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THE TIME-OPTIMAL FEEDBACK CONTROL FOR SINGULARLY PERTURBED LINEAR SYSTEMS

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Abstract

Using some recent results concerning the time-optimal feedback control ([3]-[5],[10]-[11]), it is proved that for some classes of singularly perturbed linear control systems, the time-optimal feedback control of the "reduced" system obtained by neglecting the small time constants has stabilizing properties.

1. The problem

We consider the singularly perturbed linear control system described by the equations:

$$(1.1) \qquad \frac{dx}{dt} = A_{11}x + A_{12}y + B_1u \quad , \; x \in R^n, \; y \in R^m, \; u \in U \subset R^p$$

$$(1.2) \qquad \varepsilon \frac{dy}{dt} = A_{21}x + A_{22}y + B_2u \quad , \; \varepsilon > 0$$

where the matrices A_{11}, A_{12}, B_1, A_{21}, A_{22}, B_2 and the control space, U, satisfy the following conditions:

(1.3) $U \subset R^p$ is a compact, convex polyhedron containing the origin, $0 \in R^p$, in its interior;

(1.4) $A_{22} \in L(R^m, R^m)$ is a stable matrix (i.e. it has only eigenvalues with negative real part) so there exist α, $k > 0$ such that:

(1.5) $\| \exp(A_{22}t) \| \leqslant k(\exp(-\alpha t))$ for $t \geqslant 0$

If we denote:

(1.6) $A = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B = B_1 - A_{12}A_{22}^{-1}B_2$

(A_{22} is invertible since it is stable) then we assume that:

(1.7) <u>The pair (A,B) is completely controllable</u> (i.e. $\operatorname{rank}(B, AB, \ldots A^{n-1}B) = n$)

<u>The control problem</u> we are considering is the so called time-optimal control problem:

<u>Problem 1: find a feedback control</u>, $v : R^n x R^m \longrightarrow U$, <u>such that any solution of the system:</u>

$$(1.8) \quad \begin{cases} \dfrac{dx}{dt} = A_{11}x + A_{12}y + B_1 v(x,y) \\ \varepsilon \dfrac{dy}{dt} = A_{21}x + A_{22}y + B_2 v(x,y) \end{cases}$$

<u>reaches the origin $(0,0) \in R^n x R^m$</u> (or a given neighborhood of (0,0)) <u>in minimal time</u>.

The "small parameter", $\varepsilon > 0$, in the left-hand side of the equation (1.2) makes the second component, y, of the state (x,y) of the system to be very fast and therefore dificult to measure or to control it (either technologically or even computationally).

In the engeneering practice it is often used only the "reduced" system defined by (1.1) and the (algebraic !) equation

(1.9) $0 = A_{21}x + A_{22}y + B_2 u$

obtained from (1.2) for $\varepsilon = 0$.

Using the hypothesis (1.4) and the notation (1.6) the system (1.1), (1.9) becomes:

(1.10) $\dfrac{dx}{dt} = Ax + Bu$

The problem 1 is considered now for the system (1.10) but the obtained time-optimal feedback control is used for the "real" system, (1.1)-(1.2). Intuitively, the time-optimal feedback control of the reduced system, (1.10), should stabilize also the real system, (1.1)-(1.2), and, roughly speaking, the problem we are trying to solve in what follows is to find conditions under which this happens.

In order to state more precisely this problem we recall that from the now classical textbooks in Control Theory, [1], [9], etc. , it is well known that if the hypotheses (1.3) and (1.7) are satisfied then for the control system (1.10), the controllability set to the origin, $G \in R^n$, is an open convex neighborhood of $0 \in R^n$ and there exists a time-optimal feedback control, $v(.):G \longrightarrow U$, with values in the set of the vertices of the polyhedron U. This means that for any point $x \in G$, the time-optimal trajectory of (1.10) through x, $\tilde{\varphi}(.;x)$ is a classical (Carathéodory) solution of the right-hand side discontinuous differential system:

$$(1.11) \qquad \frac{dx}{dt} = Ax + Bv(x)$$

and reaches the origin in the minimal time $T(x) \geqslant 0$. Moreover, the minimal-time function, $T(.):G \longrightarrow R_+$, is continuous ([1] ,[8],[9])

The problem we are trying to solve in the next sections may be stated as follows:

<u>Problem 2: if $v(.):G \longrightarrow U$ is the time-optimal feedback control of the reduced system (1.10), then find conditions under which any solution of the differential system:</u>

$$(1.12) \quad \begin{cases} \dfrac{dx}{dt} = A_{11}x + A_{12}y + B_1 v(x) \\ \varepsilon \dfrac{dy}{dt} = A_{21}x + A_{22}y + B_2 v(x) \end{cases}$$

<u>or, possibly, only its first component, reaches a given neighborhood of the origin</u> (and in a time close to the minimal one associated to the first component of the initial state by the system (1.11)).

