DEFORMATIONS WITH FIBRE CONSTANCY

YING CHEN, CEZAR JOIŢA, AND MIHAI TIBĂR

Abstract. Deformations which split the singularities into simpler ones while preserving the general fibres are fundamental tools for studying the local topology of holomorphic function germs. We define such "admissible deformations" in the general setting of deformations of real analytic map germs, and find conditions under which the conservation of the general fibres holds.

1. Introduction

Deforming a function germ is the most efficient method for capturing information on the topology of its general fibre, including its monodromy. In the case of holomorphic functions with isolated singularities, Brieskorn [Br] showed in this way that the Milnor number of the function germ F_0 is equal to the number of critical points in any deformation F_t with only Morse singularities. If one enlarges the setting by considering holomorphic functions with isolated singularities on singular space germs, then one needs to define "allowable deformations" in order to obtain convenient conservation properties, see for instance [JiT], [MT] where the Brieskorn principle is extended. In another direction, considering holomorphic functions F_0 with *non-isolated singularities*, Siersma and his school studied the general fibre of F_0 by using a class of "admissible deformations", see e.g. [Si], [Sc], [dJ], [Pe], [Za], [Fe], and some more recent papers such as [FM], [ST], [MPT]. These deformations F_t have the property that the general fibre of F_0 is preserved as being identifiable with the global general fibre of the deformation within a fixed ball neighbourhood of the origin. This property will be defined in a precise manner by Definition 1.2 below, and will be called here *fibre constancy*. Two very recent papers define several natural classes of deformations for which the "fibre constancy" holds, cf [Hof] and [JST].

We address here the problem of selecting meaningful classes of "admissible deformations" in the more general setting of analytic map germs $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^m, 0)$ in both real and complex settings, i.e. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. This is a more delicate task especially in the real setting where there are several general fibres, unlike the complex setting where there is only one.

To set the notations and introduce our main definitions and statements, let $F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^m, 0)$, $n \geq m \geq 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a $\mathbb{K}$-analytic map germ regarded as a one-parameter deformation $F_t(x)$ of the map germ $F_0(x) := F(x, 0)$. We consider the associated map germ $\widetilde{F} = (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^m \times \mathbb{K}, 0)$, $\widetilde{F}(x, t) := \big(F(x, t), t\big)$.

For simplicity, we fix $\mathbb{K} = \mathbb{R}$ in the next definition. Let ρ denote the square of the Euclidean distance function in $\mathbb{R}^{n+1}$.

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Definition 1.1 (Tame deformations).
We say that $F_t(x) = F(x,t)$ is a *tame deformation* of F_0 if the following condition holds (called *ρ-regularity*, cf Definition 2.1):

$$M(\widetilde{F}) \bigcap F_0^{-1}(0) \cap \operatorname{Sing} F_0 \subset \{(0,0)\}, \tag{1.1}$$

where $M(\widetilde{F}) := \overline{\operatorname{Sing}(\widetilde{F},\rho) \setminus \operatorname{Sing}\widetilde{F}}$.

Tame deformations insure that the image of $\widetilde{F}$ is well-defined as a set germ at $(0,0) \in \mathbb{R}^m \times \mathbb{R}$, and that the image $\widetilde{F}(\operatorname{Sing}\widetilde{F})$ is a well-defined set germ at $(0,0) \in \mathbb{R}^m \times \mathbb{R}$, in which case one usually calls it *discriminant* and denotes it by $\operatorname{Disc}\widetilde{F}$ (see Proposition 2.4). The ρ-regularity of Definition 1.1 also implies (cf. [ART, Proposition 4.2]) the key property that $\widetilde{F}$ has a locally trivial fibration outside its discriminant, which one calls *Milnor-Hamm fibration*, see Definition 2.5.

With this preparation at hand, we may now state our announced definition (see also [JST, Definition 1.1]):

Definition 1.2 (Deformations with fibre constancy).
Let $F_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, $n \geq m \geq 2$ be a non-constant analytic map germ. We say that the deformation F of F_0 is a *deformation with fibre constancy* if $\widetilde{F}$ has a Milnor-Hamm fibration.

This tells the following: if $B_r \subset \mathbb{R}^n \times \mathbb{R}$ and by $B_\delta \subset \mathbb{R}^m \times \mathbb{R}$ are those balls of radii $r > 0$ and $\delta > 0$, centred at the respective origins, which occur in Definition 2.5 for $G := \widetilde{F}$, then we have the diffeomorphism of fibres:

$$B_r \cap F_0^{-1}(a) \simeq B_r \cap \widetilde{F}^{-1}(\lambda_a)$$

for any $a \in (B_\delta \cap \{t = 0\}) \setminus \operatorname{Disc} F_0$ and any $\lambda_a \in B_\delta \setminus \operatorname{Disc}\widetilde{F}$ which belongs to the same connected component[1] of $B_\delta \setminus \operatorname{Disc}\widetilde{F}$ as a. In particular, the fact that $\widetilde{F}$ has a locally trivial fibration over the complement of the discriminant $\operatorname{Disc}\widetilde{F}$ implies that F_0 has a locally trivial fibration over the complement of its own discriminant $\operatorname{Disc} F_0 = \{t = 0\} \cap \operatorname{Disc}\widetilde{F}$.

The above definition tacitly assumes that the involved maps have well defined images and discriminants. Throughout the paper we will take care that our hypotheses insure this property too.

Let us briefly explain an application of the above defined *fibre constancy*. Let $F_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ define a *complete intersection* with non-isolated singular locus $\operatorname{Sing} F_0 := \Sigma_0$ of dimension 1, and consider a deformation $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^m, 0)$ with fibre constancy in the sense of Definition 1.2. Therefore F_0 has a Milnor-Hamm fibration, and thus the topology of its Milnor fibre can be studied by extending the technique developed in the paper [ST] for "admissible deformations" of a function germ. More precisely, the singular set $\operatorname{Sing} F_0$ deforms into $\operatorname{Sing} F_t$ which is a disjoint union of a 1-dimensional singular set Σ_t and a finite set P_t of isolated singularities. Outside a certain finite set $Q_t \subset \Sigma_t$ of "special points", on any connected component of $\Sigma_t \setminus Q_t$, the map F_t has the transversal type of an

[1] Let us remark the equality $\operatorname{Disc} F_0 = \{t = 0\} \cap \operatorname{Disc}\widetilde{F}$.

