TAME DEFORMATIONS OF HIGHLY SINGULAR FUNCTION GERMS

CEZAR JOIŢA, MATTEO STOCKINGER, AND MIHAI TIBĂR

Abstract. We give analytic and algebraic conditions under which a deformation of real analytic functions with non-isolated singular locus is a deformation with fibre constancy.

1. Introduction

Deformations of function germs is a classical topic with abundant results, and has a huge impact in geometry, algebra and topology as well as in many more applied fields. Recently, motivated by solving a technical glitch within a Floer Homology problem, Ciprian Manolescu [Ma] asked the following:

Question 1. *Consider a family of highly singular real-analytic function germs $F_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, such that the Jacobian ideal (∂F_t) of F_t is independent of t in a small disk at 0. Is the homology of the Milnor fibres of F_t "constant" in some sense?*

In the *complex setting*, for holomorphic function germs with *isolated singularities* (i.e. the very special case of the 0-dimensional singular locus), the constancy of the Jacobian ideal (∂F_t) obviously implies that the Milnor number $\mu(F_t) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\partial F_t)}$ is constant, and thus the homology of the Milnor fibre is constant too since it is concentrated in the group $\tilde{H}_{n-1}(F_t, \mathbb{Z})$, which is free of rank precisely $\mu(F_t)$. Moreover, by the Lê-Ramanujam [LR], Timourian [Tim], and King [Ki] results, the constancy of the Milnor number implies that the deformation F_t is topologically trivial with the exception of the surface case (i.e. $n = 3$) which remains open.

In the case of complex non-isolated singularities, this question seems to be open in general. There is nevertheless a rich literature on equisingularity problems for families of function germs, dealing with various algebraic-geometric sufficient conditions, starting with the Whitney equisingularity studied by Teissier [Te1, Te2], extended by Gaffney [Ga], Houston [Hou] and many others.

In the real setting there are results by King [Ki] on the topological triviality of one-parameter families of function germs with *isolated singularity*. While the non-isolated case remains widely open in general, Parusinski's paper [Pa2] goes beyond "isolated singularity" in the setting of families of the form $F(x, t) = f(x) + tg(x)$ subject to the

2010 *Mathematics Subject Classification.* 14B07, 32S30, 14D06, 32S55, 14P15.

Key words and phrases. deformations, fibrations, fibre constancy.

The authors acknowledge the partial support of the Labex CEMPI (ANR-11-LABX-0007-01). C. Joiţa and M. Tibăr acknowledges support from the project "Singularities and Applications" - CF 132/31.07.2023 funded by the European Union - NextGenerationEU - through Romania's National Recovery and Resilience Plan, and support by the grant CNRS-INSMI-IEA-329.

condition:

(1) $|g(x)| \ll \big\|\partial f(x) - t\partial g(x)\big\|$ whenever $(x,t) \to (x_0, 0)$, where $x_0 \in \{f=0\} \cap \{g=0\}$,

and, remarkably, shows that such a family is *topologically trivial*, cf [Pa2].

In this paper we are not interested in the problem of the topological triviality of families of function germs. Instead, we will answer *Question 1* in a different way, by focussing on the constancy of the fibration (2) in a fixed ball, as explained in the following.

Definition 1.1. Let $F : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K}^m, 0)$, $n \geq m \geq 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a $\mathbb{K}$-analytic map germ, regarded as a 1-parameter deformation $F_t(x) = F(x,t)$ of the function germ $F_0 = f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ having a singular locus of dimension ≥ 1.

We say that the deformation F of the singular function germ f is *a deformation with fibre constancy* if for any small enough radius $r > 0$, there exist $0 < \delta := \delta(r) \ll r$, and $0 < \eta := \eta(r, \delta) \ll \delta$ such that the restriction:

$$(F_t)_{|} : B_r \cap F_t^{-1}(\partial D_\delta) \to \partial D_\delta. \tag{2}$$

is a locally trivial fibration, which is independent, modulo isotopy, of r and δ, and of the parameter t with $|t| \leq \eta \ll \delta$.

In the real setting, D_δ denotes an interval centred at the origin and its boundary $\partial D_\delta = \{a_-, a_+\}$ consists of two points. In this setting, the independency asked by Definition 1.1 means the invariance of the two fibres[1], $B \cap F_t^{-1}(a_-)$ and $B \cap F_t^{-1}(a_+)$. One may then ask:

Question 2. *What natural (and minimal) condition implies that the deformation F is a deformation with fibre constancy?*

The problem posed by *Question 2* is not a classical equisingularity problem and makes really sense in the case of f with non-isolated singularities. If the function germ F_0 has an isolated singularity, then for $t \neq 0$ this may either split into several isolated singularities of the restriction map $(F_t)_{|}$ from (2) above, or not split at all. Even if the isolated singularity of F_0 splits, the tube fibrations (2) still exists precisely because the splitting is concentrated at the origin only (i.e. because of the stated choice of a much smaller range of the parameter t, namely $|t| \ll \delta$), see e.g. [JiT] for the study of a slightly more general setting of isolated singularities. In case it does not split, then one has stronger results: the above cited ones by Lê-Ramanujam [LR] and Timourian [Tim] over $\mathbb{C}$, and by King [Ki] over $\mathbb{R}$, yield the topological triviality of the family of function germs F_t, or what one calls topological equisingularity.

In the setting of *non-isolated holomorphic functions*, the concept of "deformation with fibre constancy" (Definition 1.1) appeared under the name of "admissible deformations" in the studies of the topology of certain classes of function germs with non-isolated singular locus, starting with the seminal paper by Siersma [Si1] on "line singularities", and the follow-up more general studies of the Milnor fibre of function germs with 1-dimensional and 2-dimension singular loci, see e.g. [Si2], [Pe], [dJ], [Za], [MS], [Ne], [Fe], [FM], [ST2]

[1] Of which one may be empty, but not both empty since we assume that the function germ F_0 is non-constant.

etc. The very recent paper [Hof] gives algebro-geometric criteria which are sufficient to establish admissibility for complex-analytic deformations.

In both settings, complex and real, our general answer to *Question 2* is (Theorem 3.7): *"tame deformations" are deformations with fibre constancy.* This "tameness" condition, introduced in Definition 3.5, is based on the transversality of all small enough spheres to the fibres of the deformation $(F_t)_|$ defined at (2) above an appropriate small disk D_δ. This derives from a principle established by Milnor [Mi] and which occurs, mostly under the name of *ρ-regularity*, in several studies of fibrations of real analytic map germs on smooth or singular base spaces, such as [ACT1], [ACT2], [AT1], [AT2], [ART], [JoT1], [CJT], and see [Ti2] for older references.

Searching for conditions which imply tameness, we focus on two of them, one analytic and the other algebraic. Let us introduce the first one. It is inspired by Parusinski's condition[2] [Pa2] evoked above as (1), although much weaker than that.

Definition 1.2. Let $F_t(x) = F(x,t)$ be a C^1-family of analytic function germs $F_t : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$. We consider the following condition:

$$(3)\qquad \begin{cases} \text{For any } x_0 \in \operatorname{Sing} F_0 \setminus \{0\}, \text{ there is } c_{x_0} > 0 \text{ such that} \\ \left|\dfrac{\partial F}{\partial t}\right| \le c_{x_0} \left\| \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n} \right\| \ \text{ when } (x,t) \to (x_0, 0), \ \text{for } (x,t) \notin \operatorname{Sing} \widetilde{F}. \end{cases}$$

where $\widetilde{F}(x,t) := \big(F(x,t), t\big)$, where $\widetilde{F}(x,t)$ is the map germ defined at (5).

Note that condition (3) excepts[3] the origin $(0,0)$. This allows the splitting of the singular locus of F_0 out of the origin (but only at the origin).

Condition (3) may be interpreted as an integral dependence as follows (see Proposition 5.3): *for any $x_0 \neq 0$ close enough to 0, the function germ $\frac{\partial F}{\partial t}$ belongs to the integral closure* $\overline{\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)}$. This may also be contrasted to Teissier's condition (c) [Te1], [Te2] in the complex setting: $\frac{\partial F}{\partial t}$ *belongs to the integral closure* $\overline{\mathfrak{m}_x\left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)}$, *where $\mathfrak{m}_x$ denotes the maximal ideal in $\mathcal{O}_n$*, which has been used to study families of holomorphic functions with *isolated singularity*. Still in case of *complex isolated singularities*, Teissier showed in [Te1], [Te3], that this is also equivalent to the *Whitney equisingularity* of the family of function germs F_t, which is known to imply its topological triviality.

Here we go beyond the isolated singularity setting, and prove the following:

Theorem 1.3. *Let $F(x,t) = F_t(x)$ be a $\mathbb{K}$-analytic deformation of F_0 which satisfies condition* (3) *of Definition 1.2.*

Then the deformation F of F_0 is a deformation with fibre constancy in the sense of Definition 1.1. Moreover, $\widetilde{F}$ has a Milnor-Hamm tube fibration (7)*, cf Definition 3.1.*

[2]Parusinski's condition (1) had extended a condition used in the classical case of isolated singularities by Teissier [Te1] and Lê-Saito [LS], and also extended a condition used before for studying the fibres of polynomial functions at infinity ([Pa1], [ST1], [Ti1], [Ti2], [Pa2, §3] etc).