Since the time-optimal feedback control of (1.10) , $v(.):G \longrightarrow U$, is discontinuous, we can consider only Filippov solutions ([7]) for the system (1.12). On the other hand, in order to solve the problem 2 we have to estimate the distance between the first component of a solution of (1.12) and the corresponding optimal trajectory of (1.10) which is a classical solution of (1.11). Therefore we have to assume that the optimal trajectories of (1.10) coincide with the Filippov solutions of (1.11) (hence the Filippov solutions of (1.11) are unique).

In [3] - [5] and [10] - [11] , some classes of linear, time-optimal control systems of the form (1.10) are studied, for which the fundamental hypothesis:

(1.13) <u>the optimal trajectories of (1.10) coincide with the corresponding Filippov solutions of (1.11),</u>

is satisfied.

In what follows (section 3) we shall prove that if the reduced system (1.10) satisfies the fundamental hypothesis (1.13) then the first component of any solution of (1.12) reaches a given neighborhood of the origin in a time no longer than the minimal one associated via (1.11) to the initial state if the parameter $\varepsilon > 0$ is small enough and if we restrict ourselves to a certain compact neighborhood of the origin. Moreover, we show that if the control variable does not influence the equation (1.2) (i.e. if $B_2 = 0 \in L(R^p, R^m)$) then also the second component of any solution of (1.12) reaches a given

neighborhood of the origin.

In order to prove the result we need a generalization of the theorem 1 in [4] concerning the continuous dependence of Filippov solutions on the right-hand side of the equation; we prove this preliminary result in section 2.

2. Stability of Filippov solutions with respect to perturbations of the right-hand side of the equation.

Using theorem 11 in [7] we prove first the following result:

Lemma 2.1

Let us consider the differential system (1.11) such that for any $x \in G \subset R^n$ there exists a unique Filippov solution $\tilde{\varphi}(.;x) : [0,\infty) \longrightarrow G$, of (1.11) through x.

Then, for any compact subset $G_0 \subset G$ and any reals T_0, $\eta_0 > 0$ there exists $\delta > 0$ (depending on G_0, T_0, η_0) such that for any measurable map, $\xi(.) : [0,T_0] \longrightarrow R^n$, that satisfies:

$$(2.1) \qquad \int_0^{T_0} \| \xi(t) \| \, dt < \delta ,$$

for any $x \in G_0$ and any Filippov solution through x, $\varphi_\xi(.;x)$ of the differential system:

$$(2.2) \qquad \frac{dx}{dt} = Ax + Bv(x) + \xi(t)$$

the following estimation holds:

$$(2.3) \qquad \| \varphi_\xi(t;x) - \tilde{\varphi}(t;x) \| < \eta_0 \qquad \text{for } t \in [0,T_0]$$

Proof:

By contradiction: let us suppose that the statement in the lemma is false. Then, there exist a compact subset $G_0 \subset G$

and the reals $T_o, \eta_o > 0$ such that for any positive integer $k \in N$ there exist $t_k \in [0,T_o]$, $x_k \in G_o$, a measurable map, $\xi_k(.) : [0,T_o] \longrightarrow R^n$ and a Filippov solution, $\Psi_k(.;x_k) : [0,T_o] \to G$ through x_k of the differential system:

$$(2.4) \qquad \frac{dx}{dt} = Ax + Bv(x) + \xi_k(t)$$

such that the following estimations hold:

$$(2.5) \qquad \int_0^{T_o} \| \xi_k(t) \| \, dt \leq 1/k$$

$$(2.6) \qquad \| \Psi_k(t_k;x_k) - \tilde{\varphi}(t_k;x_k) \| > \eta_o$$

We note that the existence of the Filippov solutions of the systems (1.11), (2.2) and (2.4) on the whole interval $[0,T_o]$ follows from the theorems 4 and 5 in [7]. Since the map $f_\xi(.,.) : Gx[0,T_o] \longrightarrow R^n$ defined by: $f_\xi(x,t) = Ax + Bv(x) + \xi(t)$ is measurable (according to the theorem 10 in [8] the time-optimal feedback control $v(.):G \longrightarrow U$ is measurable) and since $\| f_\xi(x,t) \| \leq \| A \| \max \{ \| x \| \mid x \in G_o \} + \max \{ \| Bu \| \mid u \in U \} + \| \xi(t) \|$, it follows that the hypotheses of the theorems 4 and 5 in [7] are satisfied for the systems (1.11), (2.2) and (2.4).