ICIS (isolated complete intersection singularity), and the corresponding transversal Milnor fibre is endowed with a *Milnor monodromy* and with a *vertical monodromy*[2]. Then, as described in [ST], one can patch together these data in order to build the homology of the Milnor fibre of F_0.

Our study focusses on defining deformations with fibre constancy in terms of the *partial Thom regularity* (also denoted by *∂-Thom regularity*, cf Definition 2.2). We first show that ∂-Thom regularity implies ρ-regularity, which in turn implies the existence of the Milnor-Hamm fibration needed in Definition 1.2, cf [ART], [JoT1].

Section 3 contains our main results. We show how to control the key ∂-Thom regularity by using inequalities of Łojasiewicz type for map germs in Theorem 3.1, and by using a Parusiński type inequality in Theorem 3.5. In each case, proofs are new and also radically different with respect to what had been done before in some particular contexts.

Section 4 treats the composition of deformations in very large generality. We discuss and provide a general answer, cf Theorem 4.1 and Example 4.2, to a problem addressed in [AG].

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2. Terminology and preliminary results

Let $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a $\mathbb{K}$-analytic map germ, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We will denote by $V(G)$ the central fibre $G^{-1}(0)$.

Definition 2.1 (ρ-regular map)**.**
Let ρ denote the square of the Euclidean distance function in $\mathbb{K}^n$. Let

$$M(G) := \overline{\mathrm{Sing}\,(G, \rho) \setminus \mathrm{Sing}\,G}$$

be the *Milnor set of* G. We say that G is a *ρ-regular map* map germ if:

$$M(G) \cap V(G) \cap \mathrm{Sing}\,G \subset \{(0, 0)\}. \tag{2.1}$$

We have recently considered the question what conditions insure the tameness in the composition of map germs.

Definition 2.2 (∂-Thom regularity)**.**
Let $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a $\mathbb{K}$-analytic map germ. We say that G is *∂-Thom regular* if there exists a Whitney (a)-stratification $\mathcal{W}$ of some open ball B centred at $0 \in \mathbb{K}^n$ such that $B \setminus G^{-1}(G(B \cap \mathrm{Sing}\,G))$ and $\{0\}$ are strata, that $B \cap V(G)$ and $B \cap V(G) \cap \mathrm{Sing}\,G$ are unions of strata, and that the pair of strata

$$\Big(B \setminus G^{-1}\big(G(B \cap \mathrm{Sing}\,G)\big), W\Big)$$

[2] For all this terminology, the reader is referred to [ST] and its included references.

satisfies the Thom (a_G)-regularity condition for any stratum $W \subset B \cap V(G) \setminus \{0\}$.

Comparing to [JoT1, Definition 5.7], we observe that the above definition is a particular case of the ∂-Thom regularity used in [JoT1] in the setting where the singular locus itself has a stratification and one deals with a singular fibration.

2.1. **The image problem for map germs.** The image of a map germ is not necessarily well-defined as a set germ. We refer to [ART], [JoT1], [JoT2], [JoT3] for details, examples, and recent results.

Let us first recall the following notion: for $U, V \subset \mathbb{K}^n$ subsets containing the origin, the set germs $(U, 0)$ and $(V, 0)$ are equal if and only if there exists some open ball $B_\varepsilon \subset \mathbb{K}^n$ centred at 0 of radius $\varepsilon > 0$ such that $U \cap B_\varepsilon = V \cap B_\varepsilon$.

Definition 2.3. [ART, Definition 2.2], [JoT1] Let $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $n \geq p > 0$, be a continuous map germ, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. We say that the image by G of a set $K \subset \mathbb{K}^n$ containing 0 is a well-defined set germ at $0 \in \mathbb{K}^p$ if the set germ $\big(B_\varepsilon \cap G(K), 0\big)$ is independent of the small enough radius $\varepsilon > 0$.

Proposition 2.4. *Let $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a $\mathbb{K}$-analytic map germ which is ∂-Thom regular. Then G is ρ-regular, and the images* ImG *and* $G(\mathrm{Sing}\, G)$ *are well-defined as set germs at $0 \in \mathbb{K}^p$, cf Definition 2.3.* □

Proof. The existence of a ∂-Thom stratification on some open ball B implies that there is $R > 0$ such that, for any positive $r \leq R$, the sphere $S_r \subset \mathbb{K}^n$ is transversal to all positive dimensional strata $W \in \mathcal{W}$ such that $W \subset V(G)$. It follows that the sphere S_r is transversal to the smooth nearby fibres of G, and therefore G is ρ-regular.

The ρ-regularity implies that ImG and $G(\mathrm{Sing}\, G)$ are well-defined set germs, as shown in [JoT1, Theorem 4.5 (a)]. □

Notation. If the image $G(\mathrm{Sing}\, G)$ is a well-defined set germ at the origin, then we will denote it by Disc$G$, and usually call it "the discriminant of G". We say that the map germ G is *nice* if it has well defined image and discriminant as set germs at $0 \in \mathbb{K}^p$.

Definition 2.5 (*Milnor-Hamm fibration*)**.** Let $G : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ. We say that G has *Milnor-Hamm fibration* if, for any $\varepsilon > 0$ small enough, there exists $0 < \delta \ll \varepsilon$ such that the restriction:

$$G_| : B^n_\varepsilon \cap G^{-1}(B^p_\delta \setminus \mathrm{Disc}\, G) \to B^p_\delta \setminus \mathrm{Disc}\, G \tag{2.2}$$

is a C^∞ locally trivial fibration over each connected component of $B^p_\delta \setminus \mathrm{Disc}\, G$, such that it is independent of the choice of ε and δ, up to diffeomorphisms.

It has been shown in [ART, Proposition 4.2] that: *if G is ρ-regular then G has a Milnor-Hamm fibration.* By Proposition 2.4, we then get the following consequence:

Corollary 2.6. *If $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is a $\mathbb{K}$-analytic map germ which is ∂-Thom regular, then G has a Milnor-Hamm tube fibration.* □

3. ∂-Thom regularity of deformations

Let $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ be some analytic map germ, $n \geq m \geq 2$, and let $V := \psi^{-1}(0)$. We discuss here several ways of checking the partial Thom regularity that we assume in the statement of the main theorem.