[3]See the examples in §6. Typically such deformations do not satisfy condition (3) at the origin $(0,0)$.

The proof follows from Theorem 3.7 after showing in Theorem 4.3 that a nice deformation is tame (Definition 3.5). The tameness follows from the proof of the existence of the Milnor-Hamm tube fibration which is based on checking the *Thom regularity condition*, a well-known method employed e.g. in [Hi], [LS], [Sa1, Sa2], [PT], [AT2], [ACT1] etc. Here we use a less demanding condition called *∂-Thom regularity* (see §4.1 for details), and that the ∂-Thom regularity implies the ρ-regularity (Proposition 4.2).

Our second condition is algebraic. We show by Theorem 5.6 that if the Jacobian ideal inclusion $(\partial F_t) \subset (\partial F_0)$ holds for any t close enough to 0, then condition (3) holds, hence the deformation F is tame, and thus it is "a deformation with fibre constancy" by our general Theorem 4.3. To a certain extent, this comes closer to Manolescu's *Question 1*. Moreover, in §5 we slightly extend the setting by replacing the ideal (∂F_0) in the above inclusion by its *integral closure* $\overline{(\partial F_0)}$.

Schematically, this is what we prove here:

(4) $\boxed{\text{deformation with fibre constancy}} \Longleftarrow \boxed{\text{tame deformation}} \Longleftarrow \boxed{(3)} \Longleftarrow \boxed{(\partial F_t) \subset \overline{(\partial F_0)}}$

Acknowledgements. We thank Ciprian Manolescu for bringing to our attention *Question 1*. We also thank him and Laurenţiu Păunescu for enriching observations which impelled our study. Special thanks go to the referee for very interesting remarks which helped us improve the paper.

C. Joiţa and M. Tibăr acknowledge support of the project "Singularities and Applications" - CF 132/31.07.2023 funded by the European Union - NextGenerationEU - through Romania's National Recovery and Resilience Plan, and support by the grant CNRS-INSMI-IEA-329.

2. Image and discriminant of a deformation

Let $F_0 : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be a non-constant analytic function germ. Let $F_t(x) := F(x,t)$ be an analytic deformation of the analytic function germ $F_0(x) := F(x, 0)$ such that $F(0,t) = 0$ for any t in some small neighbourhood of $0 \in \mathbb{K}$. Let

$$\widetilde{F} := (F(x,t), t) : (\mathbb{K}^n \times \mathbb{K}, 0) \to (\mathbb{K} \times \mathbb{K}, 0) \tag{5}$$

be the analytic map germ defined by the deformation F. The singular set $\operatorname{Sing} \widetilde{F}$ is the zero locus $Z\big(\frac{\partial F}{\partial x_1}(x,t), \ldots, \frac{\partial F}{\partial x_n}(x,t)\big)$. As a set germ at the origin $(0,0)$, it contains the union of set germs $\bigcup_{t\in\mathbb{K}} \operatorname{Sing} F_t$, but may contain other irreducible components, see e.g. Example 6.1. Note that $\operatorname{Sing} \widetilde{F} \cap \{t = 0\} = \operatorname{Sing} F_0$, and that we have the inclusion $\operatorname{Sing} F \subset \operatorname{Sing} \widetilde{F}$, which may be a strict inclusion even when restricted to the slice $\{t = 0\}$, like in the example: $F(x,y,t) = x^2 + ty$.

We will consider here the class of deformations F with the property that the inclusion $\operatorname{Sing} F \subset \operatorname{Sing} \widetilde{F}$ restricts to an equality in the slice $\{t = 0\}$, namely deformations F such that:

$$(\operatorname{Sing} F)_{|t=0} = (\operatorname{Sing} \widetilde{F})_{|t=0}. \tag{6}$$

We refer to Proposition 5.5 for the relations of (6) with other conditions. See also the Examples section §6 for more comments.

2.1. **The image problem.** Let us start by observing that analytic function germs have well-defined images as set germs, either over $\mathbb{C}$ or over $\mathbb{R}$. This is the case for our function germs F_t, for any t. However this is no more the case for the image of a map germ $G : (\mathbb{K}^m, 0) \to (\mathbb{K}^2, 0)$ with $m \geq 2$, as pointed out in [JoT1], see also [ART]. The map G may be locally open, i.e. one may have the equality of set germs $(\operatorname{Im} G, 0) = (\mathbb{K}^2, 0)$, but certain map germs, for instance $G(x, y) = (x, xy)$, do not even have well-defined image as a set germ at the origin. For the problem "when a map germ has a well defined image as a set germ" we refer to [JoT2], see also [ART] and [JoT1].

In our setting, the map germ $\widetilde{F}$ associated to an analytic deformation $F(x,t)$ is somewhat more special.

Proposition 2.1. *In case $\mathbb{K} = \mathbb{C}$, the map germ $\widetilde{F}$ is locally open.*
In case $\mathbb{K} = \mathbb{R}$ we have:

(a) *If F_0 is locally open then $\widetilde{F}$ is a locally open map germ. In particular this is the case if $\operatorname{Sing} F_0 \neq F_0^{-1}(0)$.*
(b) *If F_0 is not locally open then for any radius $r > 0$, the image $\widetilde{F}(B_r)$ contains either the set germ $(\mathbb{R}_{\geq 0} \times \mathbb{R}, 0)$ or the set germ $(\mathbb{R}_{\leq 0} \times \mathbb{R}, 0)$.*

Remark 2.2. In the above point (b) one may not have equality, therefore the word "contains" is important. Unlike the complex setting, in the real setting the map $\widetilde{F}$ might not have a well-defined image as a set germ at $0 \in \mathbb{R}^2$. This may happen in Proposition 2.1(b), as shown by the following very simple example: $F(x, y, t) = x^2 + t(x + y)$ is a deformation of $F_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $F_0(x, y) = x^2$.

Note that if we view this example over $\mathbb{C}$ then the image of $\widetilde{F}$ is a well-defined set germ, by Proposition 2.1.

Proof of Proposition 2.1.
Case $\mathbb{K} = \mathbb{C}$. The zero set $\widetilde{F}^{-1}(0, 0) = F_t^{-1}(0)$ has codimension 2 in the source $\mathbb{C}^n \times \mathbb{C}$ of the map $\widetilde{F}$. Therefore [JoT2, Theorem 1.1(a)(i)] applies here, and tells that $\widetilde{F}$ is locally open.
Case $\mathbb{K} = \mathbb{R}$. Let $B_r \subset \mathbb{R}^n \times \mathbb{R}$ denote the open ball of radius $r > 0$ centred at the origin, and let $B'_{r'} \subset \mathbb{R}^n$ such a ball in $\mathbb{R}^n$. We need the following result:

Lemma 2.3. *If $\operatorname{Im} F_0 \supset \mathbb{R}_{\geq 0}$ as set germs, then for any $r > 0$, we have $\widetilde{F}(B_r) \supset \mathbb{R}_{\geq 0} \times \mathbb{R}$. The similar statement holds if we replace $\mathbb{R}_{\geq 0}$ by $\mathbb{R}_{\leq 0}$.*

Proof. The proof is elementary and uses only the continuity of the maps; we provide it for completeness.

Our deformation of F_0 presents as $F(x,t) = F_0(x) + tG(x,t)$ with G analytic, hence continuous, where $G(0,t) = 0$ for all t close enough to 0. Let us fix some small enough radius $r > 0$, such that $|G(x,t)| \le 1$ on B_r. Let $r' > 0$ and $\eta' > 0$ such that $B'_{r'} \times (-\eta', \eta') \subset B_r$.

By our hypothesis $\operatorname{Im} F_0 \supset (\mathbb{R}_{\ge 0}, 0)$, there exists $\varepsilon_0 > 0$ such that $F_0(B'_{r'}) \supset [0, \varepsilon_0)$. Setting $\varepsilon := \frac{\varepsilon_0}{4}$ and $\eta := \min\{\eta', \varepsilon\}$, we will show that $\widetilde{F}(B_r) \supset [0,\varepsilon) \times (-\eta, \eta)$, as follows.

Let $(\alpha, \beta) \in [0,\varepsilon) \times (-\eta,\eta)$. We have to show that there exits $x \in B'_{r'}$ such that $F(x,\beta) = \alpha$. Let $x_1 \in B'_{r'}$ such that $F_0(x_1) = \frac{\varepsilon_0}{2}$. Since $|G(x_1,\beta)| \le 1$ and $|\beta| < \varepsilon \le \frac{\varepsilon_0}{4}$, we get:

$$F(x_1,\beta) = \frac{\varepsilon_0}{2} + \beta G(x_1,\beta) > \frac{\varepsilon_0}{4} = \varepsilon > \alpha.$$

Since we also have $F(0,\beta) = 0 \le \alpha$, and since F is continuous, we conclude that there exists a point x on the segment joining the origin with x_1, hence in $B'_{r'}$, such that $F(x,\beta) = \alpha$. Our claim is proved.