Since $G_o \subset G$ and $[0,T_o] \subset R$ are compact subsets, we may suppose that the sequences $\{x_k\} \subset G_o$ and $\{t_k\} \subset [0,T_o]$ are convergent (we may, possibly, take some convergent subsequences) and therefore there exist $x_o \in G_o$ and $t_o \in [0,T_o]$ such that $x_k \longrightarrow x_o$ and $t_k \longrightarrow t_o$ as $k \longrightarrow \infty$.

Obviously, the hypotheses of theorem 11 in [7] are satisfied by the systems (1.11) and (2.4) and hence,

according to this theorem, $\Psi_k(\cdot;x_k) \longrightarrow \tilde{\varphi}(\cdot;x_o)$ uniformly on $[0,T_o]$ as $k \longrightarrow \infty$, and this contradicts (2.6) so the lemma is proved.

For any $t \geqslant 0$ we denote by $\mathcal{R}(t) \subset G$ the reachability set of (1.10) at a time not greater than t, that is, the set of points $x \in R^n$ which may be steered to the origin by measurable controls with values in U in a time not greater than t. It is well known ([1],[9]) that for any $t > 0$, $\mathcal{R}(t)$ is a compact, convex neighborhood of the origin $0 \in R^n$ and the map $t \longmapsto \mathcal{R}(t)$ is increasing with respect to the set inclusion and continuous in the Pompeiu-Hausdorff topology of the compact subsets of R^n. Obviously, $G = \bigcup\{\mathcal{R}(t) \mid t \geqslant 0\}$.

We prove now the following extension of the theorem 1 in [4]:

Theorem 2.2

Let us consider the control system (1.10) such that the hypotheses (1.3), (1.7) and (1.13) are satisfied.

Then, for any reals $\tau, \eta > 0$ there exists $\delta > 0$ (depending on τ and η) such that for any measurable map, $\xi(\cdot) : [0,\infty) \longrightarrow R^n$ that satisfies:

$$(2.7) \qquad \int_{k\tau}^{(k+1)\tau} \|\xi(t)\| \, dt < \delta \quad \text{for} \quad k = 0,1,2,\ldots$$

and for any $x \in \mathcal{R}(\tau)$, any Filippov solution through x, $\varphi_\xi(\cdot;x)$, of the system (2.2) satisfies:

$$(2.8) \qquad \|\varphi_\xi(t;x)\| < \eta \quad \text{for} \quad t \geqslant T(x)$$

where $T(\cdot) : G \longrightarrow R_+$ is the minimal-time function of (1.10).

Proof:

We take $v(0) = 0 \in U$ and therefore we may consider that

any time-optimal trajectory, $\tilde{\varphi}(.;x)$ of (1.10) (which, according to the hypothesis of the theorem is a Filippov solution of (1.11)) is defined on the whole real half-line $[0,\infty)$ by $\tilde{\varphi}(t;x) = 0 \in R^n$ for $t \geqslant T(x)$.

We note first that since $\mathcal{R}(\tau)$ is a neighborhood of the origin and the minimal-time function , $T(.) : G \longrightarrow R_+$, is continuous, it follows that for any τ , $\eta > 0$ there exists $\eta_0 > 0$, $\eta_0 < \eta/2$ such that:

(2.9) $\bar{B}_{\eta_0}(0) \subset \mathcal{R}(\tau)$ and $\|\tilde{\varphi}(t;x)\| \leqslant \eta/2$ for $t \in [0,\tau]$ and $x \in \bar{B}_{\eta_0}(0)$

where $\bar{B}_{\eta_0}(0)$ denotes the closed ball of radius η_0 centered at $0 \in R^n$.