The idea is to control the growth of functions by inequalities.

3.1. **Thom regularity via the Łojasiewicz inequality.** To prove that the Thom (a_f)-regularity holds for functions f, Hamm and Lê showed in [HL] how to use the existence of the Łojasiewicz inequality for $\mathbb{K}$-analytic function germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, namely: there exists some $0 < \theta < 1$, such that for any x in some neighbourhood of 0 one has

$$\|f(x)\|^\theta \leq \|\operatorname{grad} f(x)\|, \tag{3.1}$$

where $\operatorname{grad} f(x)$ denotes here the conjugate of the complex gradient.

In the setting of real analytic map germs $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$, Hamm and Lê's method of proof may still work when assuming a Łojasiewicz type inequality. Massey in [Ma] points out that if the following condition:

$$\|\psi(x)\|^\theta \leq K\nu(x), \tag{3.2}$$

holds for some $0 < \theta < 1$ and some $K > 0$ in some neighbourhood of 0, where:

$$\nu(x) := \min_{\|a\|=1} \|\sum_i a_i \operatorname{grad} \psi_i(x)\| \tag{3.3}$$

is the Rabier distance function, cf [Ra]. Massey shows that this implies the existence of a Thom (a_ψ)-regular stratification of $(V, 0)$.

3.2. **Łojasiewicz type inequality in case of deformations of maps.**

Let $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^m, 0)$ be an analytic map germ viewed as one-parameter deformation $F_t(x) := F(x, t) = (f_1(x, t), \ldots, f_m(x, t))$, and let $\widetilde{F}(x, t) := (F(x, t), t)$ be the associated map germ.

Let us set the notation:

$$\nu_{F_t}(x) := \min_{\|a\|=1} \left\| \sum_{i=1}^m a_i \operatorname{grad_x} f_i(x, t) \right\|, \tag{3.4}$$

where $\operatorname{grad_x} f_i := \left(\overline{\frac{\partial f_i}{\partial x_1}}, \ldots, \overline{\frac{\partial f_i}{\partial x_n}}\right)$ thus contains the partial derivatives with respect to the variables x only.

We show a condition under which $\widetilde{F}$ is ∂-Thom regular.

Theorem 3.1. *Let $F_t(x) = F(x, t)$ be a deformation of the map germ $F_0 := F(x, 0) : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$. Assume that the following condition holds:*

$$\left\{ \begin{array}{l} \textit{There exists } 0 < \theta < 1 \textit{ such that for any } x_0 \in F^{-1}(0) \cap \operatorname{Sing} F_0 \setminus \{0\} \\ \textit{there is a constant } c(x_0) > 0 \textit{ for which the following inequality holds:} \\ \|F(x, t)\|^\theta \leq c(x_0)\nu_{F_t}(x) \quad \textit{when } (x, t) \to (x_0, 0),\ (x, t) \notin \operatorname{Sing} \widetilde{F}. \end{array} \right. \tag{3.5}$$

Then the associated map germ $\widetilde{F}$ is ∂-Thom regular.

Remark 3.2. It may happen that $\widetilde{F}$ is ∂-Thom regular without F being ∂-Thom regular. For instance, let $F(x,y,t) = (f_1, f_2) := (x, y(x^2+y^2)+xt^2)$, and let $\widetilde{F} : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$, $\widetilde{F}(x,y,t) = (x, y(x^2+y^2)+xt^2, t)$ be the associated map. Since $V(\widetilde{F}) = \{(0,0,0)\}$, the map germ $\widetilde{F}$ is ∂-Thom regular by definition. According to [ART, Example 5.2], $F(x,y,t) = (x, y(x^2+y^2)+xt^2)$ is not ∂-Thom regular.

Remark 3.3. The inequality within (3.5) is a Lojasiewicz type condition. In case of map germs but without reference to deformations, Massey used such a condition in [Ma] over a full neighbourhood of the origin, and with the restriction $\mathrm{Sing}\, F \subset F^{-1}(0)$. In the setting of deformations of map germs, Massey's condition appears to be too rigid, since it does not allow deformations where $\mathrm{Sing}\, F_0$ splits outside the central fibre $F^{-1}(0)$ and which are precisely the object of many papers in the literature, e.g. [Si], [dJ], [Pe], [Za].

In contrast, our theorem includes such splittings since it concerns deformations of map germs without restrictions on the singular locus; observe that condition (3.5) refers only non-singular points $(x,t) \notin \mathrm{Sing}\, \widetilde{F}$. Our method of proof builds on the idea used by Hamm and Lê in [HL] to prove the Łojasiewicz classical inequality for analytic functions.

Proof. Let $F(x,t) = (f_1(x,t), \ldots, f_m(x,t))$, and let $\widetilde{F}(x,t) := (F(x,t), t)$. We consider the germ at the origin of the following analytic set of dimension $n+1$:

$$X = \{f_1(x,t) - s_1^L = \cdots = f_m(x,t) - s_m^L = 0\} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$$

where $L \in \mathbb{N}$ is sufficiently large such that $0 < \theta < \frac{L-1}{L}$. Let $\mathcal{W}$ be a Whitney (a)-regular stratification $\mathcal{W}$ of the set germ X, such that $\mathrm{Sing}\, X$ is a union of strata. The stratified singular locus $\mathrm{Sing}_{\mathcal{W}}\pi$ of the function germ $\pi : (X, 0) \longrightarrow (\mathbb{R}, 0)$, $(x,t,s) \mapsto t$, is a closed analytic closed set, and it is included in the fibre $X \cap \{t = 0\}$. As proved by Hironaka, the Whitney stratification $\mathcal{W}$ may be refined into a Whitney stratification which is also Thom (a_π)-regular and such that $\mathrm{Sing}_{\mathcal{W}}\pi$ is a union of strata. This shows that there exists a ∂-Thom stratification $\mathcal{S}$ of $\mathrm{Sing}_{\mathcal{W}}\pi$, i.e. satisfying the following condition: *for any stratum $S \in \mathcal{S}$, the pair $\big(X \setminus (\{t=0\} \cup \mathrm{Sing}\, X), S\big)$ is Thom (a_π)-regular.*

We consider now the slices $S \cap \{s = 0\}$ of all the strata $S \subset \mathrm{Sing}_{\mathcal{W}}\pi$. Since these slices are not necessarily non-singular, one needs to refine the partition into a Whitney (a)-regular stratification $\mathcal{S}'$ of $X \cap \{t = 0\} \cap \{s = 0\}$. Since $\mathcal{S}'$ is a refinement of $\mathcal{S}$, it follows that, for any stratum $W \in \mathcal{S}'$, the pair

$$\Big(X \setminus (\{t = 0\} \cup \mathrm{Sing}\, X), W\Big) \tag{3.6}$$

is Thom (a_π)-regular.