Mutatis mutandis, the same proof applies if we replace $\mathbb{R}_{\ge 0}$ by $\mathbb{R}_{\le 0}$. □

(b). If F_0 is not locally open, then its image is either the set germ $(\mathbb{R}_{\ge 0}, 0)$ or the set germ $(\mathbb{R}_{\le 0}, 0)$. We may thus apply Lemma 2.3 to conclude.

(a). The first statement of (a) is also a direct consequence of Lemma 2.3.

Let us prove the second statement of (a). We have $\operatorname{Sing} F_0 = \operatorname{Sing} \widetilde{F} \cap \widetilde{F}^{-1}(0,0)$ and by our hypothesis this is included but not equal to $F_0^{-1}(0) = \widetilde{F}^{-1}(0,0)$ which is the central fibre of the map $\widetilde{F}$. We may therefore apply [JoT1, Lemma 2.5] to the map $\widetilde{F}$ and conclude that it is a locally open map. □

2.2. **The discriminant of a deformation.** The image by G of the singular locus $\operatorname{Sing} G$ of a map germ $G : (\mathbb{K}^m, 0) \to (\mathbb{K}^s, 0)$, $m \ge s \ge 2$, supports the same discussion. It has been pointed out in [JoT1] that this might not be well-defined as a set germ, and moreover, this may happen even if $\operatorname{Im} G$ is a well defined set germ.

Nevertheless, the case of a deformation F turns out to be more special; the next result tells that the image $\widetilde{F}(\operatorname{Sing} \widetilde{F})$ is a well-defined set germ.

Proposition 2.4. *If the dimension of the target of the non-constant analytic map germ $G : (\mathbb{K}^m, 0) \to (\mathbb{K}^s, 0)$ is $s = 2$, then $G(\operatorname{Sing} G)$ is a well-defined set germ. It is either empty, or one point (the origin), or it is as follows: in the case $\mathbb{K} = \mathbb{C}$ it is a complex analytic curve, whereas in case $\mathbb{K} = \mathbb{R}$ it is a semi-analytic curve.*

In particular, the image $\Delta := \widetilde{F}(\operatorname{Sing} \widetilde{F})$ is a well-defined set germ, that we will call "discriminant of the deformation F".

Proof. By [JoT1, Theorem 3.2], for any non-constant holomorphic map germ G of target $\mathbb{C}^2$, the image $G(\operatorname{Sing} G)$ is a well-defined complex analytic set germ. Since the complement of this image is dense (by Sard's Theorem), the image cannot have dimension 2.

In the setting of real deformations we do as follows: for any fixed ball B_r, the image $G(B_r \cap \operatorname{Sing} G)$ is a subanalytic subset of $\mathbb{R}^2$, which is in fact semi-analytic as proved by Łojasiewicz [Lo] when the target is of dimension 2. This set is included in the complex discriminant of G viewed as a holomorphic function, which is a well-defined germ of a complex curve, as shown above. It then follows that $G(B_r \cap \operatorname{Sing} G)$ is a real curve and that it does not depend on the radius r regarded as a germs at the origin of $\mathbb{R}^2$.

This proof applies to $G := \widetilde{F}$, which shows our second claim. □

3. Tame deformations

In the preceding section we have seen that the real setting is different from the complex one in what concerns the image of the map $\widetilde{F}$. However we are interested here in certain general fibres of the deformation, and therefore we will adapt the study to this more particular interest.

Notation $\mathcal{H}_F$. In the complex setting, $\mathcal{H}_F$ will denote the full target $\mathbb{C}^2$. In the real setting, according to the 3 situations in Proposition 2.1(a) and (b), we have:

(a) $\mathcal{H}_F := \mathbb{R}^2$ if $\widetilde{F}$ is locally open.

(b) $\mathcal{H}_F := (\mathbb{R}_{\geq 0} \times \mathbb{R}, 0)$, or $\mathcal{H}_F := (\mathbb{R}_{\leq 0} \times \mathbb{R}, 0)$, if $\widetilde{F}$ is not locally open but $\widetilde{F}(B_r)$ contains this half-plane for any $r > 0$, respectively.

3.1. **Local tube fibration.** We have seen that the discriminant $\Delta := \widetilde{F}(\operatorname{Sing} \widetilde{F})$ is a well-defined set germ, cf Proposition 2.4.

Definition 3.1. We say that $\widetilde{F}$ *has a local tube fibration*, also called *Milnor-Hamm fibration*, if for any small enough $r > 0$ there exists a radius $\delta \ll r$, and a radius $\eta \ll \delta$ such that the restriction:

$$\widetilde{F}_| : \overline{B_r} \cap \widetilde{F}^{-1}(\mathcal{H}_F \cap (D_\delta \times D_\eta) \setminus \Delta) \to \mathcal{H}_F \cap (D_\delta \times D_\eta) \setminus \Delta \tag{7}$$

is a locally trivial fibration which is independent of the chosen constants up to isotopy.

The above definition sounds a bit different from the general definition of the Milnor-Hamm fibration in [ART] since here we do not assume that the map germ $\widetilde{F}$ is "nice", i.e., both $\operatorname{Im} \widetilde{F}$ and $\widetilde{F}(\operatorname{Sing} \widetilde{F})$ to be well-defined as set germs at the origin. Let us explain what happens in our particular situation.

In the holomorphic setting, the map $\widetilde{F}$ is automatically nice, by Proposition 2.1. If the local tube fibration (7) exists then the fibre $B_r \cap F_t^{-1}(s)$ is diffeomorphic to the fibre $B_r \cap F_0^{-1}(s)$, for any $(s,t) \in (D_\delta \times D_\eta) \setminus \Delta$ because this later set is connected. This means that one has a single fibre of (7).

In the real analytic setting, $\mathcal{H}_F$ is included in the target of $\widetilde{F}$ by Proposition 2.1, and therefore the target of the map $\widetilde{F}_|$ defined by (7) is independent of the radius of the ball B_r. As Δ is well-defined, it follows that $\operatorname{Im} \widetilde{F}_|$ is well-defined as a set germ, and therefore $\widetilde{F}_|$ is a "nice map" in the sense of [ART]. The set $\mathcal{H}_F \cap (D_\delta \times D_\eta) \setminus \Delta$ consists of finitely many connected components $A_1, \ldots, A_\zeta$, each of them having a unique fibre $B_r \cap F_t^{-1}(s)$ for $(s,t) \in A_j \subset \mathcal{H}_F \cap (D_\delta \times D_\eta) \setminus \Delta$, up to diffeomorphisms.

In particular we have:

Corollary 3.2. *If the local tube fibration (7) exists, then the deformation $F(x,t)$ is a deformation with fibre constancy (in the sense of Definition 1.1).*

Proof. Let $B'_r \subset \mathbb{K}^n$ be a small enough ball centred at the origin of radius $r > 0$ such that it is a Milnor ball for the function germ F_0, and let $0 < \delta \ll r$ be a corresponding Milnor disk. As above, let $B_r \subset \mathbb{K}^n \times \mathbb{K}$ be the ball centred at the origin of the same radius r.

Since $\operatorname{Sing} \widetilde{F}$ intersects the slice $\{t = 0\}$ at $\operatorname{Sing} F_0 \subset F^{-1}(0)$, it follows that for small enough $0 < \eta \ll \delta$ we have:

$$\Delta \cap \mathcal{H}_F \cap (\partial D_\delta \times D_\eta) = \emptyset.$$

This shows that (2) is a sub-fibration of (7) for any $t \in D_\eta$. □

Remark 3.3. The fibre $B_r \cap F_0^{-1}(s)$ is precisely a Milnor fibre of F_0, since we have assumed that the ball B_r is a Milnor ball for F_0, and it is diffeomorphic to $B_r \cap F_t^{-1}(s)$. Nevertheless $B_r \cap F_t^{-1}(s)$ is not necessarily the Milnor fibre of the function germ F_t. Indeed, the function germ F_t may require the choice of a smaller radius $r' < r$ such that $B_{r'}$ is Milnor ball for F_t at 0. This problem is well-known in deformations (see e.g. [JiT]): for each $t \in D_\eta$ there is a "maximum radius" r_t of the Milnor ball of the function germ F_t, and it is possible that $\lim_{t\to 0} r_t = 0$, whereas $r_0 := r$ is a well-defined positive value.

Definition 3.4. (Milnor set)
Consider the square of the Euclidean distance function from the origin $\rho : \mathbb{K}^{n+1} \to \mathbb{R}_{\geq 0}$, and the map

$$(\widetilde{F}, \rho)_| : \widetilde{F}^{-1}((D_\delta \times D_\eta) \setminus \Delta) \to \mathbb{K}^2 \times \mathbb{R}_{\geq 0}$$

The analytic set

$$M(\widetilde{F}) := \overline{\operatorname{Sing}(\widetilde{F}, \rho)_| \setminus \widetilde{F}^{-1}(\Delta)}$$

will be called here the *Milnor set* of $\widetilde{F}$.

