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1. Introduction. In [12] J.Wermer has proved that the algebra of all continuous functions on the unit circle $|z|=1$ which can be extended to the unit disc $|z|\leq 1$ so as to be analytic in the interior is a maximal subalgebra of the Banach algebra $C(\mathbb{T})$ of all continuous, complex-valued functions on the unit circle.

In [1] Arens and Singer have presented a generalisation of part of the theory of analytic functions in the unit disc, established by observing the role played in the classical theory by the group of integers and replacing this group by a locally compact abelian group G possessing a suitably distinguished semigroup G_+.

Further, in [6] K Hoffman and I.M. Singer have extended to this context the maximality theorem of Wermer. In this paper we extend the same theorem to the context of C^*-crossed products. We mention that the analogous study for W^*-crossed products was made succesfuly in [8] for the case $G=\mathbb{Z}$. In § 4 we solve the same question for a C^*-dynamical system associated with a von Neumann algebra with a homogeneous periodic state.

2. Preliminaries and notations

2A. Dynamical systems and spectra. Let (A,G,α) a C^*-dynamical system with G abelian, i.e. a C^*-algebra A and an abelian locally compact group G of $*$-automorphisms of A with the property that for each a A, the function $g \rightarrow \alpha_g(a)$ is continuous.

We define a representation $\alpha(\cdot)$ of $L^1(G)$ into the bounded operators on A by $\alpha(f)a=\int f(g)\alpha_g(a)dg$ $(a\in A)$ where $f\in L^1(G)$. For $f\in L^1(G)$ we put $Z(f)=\{p\in\hat{G}\mid \hat{f}(p)=0\}$ where $\hat{G}$ is the dual of G and $\hat{f}$ is the Fourier transform of f.

Let $Sp\,\alpha$ be defined as $\bigcap\{Z(f)\mid f\in L^1(G),\ \alpha(f)=0\}$. If $a\in A$ let $Sp_\alpha(a)=\bigcap\{Z(f)\mid f\in L^1(G),\ \alpha(f)a=0\}$.

We refer the readers to [3] for the elementary properties of spectra and spectral subspaces.

Throughout this paper we suppose G discrete and hence $\hat{G}$ compact. Suppose that there exists a subsemigroup $G_+\subset G$ with the following properties:

$$G_+\cup(-G_+)=G$$

$$G_+\cap(-G_+)=(0).$$

Let $(B,\hat{G},\beta)$ be a C^*-dynamical system. Denote $\mathcal{A}(G,\beta)=$ $=\{b\in B\mid Sp_\beta(b)\subset G_+\}$. By the results in [3], $\mathcal{A}(G,\beta)$ is a norm-closed non-selfadjoint subalgebra of B.

Now, for each $g\in G$, we consider the weak integration:

$$\mathcal{E}_g(b)=\int_{\hat{G}}\langle g,p\rangle\beta_p(b)\,dp\quad,\quad b\in B$$

where dp is the normalised Haar measure on B.

Then $\mathcal{E}_g$ is a bounded linear mapping from B onto $B_g=\{b\in B\mid \beta_p(b)=\overline{\langle g,p\rangle}b,\ \ p\in\hat{G}\}$. We have also the following properties:

$$\mathcal{E}_{g_1}\circ\mathcal{E}_{g_2}=\delta_{g_1g_2}\mathcal{E}_{g_1}\quad,\quad g_1,g_2\in G$$

(Here $\delta_{g_1g_2}$ is the Kronecker symbol)

$$\mathcal{E}_g(a_1ba_2)=a_1\mathcal{E}_g(b)a_2\quad,\quad a_1,a_2\in B_0,\ b\in B.$$

Clearly $B_0=\mathcal{A}(G,\beta)\cap\mathcal{A}(G,\beta)^*$ is the algebra of all fixed points with respect to β, and $\mathcal{E}_0$ is a faithful, β-invariant projection of norm one from B onto B_o. Clearly, [illegible] is multiplicative on [illegible].