We have the equality $\mathrm{Sing}\, \widetilde{F} \cap V(\widetilde{F}) \times \{0\}^m = \mathrm{Sing}\, X \cap \mathrm{Sing}\, \pi \cap \{s = 0\}$, where $\mathrm{Sing}\, \pi \subset \{t = 0\}$. Thus $\mathcal{S}'$ is a stratification of the set $\mathrm{Sing}\, \widetilde{F} \cap V(\widetilde{F})$, and we will show that it is a partial Thom stratification of $\widetilde{F}$, i.e. that the pair $\big(B_\varepsilon^{n+1} \setminus \widetilde{F}^{-1}(\mathrm{Disc}\,(\widetilde{F}), W\big)$ is $(a_{\widetilde{F}})$-regular for all $W \in \mathcal{S}'$.

We consider sequences of points $(x,t) \subset B_\varepsilon^{n+1} \setminus \widetilde{F}^{-1}(\mathrm{Disc}\,(\widetilde{F}))$ such that $x \to x_0$ and $t \to 0$ where $(x_0, 0) \in \mathrm{Sing}\, \widetilde{F} \cap V(\widetilde{F})$, and such that $(x_0, 0) \in W$ for a positive dimensional stratum $W \in \mathcal{S}'$. Note that for a corresponding triple $(x,t,s) \in X$, the variable s converges $0 \in \mathbb{R}^m$, since $F(x_0, 0) = 0$.

By (3.6), the inclusion:

$$T := \lim_{(x,t,s)\to(x_0,0,0)} T_{(x,t,s)}\big(\pi^{-1}(t)\cap X\big) \supset T_{(x_0,0,0)}W \tag{3.7}$$

holds, where $\dim T_{(x_0,0,0)}W \geq 1$, and by tacitly assuming that the limit T exists in the appropriate Grassmannian. Note that we have $\dim T = n$.

Lemma 3.4. *Under the hypotheses of Theorem 3.1, one has the equality:*

$$T = A \times \{0\}_1 \times \mathbb{R}^m \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m,$$

where $A \subset \mathbb{R}^n$ is some linear subspace of dimension $n-m$.

Proof. The normal space to $T_{(x,t,s)}\big(\pi^{-1}(t)\cap X\big)$ is spanned by the vectors

$$V^i(x,t,s) := \Big(\frac{\partial f_i}{\partial x_1}(x,t),\ldots,\frac{\partial f_i}{\partial x_n}(x,t),0,\ldots,0,Ls_i^{L-1},0,\ldots,0\Big) \in (\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m)^*,$$

for $i = 1,\ldots,m$.

We consider paths $\gamma(\lambda) := (x_\lambda, t_\lambda, s_\lambda) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m$ depending on a parameter $\lambda \in \mathbb{R}$, such that $(x_\lambda, t_\lambda, s_\lambda) \to (x_0,0,0)$ when $\lambda \to 0$.

We claim that we obtain the cotangent space T^* dual to T by first considering the limits, along all such paths $\gamma(\lambda)$, of all the linear combinations $\sum_{i=1}^m \alpha_\lambda^i V^i(x_\lambda,t_\lambda,s_\lambda)$ viewed as elements in the projective space $\mathbb{P}(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m)^*$, with coefficients α_λ^i depending of the parameter λ too, and such that $\|\alpha_\lambda\| = 1$ for any $\lambda \neq 0$. Finally the full cotangent space T^* is the set of all scalar multiples of these limits.

Let us divide $\sum_{i=1}^m \alpha_\lambda^i V^i(x_\lambda,t_\lambda,s_\lambda)$ by the positive real $\nu_{F_t}(x)$, and compute the limits. We find:

$$\lim_{\lambda\to 0}\frac{\|\sum_{i=1}^m \alpha_\lambda^i \operatorname{grad}_x f_i(x_\lambda,t_\lambda)\|}{\nu_{F_{t_\lambda}}(x_\lambda)} \geq 1 \quad\text{and}\quad \lim_{\lambda\to 0}\frac{\|\sum_{i=1}^m \alpha_\lambda^i L s_{\lambda,i}^{L-1}\|}{\nu_{F_t}(x_\lambda)} = 0$$

where the former follows by the definition of $\nu_{F_t}(x)$ as a minimum. The later limit justifies as follows: firstly, by the definition of X, we have $\|s_i^{L-1}\| = \|f_i(x,t)\|^{\frac{L-1}{L}} \leq \|F(x,t)\|^{\theta+\varepsilon}$, where $\frac{L-1}{L} = \theta+\varepsilon$ and therefore $\varepsilon > 0$ by our choice of L. Next, by applying the hypothesis (3.5) we get :

$$\|F(x,t)\|^{\theta+\varepsilon} \leq c(x_0)\nu_{F_t}(x)\|F(x,t)\|^{\varepsilon},$$

where $\|F(x,t)\|^{\varepsilon}$ converges to 0 when $x \to x_0$ and $t \to 0$.

This shows that the limit:

$$\lim_{\lambda\to 0}\frac{1}{\nu_{F_t}(x)}\sum_{i=1}^m \alpha_\lambda^i V^i(x_\lambda,t_\lambda,s_\lambda) \tag{3.8}$$

represents a nontrivial direction in $\mathbb{P}(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m)^*$ with its last m positions equal to 0, and this holds for any path $\gamma(\lambda)$ as considered above.