Indeed, if we define $\tau_1 = \max\{t > 0 \mid \mathcal{R}(t) \subset \bar{B}_{\eta/2}(0)\}$ and $\tau_2 = \min(\tau, \tau_1)$ then, since $\tau_2 \leqslant \tau_1$ we have $\mathcal{R}(\tau_2) \subset \bar{B}_{\eta/2}(0)$; since $\mathcal{R}(\tau_2)$ is a neighborhood of the origin there exists $\eta_0 > 0$, $\eta_0 < \eta/2$, such that $\bar{B}_{\eta_0}(0) \subset \mathcal{R}(\tau_2) \subset \mathcal{R}(\tau)$ (since $\tau_2 \leqslant \tau$). For any $x \in \bar{B}_{\eta_0}(0)$ we have $T(x) \leqslant \tau_2$ and since $T(\tilde{\varphi}(t;x)) = T(x) - t \leqslant \tau_2 - t \leqslant \tau_2$ for any $t \in [0,T(x)]$ it follows that $\tilde{\varphi}(t;x) \in \mathcal{R}(\tau_2) \subset \bar{B}_{\eta/2}(0)$ for any $t \in [0,T(x)]$. As $\tilde{\varphi}(t;x) = 0$ for $t \geqslant T(x)$ it follows that (2.9) holds for any $x \in \bar{B}_{\eta_0}(0)$ and $t \in [0,\tau]$.

Let τ , $\eta > 0$ be arbitrary positive real numbers , $G_0 = \mathcal{R}(\tau)$ and $\eta_0 > 0$, $\eta_0 < \eta/2$ such that (2.9) holds.

According to lemma 2.1 , for $T_0 = \tau$ and $\eta_0 > 0$ there exists $\delta > 0$ such that for any measurable map, $\xi(.):[0,\infty) \longrightarrow R^r$ that satisfies (2.7) and for any $x \in \mathcal{R}(\tau)$, any Filippov solution of (2.2) through x , $\varphi_\xi(.;x)$ verifies 2.3.

Since for any $x \in \mathcal{R}(\tau)$, $T(x) \leq \tau$ and $\tilde{\varphi}(t;x) = 0$ for $t \geq T(x)$, it follows that (2.8) holds for $t \in [T(x), \tau]$.

Further on, $x_1 = \varphi_\xi(\tau;x) \in \overline{B_{\eta_0}(0)} \subset \mathcal{R}(\tau)$ according to (2.9) and therefore we may apply again lemma 2.1 for $G_0 = \mathcal{R}(\tau)$ $T_0 = \tau$ and $\eta_0 > 0$ that satisfies (2.9).

For any measurable map, $\xi(.) : [0,\infty) \longrightarrow R^n$ that satisfies (2.7) we define the measurable map $\xi_1(.) : [0,\infty) \longrightarrow R^n$ by $\xi_1(t) = \xi(t+\tau)$ for $t \in [0, \tau]$ and obviously $\xi_1(.)$ verifies (2.1) where $T_0 = \tau$.

According to lemma 2.1, any Filippov solution, $\varphi_{\xi_1}(.;x_1)$, through $x_1 = \varphi_\xi(\tau;x) \in \overline{B_{\eta_0}(0)}$ of the system:

$$(2.10) \qquad \frac{dx}{dt} = Ax + Bv(x) + \xi_1(t)$$

verifies:

$$(2.11) \qquad \| \varphi_{\xi_1}(t;x_1) - \tilde{\varphi}(t;x_1) \| \leq \eta_0 < \eta/2 \quad \text{for } t \in [0, \tau]$$

If $\varphi_\xi(.;x) : [0,\infty) \longrightarrow R^n$ is a Filippov solution of (2.2) through x then obviously the map $t \longmapsto \varphi_\xi(t+\tau;x)$ is a Filippov solution through $x_1 = \varphi_\xi(\tau;x)$ of (2.10) and therefore $\| \varphi_\xi(t+\tau;x) - \tilde{\varphi}(t;x) \| \leq \eta_0 < \eta/2$ for $t \in [0, \tau]$. Since $\eta_0 > 0$ is chosen such that (2.9) is satisfied, it follows that $\| \tilde{\varphi}(t;x) \| < \eta/2$ for $t \in [0,\tau]$ and therefore $\| \varphi_\xi(t+\tau;x) \| \leq \| \varphi_\xi(t+\tau;x) - \tilde{\varphi}(t;x_1) \| + \| \tilde{\varphi}(t;x_1) \| \leq$ $\leq \eta_0 + \eta/2 < \eta$ for $t \in [0, \tau]$, hence (2.8) is satisfied for $t \in [\tau, 2\tau]$.