The following Lemma is a slight generalisation of [9, Lemma 1].

2.1. <u>Lemma</u> (i) For any g_1, g_2, $B_{g_1} \cdot B_{g_2} = B_{g_1+g_2}$ and $B^*_{g_1} = B_{-g_1}$

(ii) Let $b_1, b_2 \in B$. If $\mathcal{E}_g(b_1) = \mathcal{E}_g(b_2)$ for all $g \in G$ then $b_1 = b_2$.

(iii) For $b \in B$, we have $Sp_\beta(b) = \{g \in G \mid \mathcal{E}_g(b) \neq 0\}$.

(iv) For $g \in G$, $B_g = \{b \in B \mid Sp_\beta(b) = \{g\}\}$.

The following Lemma is well known and easy to prove:

2.2. <u>Lemma</u> B is linearly spanned by $\bigcup_{g \in G} B_g$ in the norm topology.

2.3. <u>Remark</u>. If $b \in B$ is such that $\mathcal{E}_{g_o}(b) = 0$ for some $g_o \in G$, then there exists a sequence $b_n = \sum_g b^n(g)$, $b^n(g) \in B_g$ which converges to b (in norm) such that $b^n(g_o) = 0$ for all n. Indeed, by Lemma 2.2. there exists a sequence $c_n = \sum_g c^n(g)$, $c^n(g) \in B_g$ which converges to b. Since $\mathcal{E}_g(c_n) = c^n(g)$ (by Lemma 1.1 (iii)) and $\mathcal{E}_g$ is bounded for all $g \in G$, we have that $\mathcal{E}_{g_o}(c_n)$ converges to $\mathcal{E}_{g_o}(b) = 0$ Therefore $b_n = c_n - \mathcal{E}_{g_o}(c_n)$ satisfies the desired property.

2B. <u>C^*-crossed products</u>

Let (A, G, α) be a C^*-dynamical system with G discrete, abelian. Assume that $A \subset B(H)$ for some Hilbert space H. Let $\mathcal{P}(G,A)$ denote the set of "trigonometric polynomials":

$$\mathcal{P}(G,A) = \{f : G \longmapsto A \mid f(g) = 0 \quad \text{for all but finitely many} \quad g \in G\}$$

Define a faithful representation of $\mathcal{P}(G,A)$ on $l^2(G,H)$ by

$$(1)\qquad (y\xi)(g)=\sum_{s\in G}\alpha_{-g}(y(s))\,\xi(g-s)\qquad y\in\mathcal{P}(G,A),\ \xi\in l^2(G,H).$$

We identify $\mathcal{P}(G,A)$ with its image in $B(l^2(G,H))$ and denote by $C^*(G,\ ,A)$ the C^*-algebra generated by $\mathcal{P}(G,A)$. It can be shown that $C^*(G,\alpha,A)$ does not depend on the representation of A on H. We say that $C^*(G,\alpha,A)$ is the crossed product of G with A.

The following element of $\mathcal{P}(G,A)$:

$$y(g)=1\ ,\quad y(s)=0\quad \text{for all } s\neq g$$

will be denoted by λ_g.

Also, the element $y\in\mathcal{P}(G,A)$:

$$y(0)=a\quad \text{for some } a\in A$$
$$y(g)=0\quad \text{for all } g\neq 0$$

will be denoted by a.

We denote by $(C^*(G,\alpha,A),\hat{G},\hat{\alpha})$ the dual system of (A,G,α) [10].

If G is ordered by a subsemigroup G_+ as in 2A, then we may apply the results in 2A to the system $(C^*(G,\alpha,A),\hat{G},\hat{\alpha})$.