In the following we will tacitly consider the germ of $M(\widetilde{F})$ at $(0, 0)$, for which we use the same notation.

Definition 3.5. (Tame deformations)
We say that the deformation $F_t(x) = F(x, t)$ of F_0 is *tame* if and only if the following condition holds:

$$M(\widetilde{F}) \cap \{t = 0\} \cap \operatorname{Sing} F_0 \subset \{(0, 0)\}. \tag{8}$$

Remark 3.6. Condition (8) is equivalent to the *ρ-regularity* of $\widetilde{F}$ considered in [ART, (6)], despite the fact that the definition of the Milnor set $M(\widetilde{F})$ itself is slightly different from the one considered in [ART].

Theorem 3.7. *If the analytic deformation $F(x, t)$ of $F(x, 0)$ is tame, then $\widetilde{F}$ has a Milnor-Hamm tube fibration (7), and the deformation $F(x, t)$ is a deformation with fibre constancy (in the sense of Definition 1.1).*

Proof. It was proved in [ART, Lemma 3.3]: *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 1$, be a non-constant nice analytic map germ. If G is ρ-regular then G has a Milnor-Hamm tube fibration.*

We have seen in §2.1 that in case of a deformation F, the condition "nice" (recalled after Definition 3.1) holds in the holomorphic setting, thus the above result applies and yields our claim.

In the real setting, we have shown in the comments after Definition 3.1 that the condition "nice" holds for the map $\widetilde{F}_|$ in (7). The arguments of the proof of [ART, Lemma 3.3] work without any modification. Indeed, the ρ-regularity (8) implies that the map (7) is a proper submersion, thus it is a locally trivial fibration over $\mathcal{H}_F \cap (D_\delta \times D_\eta) \setminus \Delta$, and the independency of the tube fibration (7) follows from the definition.

Finally we may apply Corollary 3.2, and our claim is proved in both complex or real settings.

Let us note for the record that the tameness of the deformation F is equivalent to the fact that (7) is a proper stratified submersion independently of the chosen constants. □

Although $M(\widetilde{F})$ is an analytic set, hence defined by equations, it is still not easy to compute it since one has to single out only those irreducible components which are not included in $\widetilde{F}^{-1}(\Delta)$. We will display in the following two conditions which imply "tameness", one analytic and the other algebraic.

4. A condition for the tameness

4.1. **The partial Thom stratification.** Let us recall the *∂-Thom regularity*, after [Ti1, Def. 2.1], [Ti2, A 1.1], [DRT, §6], and [ART, §4], see also [CJT]. This is a weaker condition than the usual Thom regularity, but sufficient for the existence of the Milnor tube fibration (cf Definition 3.1) in the general case of real map germs, see Proposition 4.2 below. There are however examples where the map germ G has Milnor tube fibration without being ∂-Thom regular, see [PT] and [ART].

Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, and let Δ denote its discriminant. For a given stratification of a neighbourhood of $0 \in \mathbb{R}^m$, let A, B be strata such that $B \subset \bar{A} \setminus A$. One says that the pair (A, B) satisfies the Thom (a_G)-regularity condition at $x \in B$, if the following condition holds: for any $\{x_k\}_{k\in\mathbb{N}} \subset A$ such that $x_k \to x$, if the tangent space $T_{x_k}(G_{|A})$ converges, when $k \to \infty$, to a limit H in the appropriate Grassmannian, then $T_x(G_{|B}) \subset H$.

In the case of our function germ F, it is known that there exists a *Thom (a_F)-regular stratification* of $F^{-1}(0)$, cf. Hironaka [Hi]: any real or complex function germ h admits a Thom (a_h)-regular stratification of its zero locus $h^{-1}(0)$; see also the discussion in [ART]. Moreover, in the complex analytic setting, a Whitney (b)-regular stratification of $h^{-1}(0)$ is also Thom (a_h)-regular. This was shown in [BMM, Thm. 4.2.1] with $\mathcal{D}$-module techniques; topological proofs can be found in [Pa1] and [Ti1], see also [Ti2, Thm. A 1.1.7].

Definition 4.1. (after [ART])
We say that G is *∂-Thom regular* if there is a ball B_r^m centred at $0 \in \mathbb{R}^m$ and a semi-analytic stratification $\mathcal{S} = \{S_\alpha\}$ of $B_r^m \cap G^{-1}(0)$ such that, for any stratum S_α, the pair $(B_r^m \setminus G^{-1}(\Delta), S_\alpha)$ satisfies the Thom (a_G)-regularity condition.

We also say that the stratification $\mathcal{S}$ is a *∂-Thom (a_G)-stratification.*

Let us also point out that any Thom regular stratification is obviously ∂-Thom regular. The difference is that we do not ask neither the Whitney (a)-regularity condition between couples of strata A, B inside $G^{-1}(0)$, nor the (a_G)-regularity for A outside $G^{-1}(0)$ and B inside $G^{-1}(0)$.

The ∂-Thom regularity is however sufficient to insure the existence of the Milnor-Hamm fibration:

Proposition 4.2. [ART, Prop. 4.2] *and* [JoT1, Cor. 5.8]. *If G is ∂-Thom regular, then G is ρ-regular and has a Milnor-Hamm fibration.* □

4.2. **Theorem and its proof.** Our $\mathbb{K}$-analytic deformation of F_0 has by definition the following presentation:

$$F(x,t) = f_0(x) + \sum_{j\geq 1}^{\infty} t^j f_j(x) \tag{9}$$

where, for any $j \geq 0$, f_j is a $\mathbb{K}$-analytic function germ at 0 of variable $x \in \mathbb{K}^n$, with $f_j(0) = 0$, since we have assumed that F_t is a function germ at the origin, and thus $F_t(0) = 0$, for any t close enough to 0.

Theorem 4.3. *Let $F_t(x) = F(x,t)$ be an analytic deformation of F_0 which satisfies the condition* (3) *of Definition 1.2. Then the deformation $F(x,t)$ of $F_0(x)$ is ∂-Thom regular, and therefore tame.*

Proof. Let us fix a semi-analytic Whitney stratification $\mathcal{S}$ of $\mathbb{K}^n \times \mathbb{K}$ which is Thom (a_F)-regular, thus $\operatorname{Sing} F$ is a union of strata. Let $(y,0) \in \mathbb{K}^n \times \mathbb{K}$, with $y \neq 0$, be a point on some stratum $V \in \mathcal{S}$, $V \subset \operatorname{Sing} F \subset F^{-1}(0)$. That $(y,0) \in \operatorname{Sing} F$ is equivalent to $y \in \operatorname{Sing} F_0 \cap \left\{\frac{\partial F}{\partial t}(x,0) = 0\right\} \setminus \{0\}$. Moreover, this is also equivalent to $y \in \operatorname{Sing} F_0 \setminus \{0\}$. Indeed, the inequality (3) applied at $y \in \operatorname{Sing} F_0 \setminus \{0\}$ shows that $y \in \left\{\frac{\partial F}{\partial t}(x,0) = 0\right\}$, which proves the inclusion:

$$\operatorname{Sing} F_0 \setminus \{0\} \subset \left\{\frac{\partial F}{\partial t}(x,0) = 0\right\} \setminus \{0\}. \tag{10}$$

Let us remark that this proof also shows that condition (3) implies condition (6).

Let now $T_{(x_t,t)}F^{-1}(s_t)$ denote the tangent space at some smooth point (x_t,t) of the fibre, i.e. $(x_t,t) \notin \operatorname{Sing} F$ and $s_t := F(x_t,t)$. The assumed ∂-Thom (a_F)-regularity condition at $(y,0)$ amounts to the following property: for any choice of a sequence $(x_t,t) \to (y,0)$ such that $(x_t,t) \notin \operatorname{Sing} F$, we have the inclusion:

$$\lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}F^{-1}(s_t) \supset T_{(y,0)}V, \tag{11}$$

where we may assume without loss of generality that the limit exists in the appropriate Grassmannian.

We now consider the slice of the stratification $\mathcal{S}$ by $t = 0$, consisting of the sets $V' := V \cap \{t = 0\}$ for all $V \in \mathcal{S}$. There exists the roughest semi-analytic Whitney (a)-regular stratification $\mathcal{S}'$ of the central fibre $\widetilde{F}^{-1}(0,0) = F_0^{-1}(0)$ which refines this slice stratification, thus the sets V' are unions of strata of $\mathcal{S}'$.

Lemma 4.4. *If condition (3) holds, then $\mathcal{S}'$ is a ∂-Thom stratification for the map $\widetilde{F}$.*

Proof. We consider the fibres of F_t for all t close enough to 0, and make them converge to the central fibre $F_0^{-1}(0)$. We need to prove the ∂-Thom $(\mathrm{a}_{\widetilde{F}})$-regularity condition at the point $(y,0)$, which amounts to showing that for any choice of a sequence $(x_t,t) \to (y,0)$ such that $(x_t,t) \notin \operatorname{Sing} F$ and $s_t := F_t(x_t)$, we have the inclusion:

$$\lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}(F_t^{-1}(s_t) \times \{t\}) \supset T_{(y,0)}V', \tag{12}$$

where by V' we denote the stratum of $\mathcal{S}'$ which contains $(y,0)$, in particular we have $T_{(y,0)}V \supset T_{(y,0)}V'$.

