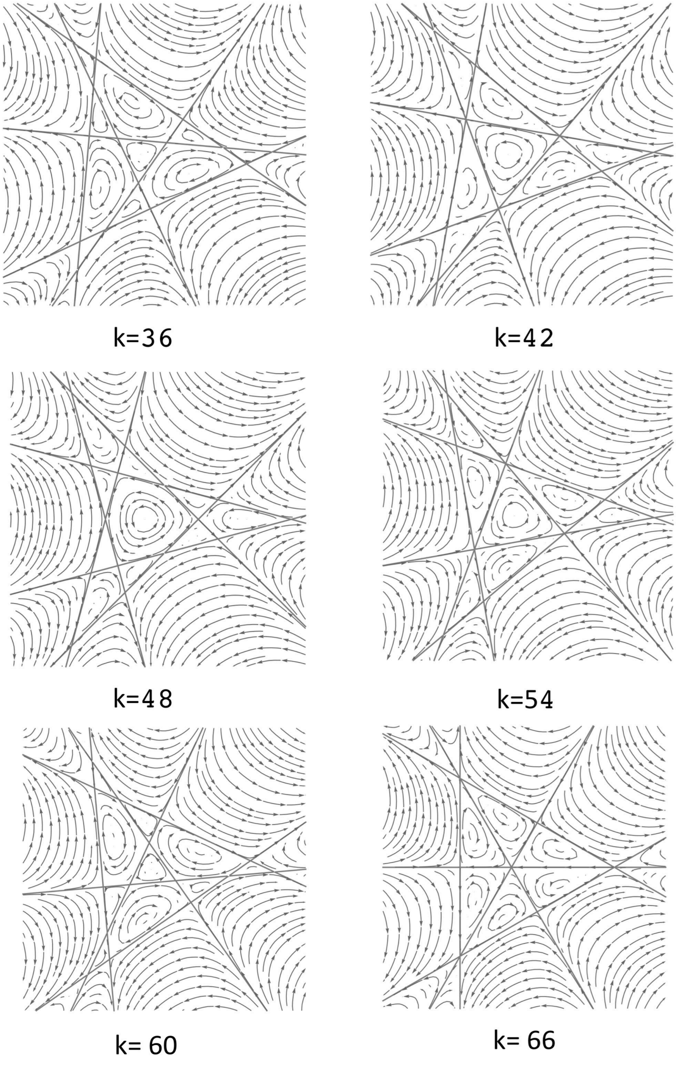

Figure 7 – Level curves $P_6(x, y, \mu) = 0$ and trajectories for $\mu = k\pi/36$, $k = 36, 42, 48, 54, 60, 66$. The curves are the same as in Figure 6 for $k = 0, 6, 12, 18, 24, 30$, respectively, but the trajectories have reversed orientations.

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