3. The main results

We say that a C^*-algebra A is simple if A has no nontrivial closed two-sided ideals. We say that A is G-simple if it has no nontrivial, closed, G-invariant two sided ideals. If G is oredered by the subsemigroup G_+, we say that G is archimedean ordered if for any $g_1, g_2\in G_+\setminus\{0\}$ there exists $n\in\mathbb{N}$ such that $ng_1>g_2$. Then, it is well known that G is isomorphic with a subgroup of $\mathbb{R}$.

Let B be a C^* algebra with unit. We say that a closed subalgebra $\mathcal{A}\subset B$ with the unit of B is a Dirichlet subalgebra if $\mathcal{A}+\mathcal{A}^*$ is norm dense in B. Let $\mathcal{E}$ be a faithful projection of norm one in B. A Dirichlet subalgebra $\mathcal{A}\subset B$ is said to be C^*-subdiagonal if the pair $(\mathcal{A},\mathcal{E})$ satisfies the following conditions:

(i) $\mathcal{E}$ is multiplicative on $\mathcal{A}$.

(ii) $\mathcal{E}(\mathcal{A})=\mathcal{A}\cap\mathcal{A}^*$

We call the C^*-subalgebra $\mathcal{A}\cap\mathcal{A}^*$ the diagonal of $\mathcal{A}$. Then using [7, Theorem 3.1], the proof of [7, Theorem 2.4] can be adapted to prove:

3.1. Theorem. Let A be a C^*-algebra with unit, and G be a discrete, commutative group of automorphisms of A.

Suppose that G is archimedean-linearly ordered. Then $\mathcal{A}(G,\hat{\alpha})$ is a maximal C^*-subdiagonal subalgebra of $C^*(G,\alpha,A)$ with respect to the projection $\mathcal{E}_o(\delta 1)$. The following Theorem give a sufficient condition for $\mathcal{A}(G,\hat{\alpha})$ to be a maximal subalgebra of $C^*(G,\alpha,A)$.

3.2. Theorem. Let A be a simple C^*-algebra with unit and b be an automorphism group of A. Suppose that G is discrete and archimedean-linearly ordered. Then $\mathcal{A}(G,\hat{\alpha})$ is a maximal subalgebra of $C^*(G,\alpha,A)$.

Proof. The proof is inspired from Cohen's proof in the classical case ([5]).Let $\mathcal{B}\subset C^*(G,\alpha,A)$ be a subalgebra which contains $\mathcal{A}(G,\hat{\alpha})$ and $\mathcal{B}\neq\mathcal{A}(G,\hat{\alpha})$. Then, there exist $b\in\mathcal{B}$ and $t_o\in G_+\setminus\{0\}$ such that $\mathcal{E}_{-to}(b)\neq 0$. It is easy to see that the set $J=\{a\in A\,|\,(\exists)\ b\in\mathcal{B}\ ,\ \lambda_{to}\mathcal{E}_{-to}(b)=a\}$ is a two-sided ideal in A.

Since A is simple, it follows that J=A. Therefore, there eixsts $b_o\in\mathcal{B}$ such that $\lambda_{to}\mathcal{E}_{-to}(b_o)=1$.

By definition of the crossed product and Remark 2.3 there exist two "trigonometric polinomials" $p,q\in\mathcal{A}(G,\hat{\alpha})$ and $h\in C^*(G,\alpha,A)$ such that

$$\lambda_{t_0} b_0 = 1 + \lambda_{t_1} p + \lambda_{-t_2} q^* + h \qquad , \ \|h\| < 1/2$$

for some $t_1, t_2 \in G_+ \setminus \{0\}$.

Let $s_0 \in G_+$ be $s_0 = \min\{t_0, t_1, t_2\}$. We denote also $b_1 = \lambda_{t_0 - s_0} b_0 \in \mathcal{B}$, $p_1 = \lambda_{t_1 - s_0} p \in \mathcal{A}(G, \hat{\alpha})$, $q_1 = q\, \lambda_{t_2 - s_0} \in \mathcal{A}(G, \hat{\alpha})$.