Since the projective cotangent space $\mathbb{P}T^*$ is the set of all directions of type 3.8, it follows that T^* (which has dimension $m+1$) is contained in $N \times \mathbb{R}^* \times \{0\}_m$, where $N \subset (\mathbb{R}^n)^*$ is some linear subspace. Therefore T equals $A \times \{0\}_1 \times \mathbb{R}^m$, for some linear subspace $A \subset \mathbb{R}^n$, and since $\dim T = n$, we get $\dim A = n-m$. □

We continue the proof of the theorem. Let us consider the projection map $p : (X, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^m, 0)$, $(x, t, s) \mapsto (t, s)$. Since (t, s) is a regular value of p, we have the following inclusion

$$T = \lim_{k\to\infty} T_{(x,t,s)}\big(\pi^{-1}(t) \cap X\big) \supset \lim_{k\to\infty} T_{(x,t,s)}\big(p^{-1}(t, s) \cap X\big) =: T^{'} \tag{3.9}$$

where $\dim T' = \dim T_{(x,t,s)}\big(p^{-1}(t, s) \cap X\big) = n - m$, and $p^{-1}(t, s) \cap X = \widetilde{F}^{-1}\big(\widetilde{F}(x, t)\big)$.

In order to complete the proof of the theorem, we need to show the inclusion $T_{(x_0,0,0)}W \subset T^{'}$. Both spaces, T' and $T_{(x_0,0,0)}W$, are included in $\mathbb{R}^n \times \{0\}_1 \times \{0\}_m$ by their definition, and they are also included in T, by (3.7) and (3.9). Since we have the identification $T = A \times \{0\}_1 \times \mathbb{R}^m$ by the above Lemma 3.4, it follows that those two spaces must be included in the intersection, which is $A \times \{0\}_1 \times \{0\}_m$.

Now, since the equality of dimensions $\dim T' = \dim A = n - m$, this inclusion must yield an equality: $T' = A \times \{0\}_1 \times \{0\}_m$. This implies the inclusion $T_{(x_0,0,0)}W \subset \mathbb{R}^n \times \{0\}_1 \times \{0\}_m \cap T$, thus we get $T_{(x_0,0,0)}W \subset T'$, which finishes the proof of our theorem. □

3.3. **Parusiński type inequality in case of map germs.** We still consider analytic map germs $F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^m, 0)$, $n > m \geq 1$, regarded as a one-parameter deformations of $F_0 := F(x, 0)$, and $F_t(x) = F(x, t)$. For simplicity, we still assume that $\mathbb{K} = \mathbb{R}$. The following result extends [JST] where one used a week Parusiński type inequality [Pa].

Theorem 3.5. *Let $F_t(x) = F(x, t)$ be an analytic deformation of F_0 such that the map germ $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^m, 0)$ is ∂-Thom regular. If F satisfies the condition*

$$\begin{cases} \text{For any } y \in F_0^{-1}(0) \cap \operatorname{Sing} F_0 \setminus \{0\} \text{ there is} \\ \text{a constant } c(y) > 0 \text{ for which the following inequality holds:} \\ \left\| \frac{\partial F}{\partial t}(x, t)\right\| \leq c(y)\nu_{F_t}(x) \quad \text{when } (x, t) \to (y, 0),\ (x, t) \notin \operatorname{Sing} \widetilde{F} \end{cases} \tag{3.10}$$

then the associated map germ $\widetilde{F}$ is ∂-Thom regular.

Proof. According to Definition 2.2, what we have to show amounts to proving the existence of a Whitney (a)-regular stratification of $\widetilde{F}^{-1}(0, 0) \cap \operatorname{Sing} \widetilde{F} \setminus \{(0, 0)\}$ in some ball B' such that it is Thom $(\mathrm{a}_{\widetilde{F}})$-regular with respect to the stratum $B' \setminus \widetilde{F}^{-1}\big(\widetilde{F}(B' \cap \operatorname{Sing} \widetilde{F})\big)$. Let us first observe the equalities of set germs at the origin:

Lemma 3.6.

$$\widetilde{F}^{-1}(0, 0) \cap \operatorname{Sing} \widetilde{F} \setminus \{0\} = F_0^{-1}(0) \cap \operatorname{Sing} F_0 \setminus \{0\} = F^{-1}(0) \cap \operatorname{Sing} F \cap \{t = 0\} \setminus \{0\}.$$

Proof. The first equality, as well as the inclusion "$\supset$" in place of the second equality are direct consequences of the respective definitions of the singular sets.

To show the reciprocal inclusion "$\subset$" in place of the second equality, we will use the condition (3.10) at the point $(y, 0) \in F_0^{-1}(0) \cap \operatorname{Sing} F_0$, where $y \neq 0$.

We still use the notation $F(x, t) = (f_1(x, t), \ldots , f_m(x, t))$. Firstly, we observe that there is some linear combination $\sum_{i=1}^m a_i \operatorname{grad_x} f_i(y, 0)$ equal to 0 since the gradients are linearly dependent at the singular point $(y, 0) \in F_0^{-1}(0) \cap \operatorname{Sing} F_0 \setminus \{0\}$, and thus Definition (3.4) tells that $\nu_{F_0}(y) = 0$.

Secondly, using the same fixed coefficients a_i, we have

$$\lim_{t\to 0, x\to y} \sum_{i=1}^{m} a_i \operatorname{grad_x} f_i(x,t) = \sum_{i=1}^{m} a_i \operatorname{grad_x} f_i(y,0),$$

by the continuity of the gradient functions. By the Definition (3.4) as a "minimum" we get $\nu_{F_t}(x) \le \| \sum_{i=1}^{m} a_i \operatorname{grad_x} f_i(x,t)\|$, and since the right hand side is equal to 0, it follows that $\lim_{t\to 0, x\to y} \nu_{F_t}(x) = 0$. It is at this moment that we apply the condition (3.10) to deduce, by continuity, that $\frac{\partial F}{\partial t}(y,0) = 0$. This shows that the last column in the Jacobian matrix of F at $(y,0)$ is zero, thus we get $(y,0) \in F^{-1}(0) \cap \operatorname{Sing} F \cap \{t=0\} \setminus \{0\}$, which ends the proof of our lemma. □

By our hypothesis, there is a semi-analytic Whitney (a)-regular stratification $\mathcal{S}$ of a ball $B \subset \mathbb{R}^n \times \mathbb{R}$ which is Thom (a_F)-regular, and such that $F^{-1}(0)\cap \operatorname{Sing} F$ is a union of strata. Let $(y,0) \in B$, $y \neq 0$, be a point on some stratum $V \in \mathcal{S}$, where $V \subset F^{-1}(0) \cap \operatorname{Sing} F$.