We will deduce (12) from the inclusion (11) via the condition (3). Let us suppose by *reductio ad absurdum* that we have:

$$\lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}(F_t^{-1}(s_t) \times \{t\}) \not\supset T_{(y,0)}V'. \tag{13}$$

Both sides are vector subspaces of $\mathbb{R}^n \times \{0\}$, the left hand side has dimension $n-1$, and the right hand side has dimension $\dim T_{(y,0)}V' \geq 1$. Our assumption (13) implies the equality:

$$T_{(y,0)}V' + \lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}(F_t^{-1}(s_t) \times \{t\}) = \mathbb{R}^n \times \{0\}. \tag{14}$$

On the other hand, both sides of (13) are included in $\lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}F^{-1}(s_t)$ and the later has dimension n.

By dimension reasons, we therefore get from (14) the equality:

$$\lim_{(x_t,t)\to(y,0)} T_{(x_t,t)}F^{-1}(s_t) = \mathbb{R}^n \times \{0\}.$$

This equality is equivalent to:

$$\lim_{(x_t,t)\to(y,0)} \frac{1}{\|\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial t}\|}\left(\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial t}\right) = (0,\ldots,0,\pm 1), \tag{15}$$

in particular we have $\lim_{(x_t,t)\to(y,0)} \frac{\left|\frac{\partial F}{\partial t}\right|}{\|\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial t}\|} = 1$.

The inequality (3) tells now that if we divide it by $\left\|(\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial t})\right\|$ then, by taking the limit, we get the inequality:

$$\lim_{(x_t,t)\to(y,0)} \left\|\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\| \Big/ \left\|\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n},\frac{\partial F}{\partial t}\right\| \geq \frac{1}{c_y} > 0$$

This tells that the first n entries in (15) cannot be all zero. We have thus shown the inclusion (12), which ends our proof that the Thom $(\mathrm{a}_{\widetilde{F}})$-regularity holds at the point $(y,0)$. □

The existence of the partial Thom stratification $\mathcal{S}'$ for the map $\widetilde{F}$ implies that there is $R > 0$ such that, for any positive $r \le R$, the sphere $S_r \subset \mathbb{K}^n$ is transversal to all positive dimensional strata of $\mathcal{S}'$. As a direct consequence of the definition (11), it follows that the sphere S_r is transversal to the smooth nearby fibres of $\widetilde{F}$ as ine (7), and therefore F is tame (and in particular, by Theorem 3.7, the tube fibration (7) exits). This ends the proof of Theorem 4.3. □

5. The Jacobian criterium

Before introducing the Jacobian criterion for tameness, let us first recall the *integral closure* of an ideal of germs of analytic functions. We denote by $\mathcal{A}_n$ the ring of germs of analytic functions at the origin on $\mathbb{K}^n$.

Definition 5.1 ([Te1], [Ga]). The integral closure of an ideal $J \subset \mathcal{A}_n$, denoted by $\overline{J}$, is the set of all $f \in \mathcal{A}_n$ such that for any analytic arc $\mu : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ one has $\mu^* f \in (\mu^* J)\mathcal{A}_1$, equivalently:

$$\mathrm{ord}_s f(\mu(s)) \ge \min\{\mathrm{ord}_s h(\mu(s)) \mid h \in J\}.$$

Remark 5.2. The above definition was introduced by Gaffney [Ga] in the real analytic setting. In the complex analytic setting, it was proved by Teissier [Te1] that the usual algebraic definition of the integral closure is equivalent to the above one.

Proposition 5.3. [Te4, 1.3.1, Proposition 1], [Ga, Proposition 4.2]
Let J be an ideal of $\mathcal{A}_n$. Then $f \in \overline{J}$ if and only if, for every set of generators $\{g_j\}$ of J, there exists a neighbourhood U of $0 \in \mathbb{K}^n$, and a constant $c > 0$, such that $|f(x)| \le c \cdot \sup_j |g_j(x)|$ for all $x \in U$. □

Corollary 5.4. *Let $f \in J$, let $\{g_j\}$ be a set of generators of J, let U be as in Proposition 5.3, and let $x_0 \in U$ such that $f(x_0) = 0$. Then, for any real analytic arc $\mu : (\mathbb{R}, 0) \to (\mathbb{R}^n, x_0)$, one has $\mathrm{ord}_s f(\mu(s)) \ge \min_j\{\mathrm{ord}_s g_j(\mu(s))\}$.*

In particular, the equality of zero sets $\mathcal{Z}(J) = \mathcal{Z}(\overline{J})$ holds. □

Coming back to our deformation $F(x,t)$, we shall work with the presentation (9) of §4.2, that we recall here:

$$F(x,t) = f_0(x) + \sum_{j \ge 1}^{\infty} t^j f_j(x). \tag{16}$$

Let us consider the following condition:

$$\left\{ \begin{array}{l} \text{For any } x_0 \in \operatorname{Sing} F_0 \cap \left\{ \dfrac{\partial F}{\partial t}(x,0) = 0 \right\}, \text{ there is } c_{x_0} > 0 \text{ such that} \\ \left| \dfrac{\partial F}{\partial t} \right| \le c_{x_0} \left\| \dfrac{\partial F}{\partial x_1}, \ldots, \dfrac{\partial F}{\partial x_n} \right\| \text{ when } (x,t) \to (x_0, 0). \end{array} \right. \tag{17}$$

Proposition 5.5. *If condition (17) holds, then one has the equality of set germs* $\operatorname{Sing}\widetilde{F} = \operatorname{Sing} F$ *and, in particular,* $\Delta = \{0\} \times \mathbb{K}$.

We have the following implications:

$$(17) \implies (3) \implies (6).$$

Proof. Let us show the equality $\operatorname{Sing} F = \operatorname{Sing}\widetilde{F}$. The inclusion of set germs at the origin $\operatorname{Sing} F \subset \operatorname{Sing}\widetilde{F}$ is a direct consequence of the definition of the singular loci. The converse inclusion follows from condition (17). Indeed, by *reductio ad absurdum*, let us suppose that this does not hold. Using the Curve Selection Lemma, one then obtains an analytic path $\eta(s) = \big(x(s), t(s)\big)$ for $s \in [0, \epsilon)$, where $t(0) = 0$ and $x(0) = 0$, such that $\eta(s) \in \operatorname{Sing}\widetilde{F} \setminus \operatorname{Sing} F$ for all $s \neq 0$. This means: $\frac{\partial F}{\partial x_i}(\eta(s)) = 0$ and $\frac{\partial F}{\partial t}(\eta(s)) \neq 0$ for all $s \neq 0$ and all $i = 1, \dots, n$. This contradicts the inequality (17) for the point $x_0 = 0$. Here ends the proof of the inclusion $\operatorname{Sing} F \supset \operatorname{Sing}\widetilde{F}$, and thus of the equality $\operatorname{Sing} F = \operatorname{Sing}\widetilde{F}$.

By restricting this equality to the slice $t = 0$, we get in particular the following inclusion of the set germs at 0:

$$\operatorname{Sing} F_0 \subset \left\{\frac{\partial F}{\partial t}(x, 0) = 0\right\}, \tag{18}$$

which shows the implication (17) $\implies$ (3).
The implication (3) $\implies$ (6) has been actually shown in the beginning of the proof of Theorem 4.3, see (10). Alternatively, let us remark that we may apply the preceding argument at any fixed point $x_0 \in \operatorname{Sing} F_0 \setminus \{0\}$ to conclude that there is a neighbourhood $U(x_0)$ within the inclusion (18) holds. This implies the inclusion (18) of set-germs at 0, which concludes our proof. □

Theorem 5.6. *Let* $F(x, t) = F_t(x)$ *be a* $\mathbb{K}$*-analytic deformation of* F_0 *such that the Jacobian ideal* (∂F_t) *is included in the integral closure* $\overline{(\partial F_0)}$ *for all* t *close enough to 0.*

Then the deformation F *of* F_0 *is a deformation with fibre constancy in the sense of Definition 1.1, and* $\widetilde{F}$ *has a Milnor-Hamm tube fibration (7), cf Definition 3.1.*

Proof. Let $I = \overline{(\partial F_0)}$ denote the integral closure of the Jacobian ideal (∂F_0). By our hypothesis we have the inclusion $(\partial F_t) \subset I$, for any t close enough to 0.

Lemma 5.7. *The following inclusions hold:*

(a) $(\partial f_j) \subset I$, *for any* $j \in \mathbb{N}$.
(b) $\big(\partial_x(\frac{\partial F}{\partial t})\big) \subset I$.

Proof. We will treat separately the two cases, $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$.