Then

$$\lambda_{s_0} b_1 = 1 + \lambda_{s_0} p_1 + \lambda_{-s_0} q_1^* + h \qquad , \ \|h\| < 1/2$$

Let $M = \| q_1 \lambda_{s_0} - \lambda_{-s_0} q_1^* \|$. Since $q_1 \lambda_{s_0} - \lambda_{-s_0} q_1^* = i.h$ where h is selfadjoint, we have for every $\delta > 0$

(1) $$\| 1 + \delta (q_1 \lambda_{s_0} - \lambda_{-s_0} q_1^* \| \leq 1 + \delta^2 M^2 .$$

Further, we have:

(2) $$\begin{aligned} \delta \cdot \lambda_{-s_0} q_1^* &= \delta (\lambda_{s_0} b_1 - 1 - \lambda_{s_0} p_1) - \delta h \\ &= \lambda_{s_0} (\delta b_1 - p_1) - \delta h - \delta \\ &= \lambda_{s_0} g - \delta h - \delta . \end{aligned}$$

where $g = \delta b_1 - p_1 \in \mathcal{B}$.

Let $q_1' \in \mathcal{A}(G, \hat{\alpha})$ be such that $q_1 \lambda_{s_0} = \lambda_{s_0} q_1'$.

From (1), (2) and the fact that $\|h\| < 1/2$, it follows that

(3) $$\| 1 + \delta + \lambda_{s_0} (-g + \delta q_1') \| \leq 1 + \delta^2 M^2 + \delta/2 .$$

If $\delta < 1/2M^2$ then, from (3) it results:

(4) $$\| 1 + \delta + \lambda_{s_0} (-g + \delta q_1') \| < 1 + \delta$$

Since obviously $\lambda_{s_o}(-g+\delta q_1') \in \mathcal{B}$, from (4) it follows that this element has an inverse k in $\mathcal{B}$. So $\lambda_{s_o}(g+\delta q_1')k=1$. From this, it follows that $\lambda_{-s_o}=(g+\delta q_1')k \in \mathcal{B}$.

Let $s \in G_+$ be arbitrary. Since G is archimedean ordered, there exists $n \in \mathbb{N}$ such that $ns_o > s$. Then

$$\lambda_{-s} = \lambda_{ns_o-s} \cdot \lambda_{-ns_o} \in \mathcal{B}.$$

It follows that $\mathcal{B}=C^*(G,\alpha,A)$.

In what follows we discuss some partial converses of the preceding Theorem.

3.3. <u>Proposition</u>. Let A be a unital C^*-algebra, and α be an $*$-automorphism of A. If $A(\mathbb{Z},\hat{\alpha})$ is a maximal subalgebra of $C^*(\mathbb{Z},\alpha,A)$, then A is simple.

<u>Proof</u>. Suppose A is not simple and let $J \subset A$ be a non-trivial two-sided ideal. We shall show that $\mathcal{A}(\mathbb{Z},\hat{\alpha})$ is not maximal. In order to do this we shall produce a subspace $\mathfrak{M} \subset C^*(\mathbb{Z},\alpha,A)$ with the following properties:

(i) $b\mathfrak{M} \subset \mathfrak{M}$ for every $b \in \mathcal{A}(\mathbb{Z},\hat{\alpha})$

(ii) There exists $g \in C^*(\mathbb{Z},\alpha,A) \setminus \mathcal{A}(\mathbb{Z},\hat{\alpha})$ such that $g\mathfrak{M} \subset \mathfrak{M}$

(iii) $\mathfrak{M}$ is not a left ideal of $C^*(\mathbb{Z},\alpha,A)$.

Let $\mathfrak{M}$ be the closure of the set of polynomials $b:\mathbb{Z} \longmapsto A$ with the property that $b(n) \in \alpha^n(J)$ for all $n<0$.