Moreover, since $T(x_1) \leq \tau$, $(\overline{B_{\eta_0}(0)} \subset \mathcal{R}(\tau))$ it follows that $\tilde{\varphi}(t;x_1) = 0$ for $t \in [T(x_1), \tau]$ and hence $x_2 = \varphi_\xi(2\tau;x) \in$ $\in \overline{B_{\eta_0}(0)} \subset \mathcal{R}(\tau)$.

By induction on $k \in N$ it follows immediately that (2.8) is satisfied for $t \in [k\tau, (k+1)\tau]$ and the theorem is completely proved.

Remark 2.3

From the theorem 2.2 it follows immediately the theorem 1 in [4]: if $\|\xi(t)\| \leq \delta/\tau$ a.e. on $[0,\infty)$ then $\xi(.)$ verifies (2.7) and therefore (2.8) holds.

3. Singularly perturbed linear control systems. The main result.

The aim of this section is to justify the use of the time-optimal feedback control of the "reduced" system, (1.10), for controlling the singularly perturbed linear control system (1.1)-(1.2). Namely, we prove the following result:

Theorem 3.1

Let us consider the control system (1.1)-(1.2) and let us suppose that the hypotheses (1.3)-(1.7) and (1.13) are satisfied.

Then, for any compact subset $G_1 \times G_2 \subset G \times R^m$ and for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $(x,y) \in G_1 \times G_2$, any Filippov solution through (x,y) of (1.12), $(\varphi_\varepsilon(.;x,y), \psi_\varepsilon(.;x,y))$, satisfies:

$$\|\varphi_\varepsilon(t;x,y)\| \leq \eta \quad \text{for} \quad t \geq T(x) \tag{3.1}$$

where $T(.):G \longrightarrow R_+$ is the minimal-time function associated to (1.10).

Moreover, if $B_2 = 0 \in L(R^p, R^m)$ then for any compact subset $G_1 \times G_2 \subset G \times R^m$ and any $\tau_0, \eta > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $(x,y) \in G_1 \times G_2$, (3.1) and

$$\|\psi_\varepsilon(t;x,y)\| \leq \eta \quad \text{for} \quad t \geq T(x) + \tau_0 \tag{3.2}$$

hold.

Proof:

In order to simplify the notations, for any $\varepsilon > 0$ and

any $(x,y) \in G \times R^m$ we denote by:

(3.3) $$(\varphi(.), \psi(.)) = (\varphi_\varepsilon(.;x,y), \psi_\varepsilon(.;x,y))$$

a Filippov solution of (1.12) through (x,y).

We denote also:

(3.4) $$\xi(t) = \xi_\varepsilon(t;x,y) = A_{21}\varphi(t) + A_{22}\psi(t) + B_2 v(\varphi(t))$$

for $t \in [0,\infty)$.

After an easy computation it follows that $(\varphi(.), \psi(.))$ is a Filippov solution through (x,y) of the differential system:

(3.5) $$\frac{dx}{dt} = Ax + Bv(x) + A_{12}A_{22}^{-1}\xi(t)$$

(3.6) $$\varepsilon\frac{dy}{dt} = A_{21}x + A_{22}y + B_2 v(x)$$

where A and B are defined by (1.6).

In order to get the estimation (3.1) we should be able to apply theorem 2.2 to the system (3.5) and therefore we have to prove that the measurable map $t \longmapsto A_{12}A_{22}^{-1}\xi(t)$ verifies (2.7).

We prove first the following two preliminary results:

<u>Lemma 3.2</u>

<u>Assume the hypotheses in theorem 3.1 be satisfied. Then, for any</u> $T_0 > 0$ <u>and any compact subset</u> $G_1 \subset G$ <u>there exists</u> $\gamma > 0$ <u>such that for any</u> $r_2, \theta > 0$ <u>there exists</u> $\varepsilon_0 > 0$ <u>such that for any</u> $\varepsilon \in (0, \varepsilon_0)$ <u>and</u> $(x,y) \in G_1 \times \overline{B_{r_2}(0)}$ <u>we have:</u>

(3.7) $$\| \psi_\varepsilon(T_0;x,y) \| \leq \gamma + \theta$$

<u>Furthermore, for any compact subset</u> $G_1 \times G_2 \subset G \times R^m$ <u>and any</u> $T_0, \varepsilon_0 > 0$ <u>there exist</u> $M_1, M_2 > 0$ <u>such that: for any</u> $(x,y) \in G_1 \times G_2$, $\varepsilon \in (0, \varepsilon_0)$ <u>and</u> $t \in [0, T_0]$ <u>we have:</u>