Let $(x_t,t) \to (y,0)$ be a continuous path such that $(x_t,t) \notin \operatorname{Sing} F$. Let $T_{(x_t,t)}F^{-1}(s_t)$ denote the tangent space at some smooth point (x_t,t) of the fibre over $s_t := F(x_t,t)$. The assumed ∂-Thom (a_F)-regularity condition at $(y,0)$ amounts to the following property: for any choice of the path $(x_t,t) \to (y,0)$ as above, we have the inclusion:

$$T := \lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}F^{-1}(s_t) \supset T_{(y,0)}V, \tag{3.11}$$

whenever the limit exists in the appropriate Grassmannian, in which case we have $\dim T = n-m+1$.

We next consider the slice of the stratification $\mathcal{S}$ by $\{t=0\}$, consisting of the sets $V' := V \cap \{t=0\}$ for all $V \in \mathcal{S}$. There exists the roughest semi-analytic Whitney (a)-regular stratification $\mathcal{S}'$ of the central fibre $\widetilde{F}^{-1}(0,0) = F_0^{-1}(0)$ which refines this slice stratification, in particular the sets V' are unions of strata of $\mathcal{S}'$.

Lemma 3.7. *Under the hypotheses of Theorem 3.5, one has $T \not\subset \mathbb{R}^n \times \{0\}$, equivalently: T is transversal to $\mathbb{R}^n \times \{0\}$ at $(y,0)$*

Proof. The normal space to the tangent space $T_{(x,t)}F^{-1}(s_t)$ is spanned by the gradient vectors $\operatorname{grad} f_i(x,t)$, for $i = \overline{1,m}$. We consider paths $\gamma(t) := (x_t,t) \in B \subset \mathbb{R}^n \times \mathbb{R}$ depending on the parameter t in some small neighbourhood of the origin $0 \in \mathbb{R}$. As shown in the proof of Lemma 3.4, the limit cotangent space T^* is the set of the scalar multiples of all the limits when $t \to 0$ of the linear combinations $\sum_{i=1}^{m} \alpha_\lambda^i \operatorname{grad} f_i(x_t,t)$ with $\|\alpha_\lambda\| = 1$.

By absurd, let us suppose that $T \subset \mathbb{R}^n \times \{0\}$. This is equivalent to $T^* \ni (0,\dots,0,1)$. Using the inequality (3.10), we find that, for any path (x_t,t), the $(n+1)$-dimensional vector:

$$\lim_{t\to 0} \frac{1}{\nu_{F_t}(x_t)} \sum_{i=1}^{m} \alpha_\lambda^i \operatorname{grad} f_i(x_t,t)$$

has on the first n entries the coordinates a vector of modulus bounded from below by 1, whereas the last entry is bounded from above by a positive constant. This tells that no such vector can be of the form $(0,\dots,0,\lambda)$, with $\lambda \neq 0$, thus this cannot be a vector in T^*, hence we get a contradiction. □

In order to show that $\mathcal{S}'$ is a ∂-Thom stratification for the map $\widetilde{F}$ (cf Definition 2.2) we need to prove the ∂-Thom $(\mathrm{a}_{\widetilde{F}})$-regularity condition at some point $(y,0)$. This amounts to showing that for any choice of a sequence $(x_t,t)\to(y,0)$ such that $(x_t,t)\notin \operatorname{Sing} F$ and $s_t := F_t(x_t)$, we must have the inclusion (where again we assume without loss of generality that the limit exists in the appropriate Grassmannian):

$$T' := \lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}(F_t^{-1}(s_t)\times\{t\}) \supset T_{(y,0)}V', \tag{3.12}$$

where V' is the stratum of $\mathcal{S}'$ which contains $(y,0)$, in particular one has $T_{(y,0)}V \supset T_{(y,0)}V'$.

Both sides of (3.12) are included in $\mathbb{R}^n\times\{0\}$ by construction, and also in T, thus they are included in the intersection $T\cap\mathbb{R}^n\times\{0\}$, which by Lemma 3.7 is a transversal intersection, and hence of dimension $n-m$. But since $T'\subset T\cap\mathbb{R}^n\times\{0\}$ and $\dim T' = n-m$, it follows that we have equality: $T' = T\cap\mathbb{R}^n\times\{0\}$. Consequently $T_{(y,0)}V'\subset T'$, and this ends our proof. □

Remark 3.8. One may be tempted to say that Theorem 3.5 implies Theorem 3.1. This turns out to be true in case of function germs, see Remark 3.10 below, but it is not true in the setting of map germs, as we will show in the following.

Let us consider the map germ $F(x,y,z,t) = (x, g(x,y,z,t)) = (x, (y(x^2+y^2+z^2))^3 + x^2(y(x^2+y^2+z^2)) + tx^k)$ with $k\geq 5$, as deformation of $F_0 := (x, g(x,y,z,0))$. Then $\operatorname{Sing} F = V(F) = \{x = y = 0\}$. By considering paths $\phi(s) = (s,0,z_0,0)$ such that $\lim_{s\to 0}\phi(s) = (0,0,z_0,0)\in \operatorname{Sing} F_0\setminus\{0\}$, we have $\mathrm{ord}_s \nu_{F_t}(\phi(s)) = 2$ and $\mathrm{ord}_s x(s) = 1$, and therefore F does not satisfy the inequality (3.5).

On the other hand, for any analytic path $\phi(s) = (x(s),y(s),z(s),t(s))$ such that $\lim\limits_{s\to 0}\phi(s) = (0,0,z_0,0)\in\operatorname{Sing} F_0\setminus\{0\}$, we have $\mathrm{ord}_s\nu_{F_t(\phi(s))}\leq 2\cdot \mathrm{ord}_s x(s)$. Since $k\geq 5$, we get: $\mathrm{ord}_s\left\|\frac{\partial F}{\partial t}\right\| = k\cdot \mathrm{ord}_s x(s) > \mathrm{ord}_s\nu_{F_t}(\phi(s))$, which implies that condition (3.10) holds, and therefore, by Theorem 3.5, $\widetilde{F} = (F,t)$ is ∂-Thom regular.