Case $\mathbb{K} = \mathbb{C}$. By the invariance of the Jacobian ideal inclusion $(\partial F_t) \subset I$, we have that $\sum_{j\geq 1}^{\infty} t^j \partial_i f_j(x) \in I$ for every $i \in \{1, \dots, n\}$, and for all t close enough to 0. Dividing out by $t \neq 0$, we thus get:

$$\partial_i f_1 + \sum_{j\geq 2}^{\infty} t^{j-1} \partial_i f_j(x) \in I. \tag{19}$$

We invoke the following classical theorem by Henri Cartan. By the reason explained in Remark 5.9 this holds only for $\mathbb{K} = \mathbb{C}$, and this is why we need to prove separately the real case of Theorem 5.6.

Theorem 5.8. [Ca, p.194]
Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$ and let I be an ideal of $\mathcal{O}_n$. Suppose that V is a neighborhood of $0 \in \mathbb{C}^n$ and $g \in \mathcal{O}(V)$. If there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of holomorphic functions $g_k \in \mathcal{O}(V)$ such that $g_k \in I$ for all k and $(g_k)_{k\in\mathbb{N}}$ converges uniformly on compact subsets of V to g, then $g \in I$.

We choose a polydisk $\{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid |x_i| < \alpha_i, |t| < \delta, i = 1,\dots,n\}$ on which F is holomorphic, i.e. for some well-chosen constants $\alpha_i > 0$. Hence each f_j is holomorphic on $V := \{x \in \mathbb{C}^n \mid |x_i| < \alpha_i, i = 1,\dots,n\}$.

We choose some sequence $t_k \to 0$ and define $g_k(x) := \sum_{j\geq 2}^{\infty} t_k^{j-1}\partial_i f_j(x)$ which, by its definition, converges to the function germ 0 as $k \to \infty$, uniformly on compact sets. Then Theorem 5.8 applies, and from (19) we may deduce that $\partial_i f_1 \in I$.

We may then apply inductively the same reasoning and obtain that $\partial_i f_j \in I$ for every $j \geq 1$. This concludes the proof of Lemma 5.7(a) over $\mathbb{C}$.

(b). From (a) it follows that the infinite series in variable t:

$$\partial_i\left(\frac{\partial F}{\partial t}\right) = \sum_{j=1}^{\infty} j t^{j-1}\partial_i f_j(x)$$

has all coefficients in I. To show that this series converges to an element of I, we apply once again Cartan's Theorem 5.8 to the finite partial sums $h_p(x,t) := \sum_{j=1}^{p} j t^{j-1}\partial_i f_j(x)$, this time as function germs in variables x and t.

This ends the proof of Lemma 5.7 over $\mathbb{C}$.

Case $\mathbb{K} = \mathbb{R}$. We start from the presentation (16), with real analytic function germs f_j, and where the Jacobian ideals (∂F_0) and (∂F_t) are with real analytic coefficients. By $I = \overline{(\partial F_0)}$ we will denote here the real integral closure.

We choose $\delta > 0$, $\alpha_i > 0$, $i \in \{1,\dots,n\}$, such that the Taylor series of F at $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ converges on $\{|x_i| < \alpha_i \mid i = 1,\dots,n\} \times \{|t| < \delta\}$. In particular the Taylor series of each f_j converges on $\{|x_i| < \alpha_i \mid i = 1,\dots,n\} \subset \mathbb{R}^n$.

Let F^c be the complexification of F (i.e. if $\sum_{J\subset\mathbb{N}^n, j\in\mathbb{N}} a_{j,J}t^j x^J$ is the Taylor series of F then $F^c(z,\lambda) = \sum_{J\subset\mathbb{N}^n, j\in\mathbb{N}} a_{j,J}\lambda^j z^J$ for $z_j, \lambda \in \mathbb{C}$) and let f_j^c be the complexification of f_j.

We consider now the polydisk $P \subset \mathbb{C}^n \times \mathbb{C}$, $P = \{(z,\lambda) \mid |z_i| < \alpha_i, |\lambda| < \delta\}$. Then F^c is holomorphic on P. Let $I_{\mathbb{C}}$ be the ideal in $\mathcal{O}_n$ generated by $\partial_z F_0^c$.

If $t \in \mathbb{R}$, $|t| < \delta$, is such that $\langle \partial_x F_t \rangle \subset I$, then it follows that $\langle \partial_z F_t^c \rangle \subset I_{\mathbb{C}}$. We notice now that in the proof of the $\mathbb{C}$-analytic case, we did not actually need the inclusion of the Jacobian ideals for all complex values of the parameter, namely some sequence convergent to 0 suffices. We obtain that $\partial f_j^c \subset I_{\mathbb{C}}$. This means that $\partial_i f_j^c$ is a linear combination with complex analytic coefficients of the generators of $I_{\mathbb{C}}$. But since the restriction to $\mathbb{R}^n$ of $\frac{\partial f_j^c}{\partial z_i}$ is $\frac{\partial f_j}{\partial x_i}$, hence a real function, it follows that, by taking the real part of each coefficient, we obtain a real linear combination equal to $\partial_i f_j$. This shows that we actually have $\partial_i f_j \in I$, and the proof of point (a) is done.

To show (b) we proceed in the same manner: we interpret the real analytic functions as complex analytic and we apply Cartan's theorem, as explained above, to get $\partial_i\big(\frac{\partial F}{\partial t}\big) \in I_{\mathbb{C}}$. Now, by the same arguments for the linear combination, we get that actually $\partial_i\big(\frac{\partial F}{\partial t}\big) \in I$. □

Remark 5.9. Theorem 5.8 is not true over $\mathbb{R}$. A counter-example can be found for instance in [ABF, Ch. 4 §6]. Namely, one constructs a sequence of real functions $\{f_k\}$ which are analytic on the real line $\mathbb{R}$, that converge uniformly on compacts to some analytic function f, and such that the germ at the origin of each f_k is in some ideal I, but that f is not in I. The problem is that the radius of convergency of the Taylor series of f_k at the origin goes to zero as $k \to \infty$. In contrast, note that the setting of our above proof insures a stronger convergence, as defined e.g. in [ABF, Definition 6.2].

We continue the proof of Theorem 5.6.

Let us show that condition (17) is satisfied. We apply Proposition 5.3 to the ideal (∂F_0) and its generators $\partial_1 F_0, \ldots, \partial_n F_0$, and for the choice of a neighbourhood U of the origin such that the conclusion of Proposition 5.3 holds.

So let us fix some point $(x_0, 0)$ with $x_0 \in \operatorname{Sing} f_0 \cap \left\{\frac{\partial F}{\partial t} = 0\right\}_{|t=0}$.

Let us consider an analytic path $\eta(s) = \big(x(s), t(s)\big)$ for $s \in [0, \epsilon)$, where $t(0) = 0$ and $x(0) = x_0$, and let us compute the limit, when $s \to 0$, of the fraction:

$$\frac{\frac{\partial F}{\partial t}(x(s), t(s))}{\|\partial_x F(x(s), t(s))\|}. \tag{20}$$

By our hypothesis on the analytic path $\eta(s)$, we get $\lim_{s\to 0} \partial_j F_{t(s)}(x(s)) = \partial_j F_0(x(0)) = \partial_j f_0(x(0)) = 0$ for all $j = 1, \ldots, n$. It follows in particular that the limit of the fraction (20), when $s \to 0$, is of type "$\frac{0}{0}$".

Let then set $\kappa := \min_{i=1}^{n}\big\{\mathrm{ord}_s \partial_i f_0(x(s))\big\}$, and note that $\kappa > 0$ since $x(0) \in \operatorname{Sing} f_0$.

Lemma 5.10. *Let $\ell \in \{1, \ldots, n\}$ such that $\mathrm{ord}_s \partial_\ell f_0(x(s)) = \kappa$. Then*

$$\mathrm{ord}_s \frac{\partial F}{\partial x_\ell}\big(x(s), t(s)\big) = \kappa.$$

Proof. By Lemma 5.7, for every $j \in \mathbb{N}$ we have $\partial_\ell f_j \in (\partial_x f_0) = I$. This implies that $\mathrm{ord}_s \partial_\ell f_j(x(s)) \geq \kappa$. Since $t(0) = 0$ we also have $\mathrm{ord}_s t^j(s) \geq 1$ for $j \geq 1$. We deduce that $\mathrm{ord}_s \sum_{j=1}^{N} t^j(s) \partial_\ell f_j(x(s)) > \kappa$ and, since $\mathrm{ord}_s \partial_\ell f_0(x(s)) = \kappa$, our claim follows. □

Our next claim is:

$$\mathrm{ord}_s \frac{\partial F}{\partial t}(x(s), t(s)) > \kappa. \tag{21}$$

To show this, we need the following basic result:

Lemma 5.11. *Let $x : [0, \varepsilon) \to \mathbb{K}^n$ be an analytic path. Let $h : (\mathbb{K}^n, x(0)) \to (\mathbb{K}, 0)$ be an analytic function germ such that $\partial_i h(x(0)) = 0$ for all $i = 1, \ldots, n$. Then:*

$$\mathrm{ord}_s h(x(s)) > \min_i \mathrm{ord}_s \partial_i h(x(s)).$$

Proof. As $h(0) = 0$, we have:

$$\mathrm{ord}_s h(x(s)) > \mathrm{ord}_s \frac{d}{ds}\Big(h(x(s))\Big) = \sum_i \partial_i h(x(s)) x'(s) \geq \min_i \mathrm{ord}_s \partial_i h(x(s)).$$

□

By Lemma 5.7(a) and Corollary 5.4 we have, for any $j \geq 0$, the inclusions of zero sets:

$$\mathcal{Z}(f_j) \supset \mathcal{Z}(\partial f_j) \supset \mathcal{Z}(I) = \mathcal{Z}(\partial f_0).$$

Therefore $h = f_j$ satisfies the hypotheses of Lemma 5.11, for any $j \geq 0$. Applying thus Lemma 5.11, we deduce:

$$\mathrm{ord}_s f_j(x(s)) > \min_i \mathrm{ord}_s \partial_i f_j(x(s)). \tag{22}$$

Next, by Lemma 5.7 again, we have $\partial_i f_j \in \overline{(\partial f_0)}$ for any $i \in \{1, \ldots, n\}$, hence we get $\mathrm{ord}_s \partial_i f_j(x(s)) \geq \kappa$ by Corollary 5.4 and our choice of U satisfying Proposition 5.3. Combining this with the inequality (22) we then get:

$$\mathrm{ord}_s f_j(x(s)) > \kappa$$

for any $j \geq 0$, which shows that our claimed inequality (21) holds indeed.

Finally, by (21) we get that the order of the numerator in (20) is $> \kappa$. By Lemma 5.10 we get that the order of the denominator in (20) is $= \kappa$. This implies that the limit of (20) is 0, and therefore condition (17) is satisfied.

By Proposition 5.5 it then follows that condition (3) is satisfied, and thus by Theorem 4.3 and Theorem 3.7, it follows that F is tame, that $\widetilde{F}$ has a Milnor-Hamm tube fibration (7), and that F is a deformation with fibre constancy (in the sense of Definition 1.1).

This finishes the proof of Theorem 5.6. □

6. Examples

By the following four examples we show here the variety of situations that may occur: the Jacobian criterion of Theorem 5.6 is satisfied in 6.2 but not in 6.1; the condition (3) is satisfied in 6.1 but not in 6.3; and finally, the tameness condition holds in 6.3 but not in 6.4. We remind here too that in our notations, the maps and the sets defined by them are all regarded as germs at the origin.

Example 6.1. Let $F : (\mathbb{K}^2 \times \mathbb{K}, 0) \to (\mathbb{K}, 0)$, $F(x, y, t) = y^2(x^2 - (y - t)^2)$. This is a *deformation of a line singularity*, in the terminology used by Siersma in [Si1] in the complex setting only. For any parameter t close enough to 0, the function germ F_t has indeed as singular locus $\mathrm{Sing}\, F_t := \{y = 0\} \subset \mathbb{K}^2$, hence the same line.

In our setting, we have the following set germs in $(\mathbb{K}^3, 0)$: $\mathrm{Sing}\, F = \{x = 0, y = t\} \cup \{y = 0\}$, and $\mathrm{Sing}\, \widetilde{F} = \{x = 0, y = t\} \cup \{x = 0, 2y = t\} \cup \{y = 0\}$.

Let us notice the strict inclusions: $\bigcup_t \operatorname{Sing} F_t = \{y = 0\} \subsetneq \operatorname{Sing} F \subsetneq \operatorname{Sing} \widetilde{F}$.

The equality $(\operatorname{Sing} F)_{|t=0} = (\operatorname{Sing} \widetilde{F})_{|t=0}$ however holds, thus the preliminary condition (6) is satisfied. The discriminant $\Delta := \widetilde{F}(\operatorname{Sing} \widetilde{F})$ is the union of the germ at 0 of the axis $\{0\} \times \mathbb{K}$ with the germ of the curve in $\mathbb{K}^2$ parametrized as $\{(-t^4/16, t)\}$.

By simple computations we see that condition (3) is satisfied, thus Theorem 1.3 holds, whereas condition (17) is not satisfied precisely at the origin, and therefore the hypothesis of the Jacobian criterion Theorem 5.6 does not hold either. In the complex setting, such a deformation of a line singularity with singular locus L was called "admissible" in [Si1], [Si2] and in more other papers, and has the property that the line $L = \operatorname{Sing} F_t$ preserves its generic transversal type (which is A_1 in the above example), while from the origin may spring, when $t \neq 0$, finitely many special points along L (i.e. with a different transversal type), as well as finitely many singular points outside L.

In the complex setting, it turns out by a proof similar to Lê-Saito's in [LS] applied at some generic point $p \in L \setminus \{0\}$, that the t-constancy of the generic transversal Milnor number implies condition (3), thus the "admissible" deformations are in fact tame, by our Theorem 4.3.

Example 6.2. Let us consider the deformation $F(x,t) = z_1^5 + z_2^5 + z_1^6 z_2^6 z_3^2 + t z_1^3 z_2^3$ of the polynomial $F_0 = z_1^5 + z_2^5 + z_1^6 z_2^6 z_3^2$ with $\operatorname{Sing} F_0 = \{z_1 = z_2 = 0\}$, either over $\mathbb{C}$ or over $\mathbb{R}$. Note that the inclusion $(\partial F_t) \subset (\partial F_0)$ doesn't hold since $z_1^3 z_2^2 \in \overline{(\partial F_0)} \setminus (\partial F_0)$ and $z_1^2 z_3^2 \in \overline{(\partial F_0)} \setminus (\partial F_0)$.

One easily checks that the inclusion $(\partial F_t) \subset \overline{(\partial F_0)}$ holds, hence our Theorem 5.6 applies.

Example 6.3. Starting from the Whitney umbrella equation $x^2 + y^2 z = 0$, let us consider the real analytic function $F_0(x, y, z) = x^3 + xy^2 z$ and its deformation $F(x, y, z; t) = (x^2 + y^2 z)(x - t)$.

The singular locus $\operatorname{Sing} F_0$, consisting of the two axes $\{x = y = 0\} \cup \{x = z = 0\}$, the first of multiplicity 4 and the second of multiplicity 1. For $t \neq 0$, the germ at $0 \in \mathbb{R}^3$ of the singular locus $\operatorname{Sing} F_t$ is $\{x = y = 0\}$ with multiplicity 1. There are two more curves in $\mathbb{R}^3 \times \{t\}$ which emerge from $\operatorname{Sing} F_0$:

$$\left\{x = \frac{2}{3}t,\ y = 0\right\} \cup \left\{x = t,\ x^2 + y^2 z = 0\right\}$$

but which are no more germs at $0 \in \mathbb{R}^3$. Their union over t, together with $\cup_t \operatorname{Sing} F_t$, constitute the singular set germ $\operatorname{Sing} \widetilde{F}$.

By straightforward computations, one establishes that F satisfies condition (6), that F does not satisfy condition (3), but that F is tame, cf Definition 3.5. Indeed, the Milnor set:

$$M(\widetilde{F}) = \{y^2 = 2z^2, 3x^2 - 2tx + 4y^5 = 0\} \cup \{y = z = 0\}$$

intersects $\{t = 0\} \cap \operatorname{Sing} F_0$ at the origin of $\mathbb{R}^3 \times \mathbb{R}$ only.

Our Theorem 3.7 then tells that this is a deformation with fibre constancy, and moreover that $\widetilde{F}$ has a Milnor-Hamm tube fibration.

Let us point out that in [Hof, Example 9.2] the author considers the same F_0 in the complex analytic setting (moreover within a 2-parameter deformation) and shows that his algebraic criterion [Hof, Theorem 1.2] applies to conclude that this is a deformation is "admissible" in the terminology of [ST2]. Through this deformation, Hof can compute the homology of the Milnor fibre of F_0 by using the special method developed [ST2] in the complex setting.

Example 6.4. Let us take a look at the deformation $F = y^2 + x^2(tz - x)$ discussed in [Hof, Example 9.3].

We have $\partial F = (-3x^2 + 2txz, 2y, tx^2;\ x^2 z)$, thus $\operatorname{Sing} F = \operatorname{Sing} \widetilde{F}$, and in particular our condition (6) is satisfied. Note that the Jacobian ideal ∂F_t is not constant for t in some neighbourhood of 0, despite the fact that the set germ $\operatorname{Sing} F_t$ is constant.

The computation of the Milnor set over $\mathbb{R}$ is easier here; we get that $M(\widetilde{F})$ contains a surface germ in the 3-space $\{y = 0\} \subset \mathbb{R}^3 \times \mathbb{R}$ of coordinates x, z, t, and the intersection of $M(\widetilde{F})$ with the slice $t = 0$ contains $\operatorname{Sing} F_0 = \{(0, 0, z, 0)\}$. This tells that $\widetilde{F}$ is not ρ-regular, which implies that the ∂-Thom regularity fails[4] along the z-axis. As we have remarked at the end of the proof of Theorem 3.7, the ρ-regularity is equivalent to the fact that the tube map (7) is a proper stratified submersion.