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(3.8) $$\| \varphi_\varepsilon(t;x,y) \| \leq M_1$$

(3.9) $$\| \psi_\varepsilon(t;x,y) \| \leq M_2$$

Proof of lemma 3.2:

We note first that according to lemma 4 in [7], for any Filippov solution $(\varphi(.), \psi(.))$ of (1.12) there exist the measurable maps $w^i(.) : [0,\infty) \longrightarrow B_iU$, $i = 1,2$, such that $(\varphi(.), \psi(.))$ is a classical (Carathéodory) solution of the differential system:

(3.10) $$\frac{dx}{dt} = A_{11}x + A_{12}y + w^1(t)$$

(3.11) $$\varepsilon\frac{dy}{dt} = A_{21}x + A_{22}y + w^2(t)$$

and therefore, from the variation of the constants formula applied to (3.10) and (3.11) we get:

(3.12) $$\varphi(t)=\exp(A_{11}t).x + \int_0^t \exp(A_{11}(t-s)).w^1(s)ds + \int_0^t \exp(A_{11}(t-s)).A_{21}\psi(s)ds,$$

(3.13) $$\psi(t)=\exp(t/\varepsilon\, A_{22}).y + 1/\varepsilon \int_0^t \exp((t-s)/\varepsilon\, A_{22})w^2(s)ds + (1/\varepsilon)\int_0^t \exp((t-s)/\varepsilon\, A_{22})A_{21}\varphi(s)ds$$

We define now the following pozitive real numbers:

(3.14) $$a_{11}=\|A_{11}\|,\ a_{12}=\|A_{12}\|,\ a_{21}=\|A_{21}\|,\ L_1=\max\{\|B_1u\|, u\in U\}$$
$$L_2 = \max\{\|B_2u\|, u\in U\},\ r_1=\max\{\|x\|, x\in G_1\},\ r_2=\max\{\|y\|, y\in G_2\}$$

Let $\alpha, k > 0$ be defined by (1.5). Then, from (3.12) and (3.13), with the usual properties of the norm, we get respectively:

(3.15) $$\|\varphi(t)\| \leq \exp(a_{11}t)r_1 + (\exp(a_{11}t)-1)L_1/a_{11} + a_{12}\int_0^t \exp(a_{11}(t-s))\|\psi(s)\|ds$$

$$(3.16)\qquad \|\Psi(t)\| \leq \exp(-\alpha t/\varepsilon)k\|y\| + (1-\exp(-\alpha t/\varepsilon))kL_2/\alpha + $$
$$+ (ka_{21}/\varepsilon)\int_0^t \exp(-(t-s)\alpha/\varepsilon)\|\varphi(s)\|ds .$$

Using (3.15), after an easy computation (3.16) becomes:

$$(3.17)\qquad \|\Psi(t)\| \leq kL_2/\alpha + \exp(-\alpha t/\varepsilon)k\|y\| + ka_{21}(a_{11}r_1+L_1)/a_{11}\alpha$$
$$.\exp(a_{11}t)+(ka_{12}a_{21}/\varepsilon)\int_0^t \exp(-\alpha(t-s)/\varepsilon).$$
$$.\left[\int_0^s \exp(a_{11}(s-\sigma))\|\Psi(\sigma)\|d\sigma\right]ds$$

Changing the order of integration into the last term in (3.17) and applying the Gronwall's lemma ([2],Ch.IV,lemma 2) we obtain:

$$(3.18)\qquad \|\Psi(t)\| \leq r(t;r_1) + \theta(t;\varepsilon)\|y\|$$

where:

$$(3.19)\qquad r(t;r_1) = kL_2/\alpha + (ka_{21}/\alpha a_{11})(a_{11}r_1+L_1)\exp(a_{11}t) +$$
$$(ka_{21}a_{12}/\alpha)(kL_2+ ka_{21}(a_{11}r_1+L_1)t/\alpha a_{11})\exp(ka_{21}a_{12}/a_{11}\alpha)$$

$$(3.20)\qquad \theta(t;\varepsilon) = k\exp(-\alpha t/\varepsilon)+ (\varepsilon a_{21}a_{12}/\alpha^2)\exp(ka_{21}a_{12}/a_{11}\alpha)$$

Obviously, if $T_o > 0$ and the compact subset $G_1 \subset G$ are given, then from (3.18) follows (3.7) as $r = r(T_o;r_1)$ defined by (3.19) depends only on T_o and G_1 and $\theta(T_o;\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$.