In case of deformations of function germs, the above theorem extends the setting of function germ deformations:

Corollary 3.9. [JST] *Let $F_t(x) = F(x,t)$ be a C^1-family of analytic function germs $F_t : (\mathbb{R}^n,0)\to(\mathbb{R},0)$ which satisfies the condition:*

$$\begin{cases} \textit{For any } y\in\operatorname{Sing} F_0\setminus\{0\}, \textit{ there is } c(y)>0 \textit{ such that for } (x,t)\notin\operatorname{Sing}\widetilde{F}: \\ \left|\frac{\partial F}{\partial t}(x,t)\right| \leq c(y)\left\|\frac{\partial F}{\partial x_1}(x,t),\dots,\frac{\partial F}{\partial x_n}(x,t)\right\| \textit{ when } (x,t)\to(y,0). \end{cases} \tag{3.13}$$

Then the deformation $\widetilde{F}$ is ∂-Thom regular. □

Remark 3.10. By applying the classical Łojasiewicz inequality to the function germ $F(x,t)$ over some neighbourhood U of $0\in\mathbb{R}^n\times\mathbb{R}$, we get the existence of $0<\theta<1$ and

$C > 0$ such that the inequality:

$$\|F(x,t)\|^\theta \le C \left\| \frac{\partial F}{\partial t}(x,t), \frac{\partial F}{\partial x_1}(x,t), \ldots, \frac{\partial F}{\partial x_n}(x,t) \right\| \tag{3.14}$$

holds over U. From (3.10) and (3.13) we then deduce:

$$\begin{cases} \text{For any } x_0 \in \operatorname{Sing} F_0 \setminus \{0\}, \text{ there is } k(x_0) > 0 \text{ such that :} \\ \|F(x,t)\|^\theta \le k(x_0) \left\| \frac{\partial F}{\partial x_1}(x,t), \ldots, \frac{\partial F}{\partial x_n}(x,t) \right\| \text{ when } (x,t) \to (x_0, 0). \end{cases} \tag{3.15}$$

While this shows the implication (3.13) $\Rightarrow$ (3.15), let us observe that the stronger condition (3.13) has the advantage upon (3.15) that it is easier to test. Remark that the converse implication is not true, see Example 3.11. For map germs, even the direct implication is not true, as seen in Remark 3.8.

In [AG] one considers the condition (3.15) for deformations of function germs F under the restriction that the associated map $\widetilde{F}$ has an isolated critical value at 0. Let us therefore point out that our Theorem 3.1 represents a far-reaching extension of the main result [AG, Theorem 3.5] since it works for deformations of map germs F, and without any special assumption on the singular locus of $\widetilde{F}$.

Example 3.11. Let $F : (\mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the deformation $F(x,y,z,t) := (x^4 + y^2z^2)(x - t) + t^2 x$, of parameter $t \in \mathbb{R}$. When $t = 0$, we have $F_0(x,y,z) = x(x^4 + y^2z^2)$ and $\operatorname{Sing} F_0(x,y,z) = \{x = y = 0\} \cup \{x = z = 0\}$. For the paths $\phi(s) = (x(s), y(s), z(s), t(s)) = \big(s, s^4, z_0, ks^3\big)$ with $z_0 \neq 0$ and $k \gg 1$, making $s \to 0$ we see that there is no $c(z_0) > 0$ such that condition (3.13) holds. Nevertheless, it appears that the condition (3.15) holds for some $\frac{4}{5} < \theta < 1$. Thus (3.15) does not imply (3.13), however $\widetilde{F}$ is ∂-Thom by Corollary 3.9.

4. Conservation of the ∂-Thom regularity in compositions of deformations

The existence of the Milnor-Hamm tube fibration for the composition of map germs is treated in the recent paper [CJT] with criteria involving the ρ-regularity condition. In case of the composition of deformations of function germs, in the aim of insuring the existence of the Milnor-Hamm tube fibration, the authors of [AG] consider the property (3.15) of Remark 3.10 within the class of deformations of function germs Q such that $\widetilde{Q}$ has isolated singular value. They leave open the question whether or not this condition is preserved by the composition of such deformation maps.

As the ∂-Thom regularity is a sufficient condition which insures the existence of the Milnor-Hamm tube fibration for the composition of deformations of function germs, our answer to the dilemma is as follows: instead of the property (3.15), which might not be preserved by compositions, one should consider the property "*$\widetilde{F}$ is ∂-Thom regular*".

Our next result shows that the property "*∂-Thom regularity*" of $\widetilde{G}$ and $\widetilde{F}$ is indeed preserved by the composition $\widetilde{H} = \widetilde{G} \circ \widetilde{F}$. Moreover, unlike [AG], we do not need to assume that $\widetilde{G}$ has an isolated critical value. Such a result turns out to hold under more general conditions for compositions of map germs.

Theorem 4.1. *If $\widetilde{F}$ and $\widetilde{G}$ are ∂-Thom regular, and if $\widetilde{F}$ has an isolated singular value, then the composed map $\widetilde{H} = \widetilde{G} \circ \widetilde{F}$ is ∂-Thom regular.*

Proof. The proof reduces to the case $\operatorname{Sing} \widetilde{H} \cap \widetilde{H}^{-1}(0)$ has positive dimension as a set germ at $0 \in \mathbb{R}^n \times \mathbb{R}$, since otherwise $\widetilde{H}$ is ∂-Thom regular by definition.

Since $\widetilde{F}$ is ∂-Thom regular, there exists a stratification $\mathcal{W}$ of the open ball $B_\varepsilon \subset \mathbb{R}^n \times \mathbb{R}$, for some $\epsilon > 0$ small enough, which verifies the conditions of Definition 2.2.

Consider a sequence $\{x_i\}_{i\in\mathbb{N}} \subset B_\varepsilon \setminus \widetilde{H}^{-1}(\operatorname{Disc}(\widetilde{H}))$ such that $x_i \to y \in V(\widetilde{H})$.

Let first $y \in V(\widetilde{F})$. We denote by $W \in \mathcal{W}$ the stratum such that $y \in W$. Remark that x_i are regular points of $\widetilde{F}$ by the assumption that $\widetilde{F}$ has an isolated singular value.