Since this example over $\mathbb{R}$ is not ρ-regular (i.e. not tame), it cannot satisfy any other criteria implying deformation with fibre constancy. In the complex setting, Hof remarks in [Hof] that this example does not satisfy his criteria, then he computes explicitly the fibres of the fibration (2) and finds that the homotopy type varies (actually one has S^2 for $t \neq 0$ and $S^1 \vee S^1$ for $t = 0$), confirming that the deformation does not have fibre constancy.

References

[ABF] F. Acquistapace, F. Broglia, J.F. Fernando, *Topics in global real analytic geometry*, Springer Monogr. Math. Springer, Cham, 2022, xvii+273 pp.

[AT1] R.N. Araújo dos Santos, M. Tibăr, *Real map germs and higher open books*, arXiv:0801.3328.

[AT2] R.N. Araujo dos Santos, M. Tibăr, *Real map germs and higher open book structures*, Geom. Dedicata 147 (2010), 177-185.

[ACT1] R.N. Araújo dos Santos, Y. Chen, M. Tibăr, *Singular open book structures from real mappings*, Cent. Eur. J. Math. 11 (2013) no. 5, 817-828.

[ACT2] R.N. Araújo dos Santos, Y. Chen, M. Tibăr, *Real polynomial maps and singular open books at infinity*, Math. Scand. 118 (2016), no. 1, 57-69.

[ART] R.N. Araújo dos Santos, M. Ribeiro, M. Tibăr, *Fibrations of highly singular map germs*, Bull. Sci. Math. 155 (2019), 92-111.

[BMM] J. Briançon, Ph. Maisonobe, M. Merle, *Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom*, Invent. Math., 117, 3 (1994), 531-550.

[Ca] H. Cartan, *Ideaux de fonctions analytiques de n variables complexes*, Ann. Sci. Éc. Norm. Supér. (3), 61 (1944), 149-197.

[CJT] Y. Chen, C. Joiţa, M. Tibăr, *Fibrations of tamely composable maps*, J. Geometry and Physics 194, (2023), Paper 105025.

[dJ] T. De Jong, *Some classes of line singularities.* Math. Z. 198 (1988), 493-517.

[4] There are examples of maps which fail to be Thom regular but they are however ρ-regular, cf [ACT1]. It remains here the question if this can happen in case of deformations.

[DRT] L.R.G. Dias, M.A.S. Ruas, M. Tibăr, *Regularity at infinity of real mappings and a Morse-Sard theorem*, J. Topol. 5 (2012), no. 2, 323-340.

[Fe] J. Fernández de Bobadilla, *Relative morsification theory*, Topology 43 (2004), no. 4, 925-982.

[FM] J. Fernández de Bobadilla, M. Marco-Buzunáriz, *Topology of hypersurface singularities with 3-dimensional critical set*, Comment. Math. Helv. 88 (2013), no. 2, 253-304.

[Ga] T. Gaffney, *Integral closure of modules and Whitney equisingularity,* Invent. Math. 107 (1992), no. 2, 301-322.

[GLPW] C. G. Gibson, E. Looijenga, A. du Plessis, K. Wirthmüller, *Topological stability of smooth mappings*, Lecture Notes in Mathematics, Vol. 552. Springer-Verlag, Berlin-New York, 1976.

[Hi] H. Hironaka, *Stratification and flatness.* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 199-265. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[Hof] A. Hof, *Milnor Fiber Consistency via Flatness*, arXiv:2212.12807, to appear in J. Topology.

[Hou] K. Houston, *Equisingularity of families of hypersurfaces and applications to mappings.* Michigan Math. J. 60 (2011), no.2, 289-312.

[JiT] G. Jiang, M. Tibăr, *Splitting of singularities*, J. Math. Soc. Japan 54 (2002), no. 2, 255-271.

[JoT1] C. Joiţa, M. Tibăr, *Images of analytic map germs and singular fibrations*, Eur. J. Math. 6 (2020), no 3, 888-904.

[JoT2] C. Joiţa, M. Tibăr, *The local image problem for complex analytic maps*, Ark. Mat. 59 (2021), 345-358.

[LR] Lê D.T., C.P. Ramanujam, *The invariance of Milnor's number implies the invariance of the topological type.* Amer. J. Math. 98(1976), no.1, 67-78.

[LS] Lê D.T., K. Saito, *La constance du nombre de Milnor donne des bonnes stratifications*, C. R. Acad. Sci. Paris Sér. A 272 (1973), 793-795.

[Lo] S. Łojasiewicz, *Triangulation of semi-analytic sets.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 18 (1964), 449-474.

[Ki] H.C. King, *Topological type in families of germs.* Invent. Math. 62 (1980/1981), no.1, 1-13.

[Ma] C. Manolescu, *private communication.*

[MS] D.B. Massey, D. Siersma, *Deformation of polar methods.* Ann. Inst. Fourier (Grenoble) 42, (1992), 737-778.

[Mat] J.N. Mather, *Notes on Topological Stability.* Bull. Amer. Math. Soc. (N.S.) 49 (2012), no. 4, 475-506.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

[Ne] A. Némethi, *Hypersurface singularities with 2-dimensional critical locus.* J. London Math. Soc. 59 (1999), no. 2, 922-938.

[PT] A.J. Parameswaran, M. Tibăr, *Thom irregularity and Milnor tube fibrations*, Bull. Sci. Math. 143 (2018), 58-72. *Corrigendum* Bull. Sci. Math. 153 (2019), 120-123.

[Pa1] A. Parusinski, *A note on singularities at infinity of complex polynomials.* Symplectic singularities and geometry of gauge fields (Warsaw, 1995), 131-141. Banach Center Publ., 39 Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1997.

[Pa2] A. Parusinski, *Topological Triviality of μ-Constant Deformations of Type $f(x) + tg(x)$*, Bull. London Math. Soc. 31 (1999), no. 6, 686-692.

[Pe] R. Pellikaan, *Deformations of hypersurfaces with a one-dimensional singular locus.* J. Pure Appl. Algebra 67 (1990), 49-71.

[Sa1] C. Sabbah, *Morphismes analytiques stratifiés sans éclatement et cycles évanescents.* C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 1, 39-41.

[Sa2] C. Sabbah, *Morphismes analytiques stratifiés sans éclatement et cycles évanescents.* Analysis and topology on singular spaces, II, III (Luminy, 1981), 286-319, Astérisque, 101-102, Soc. Math. France, Paris, 1983.

[Si1] D. Siersma, *Isolated line singularities.* Singularities, Part 2 (Arcata, Calif., 1981), 485-496, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

[Si2] D. Siersma, *Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1.*, Topology Appl. 27 (1987) 51-73.

[ST1] D. Siersma, M. Tibăr, *Singularities at infinity and their vanishing cycles*, Duke Math. J. 80 (1995), no. 3, 771-783.

[ST2] D. Siersma, M. Tibăr, *Milnor fibre homology via deformation*, in: Singularities and Computer Algebra, Festschrift for Gert-Martin Greuel on the Occasion of his 70th Birthday, pp. 306-322. Springer 2017.

[Te1] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney.* Singularités à Cargèse 1972, pp. 285-362 Astérisque, Nos. 7 et 8 Société Mathématique de France, Paris, 1973.

[Te2] B. Teissier, *Introduction to equisingularity problems.* Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), pp. 593-632 Proc. Sympos. Pure Math., Vol. 29 American Mathematical Society, Providence, RI, 1975.

[Te3] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces.* Invent. Math. 40 (1977), no.3, 267-292.

[Te4] B. Teissier, *Variétés polaires. II: Multiplicités polaires, sections planes et conditions de Whitney.* Algebraic geometry, Proc. int. Conf., La Rábida/Spain 1981, Lect. Notes Math. 961, 314-491 (1982).

[Ti1] M. Tibăr, *Topology at infinity of polynomial mappings and Thom condition*, Compositio Math. 111 (1998), 89-109.

[Ti2] M. Tibăr, Polynomials and Vanishing Cycles. Cambridge Tracts in Mathematics, no. 170. Cambridge University Press, 2007. xii+253pp.

[Tim] J.G. Timourian, *The invariance of Milnor's number implies topological triviality.* Amer. J. Math. 99 (1977), no.2, 437-446.

[Za] A. Zaharia, *Topological properties of certain singularities with critical locus a 2-dimensional complete intersection.* Topology Appl. 60 (1994), 153-171.

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucureşti, Romania and Laboratoire Européen Associé CNRS Franco-Roumain Math-Mode
Email address: `Cezar.Joita@imar.ro`

École Normale Supérieure - PSL, 45 rue d'Ulm, 75230, Paris Cedex 05, France
Email address: `mstockinger@clipper.ens.psl.eu`

Univ. Lille, CNRS, UMR 8524 – Laboratoire Paul Painlevé, F-59000 Lille, France
Email address: `mihai-marius.tibar@univ-lille.fr`