If the compact subset $G_1 x G_2 \subset G x R^m$ and T_o, $\varepsilon_o > 0$ are given then (3.18) implies (3.9) where $M_2 = r(T_o;r_1)+\theta(T_o;\varepsilon_o)r_2$

In order to prove (3.8) we note that from (3.15) and (3.9) it follows:

$$(3.21)\qquad \|\varphi(t)\| \leq r_1\exp(a_{11}t)+\exp(a_{11}t)L_1/a_{11}+\exp(a_{11}t)a_{12}M_2/a_{11}$$

From (3.21) it follows immediately (3.8) where $M_1 = (r_1 + L_1/a_{11} + a_{12}M_2/a_{11})\exp(a_{11}T_o)$ and the lemma is proved.

Lemma 3.3

Let the hypotheses of theorem 3.1 be satisfied. Then, for any compact subset $G_1 x G_2 \subset G x R^m$ and for any $\delta, T_o > 0$ there exists $\varepsilon_o > 0$ such that for any $\varepsilon \in (0, \varepsilon_o)$, any $(x,y) \in G_1 x G_2$ and any Filippov solution $(\varphi_\varepsilon(\cdot;x,y), \psi_\varepsilon(\cdot;x,y))$ of (1.12) through (x,y), the map $\xi_\varepsilon(\cdot;x,y)$ defined by (3.4) verifies:

$$(3.22) \qquad \int_0^{T_o} \| \xi_\varepsilon(t;x,y) \| dt < \delta$$

Proof of lemma 3.3

The statement in the lemma is equivalent to the following: $\lim \int_0^{T_o} \| \xi_\varepsilon(t;x,y) \| dt = 0$ as $\varepsilon \searrow 0$ uniformly with respect to $(x,y) \in G_1 x G_2$.

Integrating by parts the last term in (3.13) and using (3.12), from (3.4), after a straitforward computation we get:

$$(3.23) \qquad \xi_\varepsilon(t;x,y) = A_{22}\exp(A_{22}t/\varepsilon)y + \exp(A_{22}t/\varepsilon)A_{21}x + \\ + \exp(A_{22}t/\varepsilon)w^2(t) + \int_0^t \exp(A_{22}(t-s)/\varepsilon)A_{21}\frac{d\varphi}{ds}(s)\, ds + \\ + (1/\varepsilon)A_{22}\int_0^t \exp(A_{22}s/\varepsilon)(w^2(t-s) - w^2(t))ds$$

Using (3.8), (3.9) and the notations (3.14), from (3.23) we obtain:

$$(3.24) \qquad \| \xi_\varepsilon(t;x,y) \| \leq k(a_{22}r_2 + a_{21}r_1 + L_2)\exp(-\alpha t/\varepsilon) + \\ + ka_{21}(a_{11}M_1 + a_{12}M_2 + L_1)\varepsilon/\alpha + (ka_{22}/\varepsilon)\int_0^t \exp(-\alpha t/\varepsilon)\| w^2(t-s) - w^2(t)\| ds$$

and therefore:

$$(3.25)\qquad \int_0^{T_0} \| \xi_\varepsilon(t;x,y)\| dt \le k(a_{22}r_2+a_{21}r_1+L_2)\, \varepsilon/\alpha \; +$$

$$+ka_{21}(a_{11}M_1+a_{12}M_2+L_1)T_0\, \varepsilon/\alpha \; +(a_{22}k/\varepsilon)\int_0^{T_0}\Big[\int_0^{t} \exp(-\alpha s/\varepsilon)\|w^2(t-s)-w^2(t)\| ds\Big]dt$$

By usual procedures in singular perturbations we get:

$$(3.26)\qquad \int_0^{T_0} \| \xi_\varepsilon(t;x,y)\| dt \le k\big[a_{22}r_2+a_{21}r_1+L_2+a_{21}(a_{11}M_1+a_{12}M_2+L_1)T_0\big]\cdot$$

$$\cdot\, \varepsilon/\alpha \; + (2a_{22}L_2/\alpha)\,(T_0-\sqrt{\varepsilon})\exp(-\alpha/\sqrt{\varepsilon})+(\varepsilon/\alpha)\exp(-\alpha T_0/\varepsilon) \; +$$

$$+ (a_{22}k/\varepsilon)\int_0^{\sqrt{\varepsilon}} \exp(-\alpha s/\varepsilon)\Big[\int_0^{T_0} \|w^2(t-s)-w^2(t)\| dt\Big]ds$$

The lemma will be proved if we show that the last term in (3.26) tends to zero as ε tends to zero.