Assuming (without loss of generality) that the limits exist in the appropriate Grassmannians, and using the Thom $(\mathrm{a}_{\widetilde{F}})$-regularity, we get the first inclusion in:

$$T_y W \subset \lim_{i\to\infty} T_{x_i} \widetilde{F}^{-1}(\widetilde{F}(x_i)) \subset \lim_{i\to\infty} T_{x_i} \widetilde{H}^{-1}(\widetilde{H}(x_i)),$$

whereas the second inclusion is due to the inclusion of nonsingular fibres $\widetilde{F}^{-1}(\widetilde{F}(x_i)) \subset \widetilde{H}^{-1}(\widetilde{H}(x_i))$, for any i. This shows that the pair $\big(B^n_\varepsilon \setminus \widetilde{H}^{-1}(\operatorname{Disc}(\widetilde{H})\big), W)$ satisfies Thom $(\mathrm{a}_{\widetilde{H}})$-condition.

If now $y \in V(\widetilde{H}) \setminus V(\widetilde{F})$, then y is a regular point of $\widetilde{F}$. Since $\widetilde{H}(x_i) \notin \operatorname{Disc}(\widetilde{H})$, we also have $\widetilde{H}(x_i) \notin \operatorname{Disc}(\widetilde{G})$.

Since $\widetilde{G}$ is ∂-Thom regular by our hypothesis, we may consider an open ball $B_\delta \subset \mathbb{R}^p \times \mathbb{R}$ together with a stratification $\mathcal{S}$ which satisfy Definition 2.2 for $\widetilde{G}$ instead of $\widetilde{F}$. Therefore the point $\widetilde{F}(y) \subset V(\widetilde{G})$ belongs to a positive dimensional stratum $S \subset \mathcal{S}$. The assumed $(\mathrm{a}_{\widetilde{G}})$-regularity of the pair $\big(B_\delta \setminus \widetilde{G}^{-1}(\operatorname{Disc}(\widetilde{G})), S\big)$ yields:

$$T_{\widetilde{F}(y)} S \subset \lim_{i\to\infty} T_{\widetilde{F}(x_i)} \widetilde{G}^{-1}(\widetilde{H}(x_i)), \tag{4.1}$$

where we may again tacitly assume that the limit exist in the appropriate Grassmannian.

The map $\widetilde{F}$ is a submersion in some neighbourhood of y. By applying to (4.1) the inverse map $(T_* \widetilde{F})^{-1}$, which commutes with the limit "$\lim_{i\to\infty}$", we get the equalities $T_y \widetilde{F}^{-1}(S) = (T_y \widetilde{F})^{-1}(T_{\widetilde{F}(y)} S)$ and

$$(T_{x_i}\widetilde{F})^{-1}(\lim_{i\to\infty} T_{\widetilde{F}(x_i)} \widetilde{G}^{-1}(\widetilde{H}(x_i))) = \lim_{i\to\infty} (T_{x_i}\widetilde{F})^{-1}(\widetilde{G}^{-1}(\widetilde{H}(x_i))),$$

therefore we obtain the inclusion:

$$T_y \widetilde{F}^{-1}(S) \subset \lim_{i\to\infty} (T_{x_i}\widetilde{F})^{-1}(\widetilde{G}^{-1}(\widetilde{H}(x_i))) = \lim_{i\to\infty} (T_{x_i}\widetilde{H})^{-1}(\widetilde{H}(x_i)). \tag{4.2}$$

This shows that the pair $\big(B_\varepsilon \setminus \widetilde{H}^{-1}(\operatorname{Disc}(\widetilde{H})), \widetilde{F}^{-1}(S)\big)$ is Thom $(\mathrm{a}_{\widetilde{H}})$-regular, and this holds for all strata $S \in \mathcal{S}$ which are inside $V(\widetilde{G}) \setminus \{0\}$.

We have now a stratification of $V(\widetilde{H})$ which consists of the newly defined strata $\widetilde{F}^{-1}(S)$ for all $S \in \mathcal{S}$, $S \subset V(\widetilde{G}) \setminus \{0\}$, and all the strata $W \in \mathcal{W}'$, $W \subset V(\widetilde{F})$. What we have proved up to now is that the pair consisting of $B_\varepsilon \setminus \widetilde{H}^{-1}(\operatorname{Disc}(\widetilde{H}))$ and any of the above enumerated strata in $V(\widetilde{H})$ satisfies the Thom $(\mathrm{a}_{\widetilde{H}})$-regularity condition.

There is still one more thing to check: the Whitney (a)-regularity of the appropriate couples of strata. Since $\widetilde{F}$ is a submersion on $V(\widetilde{H}) \setminus V(\widetilde{F})$, the inverse image by $\widetilde{F}$ preserves the Whitney (a)-regularity of every pair (S', S'') of strata of $\mathcal{S}$ which are inside $V(\widetilde{G}) \setminus \{0\}$, and such that $S'' \subset \overline{S'}$. But since the Whitney (a)-regularity may not be satisfied with respect to the strata of $\mathcal{W}'$ inside $V(\widetilde{F})$, we need to refine the stratification $\mathcal{W}'$ into a Whitney (a)-regular stratification $\mathcal{W}''$ such that the pairs (S, W) are Whitney (a)-regular, for any $S \in \mathcal{S}$ and $W \in \mathcal{W}''$, and such that $W \subset \overline{S}$.

Then the germ at $0 \in \mathbb{R}^n \times \mathbb{R}$ of the stratification $\mathcal{Q} := \mathcal{S} \bigsqcup \mathcal{W}''$ is a ∂-Thom stratification for $\widetilde{H}$. This ends our proof.

□

Example 4.2. (after [AG]) Consider the maps germs $F : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, where $F(y, z, t) = y(y^2 + z^2 + t^2)$, and $G : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, where $G(y, t) = y^3 + t^2 y$. It turns out that F satisfies the condition (3.15) in a neighbourhood of the origin, whereas G has a single singularity at the origin thus satisfies (3.15) in a trivial manner. (One can also check easily that condition (3.13) is satisfied too.)

Then both $\widetilde{F}$ and $\widetilde{G}$ are ∂-Thom regular, by Theorem 3.1. Since $\widetilde{F}$ has an isolated critical value, we may now directly apply our Theorem 4.1 to conclude that $\widetilde{H} = \widetilde{G} \circ \widetilde{F}$ is ∂-Thom regular, and therefore $\widetilde{H}$ has a Milnor-Hamm fibration by Corollary 2.6. We observe that Theorem 4.1 provides a shortcut and we do not need to verify again the condition (3.15) for the composition $H = G \circ F$ as done in [AG, Example 3.7] in order to obtain a Milnor-Hamm fibration.

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