In order to prove this statement we note that according to the theorem IV.8.20 in [6], $\int_0^{T_0} \|w^2(t-s)-w^2(t)\| dt$ tends to zero as s tends to zero, and therefore, for each $h>0$ there exists $\chi(h)>0$ such that $\chi(h) \longrightarrow 0$ as $h \longrightarrow 0$ and

$$\int_0^{T_0} \| w^2(t-s)-w^2(t)\| dt \le \chi(h) \quad \text{a.e. for } s\in[0,h].$$

It follows that the last term in (3.26) is bounded by:

$$(a_{22}k/\varepsilon)\int_0^{\sqrt{\varepsilon}} \exp(-\alpha s/\varepsilon)\Big[\int_0^{T_0} \|w^2(t-s)-w^2(t)\| dt\Big]ds \le a_{22}kT_0\,\chi(\sqrt{\varepsilon})/\alpha$$

where $\chi(\sqrt{\varepsilon}) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and lemma 3.3 is proved.

<u>Proof of theorem 3.1:</u>

Let the compact $G_1 x G_2 \subset G x R^m$ and $\eta>0$ be given; denote $\sigma = \max\{T(x) \mid x\in G_1\}$; choose $r>0$ given by lemma 3.2 (associated to G_1 and ε); choose $\theta>0$ such that $r+\theta > r_2$

(r_2 being defined by (3.14)) and to r_2 and θ associate $\varepsilon_0' > 0$ by lemma 3.2; choose $\eta' < \eta$ such that $\overline{B_{\eta'}(0)} \subset G_1$ and to τ and η' associate δ' by lemma 2.1 and δ'' by theorem 2.2. Choose $\delta = (1/a_{12}\|A_{22}^{-1}\|)\min(\delta', \delta'')$ and to $\tau, \delta > 0$ and $G_1 x \overline{B_{r+\theta}(0)}$ associate ε_0'' by lemma 3.3. Choose finally $\varepsilon_0 = \min(\varepsilon_0', \varepsilon_0'')$. As already remarked, we have to show that for any $\varepsilon \in (0, \varepsilon_0)$ and any $(x,y) \in G_1 x G_2$, the map $t \longmapsto A_{12}A_{22}^{-1} \xi_\varepsilon(t;x,y)$ satisfies (2.7) with constant δ''. Since $\varepsilon_0 \leq \varepsilon_0''$ we deduce from lemma 3.3 that $\int_0^\tau \|\xi_\varepsilon(t;x,y)\|dt < \delta$ from lemma 2.1 we deduce that $x_{(1)} = \varphi_\varepsilon(\tau;x,y) \in \overline{B_{\eta'}(0)} \subset G_1$ and from lemma 3.2, $y_{(1)} = \psi_\varepsilon(\tau;x,y) \in \overline{B_{r+\theta}(0)} \subset G_2$.

By induction on k we prove that

$$\int_{(k-1)\tau}^{k\tau} \|\xi_\varepsilon(t;x,y)\|dt < \delta \quad \text{for } k=0,1,2,\ldots$$

, and thus the condition (2.7) is satisfied with constant $a_{12}\|A_{22}^{-1}\|\delta$ for the map $t \longmapsto A_{12}A_{22}^{-1} \xi_\varepsilon(t;x,y)$.

The second assertion of the theorem follows by direct estimation using the variation of the constants formula.

Remark 3.4

According to the theorem proved in [4], the hypothesis (1.13) (and therefore the statements in theorem 3.1) is satisfied if $n = p = 2$ and if the polyhedron $U \subset R^2$ and the matrix A are such that for any vertex v of U the polar cone $\Lambda_v = \{\lambda \in R^2 \mid \langle \lambda, Bv \rangle \geq \langle \lambda, Bv' \rangle$ for any vertex v' of $U\}$ corresponding to v intersects only that eigenspace of $-A^*$ which corresponds to its large t eigenvalue.

From the results proved in [5], [10], [11], it follows that the statement in theorem 3.1 is valid also for the control systems (1.1)-(1.2) for which $p = 1$, $U = [-1,+1]$ and either $n \leq 3$ or, if $n > 3$ then the matrix A (given by (1.6)) has only real eigenvalues (provided, of course, that the hypotheses (1.3)-(1.7) are satisfied).

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