Let $p \in \mathbb{N}$ and $f \in \mathfrak{M}$ a polynomial. Then $(\lambda_p \cdot f)(n)=\alpha^p f(n-p)$. Hence, if $n<0$ we have $(\lambda_p f)(n) \in \alpha^n(J)$. Also, if $a \in A$, we have $(a.f)(n)=a.f(n) \in \alpha^n(J)$. Hence $b\mathfrak{M} \subset \mathfrak{M}$ for all $b \in \mathcal{A}(\mathbb{Z},\hat{\alpha})$ and (i) is proved.

To prove (ii) let $x \in \alpha^{-1}(J)$. Then the element $x \cdot \lambda_{-1}$ satisfies (ii). Obviously $\lambda_{-1}\mathcal{M} \not\subset \mathcal{M}$, whence (iii).

Therefore $\mathcal{A}(\mathbb{Z},\hat{\alpha})$ is not maximal. It follows that A is simple.

3.4. Proposition. Let A be a unital C^*-algebra and G be a discrete, commutative, linearly ordered group of $*$-automorphisms of A.

If $\mathcal{A}(G,\hat{\alpha})$ is a maximal subalgebra of $C^*(G,\alpha,A)$, then:

(i) The order on G is archimedean .

(ii) A is G-simple.

Proof. Suppose the order on G is not archimedean. Then, there exist $t_1, t_2 \in G_+ \setminus \{e\}$ such that $nt_1 < t_2$ for every $n \in \mathbb{N}$. Then, the algebra $\mathcal{B}$ generated by $\mathcal{A}(G,\hat{\alpha})$ and λ_{-t_1} satisfies $\mathcal{A}(G,\hat{\alpha}) \subsetneq \mathcal{B} \subsetneq C^*(G,\alpha,A)$. Hence $\mathcal{A}(G,\hat{\alpha})$ is not maximal, and (i) is proved.

Now, we show that A is G-simple. Suppose A is not G-simple. Then, there exists a non-trivial G-invariant two-sided ideal $J \subset A$. Denote by $\mathcal{B} = \{b \in C^*(G,\alpha,A) \mid \varepsilon_t(b) \in J,\ t < 0\}$. Then $\mathcal{B}$ is an algebra and $\mathcal{A}(G,\hat{\alpha}) \subsetneq \mathcal{B} \subsetneq C^*(G,\alpha,A)$. Therefore A is G-simple.

3.5. Proposition. Let A be a unital C^*-algebra and G be a discrete, commutative, linearly ordered group of $*$-automorphisms of A. Suppose that

(i) A is primitive and postliminar.

(ii) $\mathcal{A}(G,\hat{\alpha})$ is a maximal subalgebra of $C^*(G,\alpha,A)$. Then we have:

(iii) G is archimedean ordered

(iv) A is a finite dimensional factor.

Proof. (iii) follows from Proposition 3.4. Let us prove (iv). Let $\hat{A}$ be the space of irreducible representations of A and Prim(A) the space of primitive ideals of A with the Jacobson topology. Then, by [4 Théorème 4.3.7] the mapping $\pi \longmapsto \ker\pi$ is a bijection between $\hat{A}$ and Prim(A). By [4 Theoreme 4.4.5] there exists a maximal open set $U \subset$ Prim (A) which is separated. Since A is primitive, it follows $(0) \in$ Prim(A). Obviously, (0) is dense in Prim(A). Therefore, $(0) \in U$. Since every open set $V \subset$ Prim(A) contains (0), it follows (0)=U. Therefore $\int(0) = \{J \in \mathrm{Prim}(A) \mid J \neq (0)\}$ is closed in Prim(A). Thus $\bigcap \int(0) = J_o \neq 0$. Now, it is easy to see that J_o is G-invariant. Since by Proposition 3.4, A is G-simple, it follows that $J_o = (0)$. This contradiction shows that A is simple. Since A is unital, primitive and postliminar, it folows that A is a finite dimensional factor.

4. Subalgebras of a von Neumann algebra with a homogeneous periodic state

Let M be a von Neumann algebra. Suppose that M has a homogeneous periodic state φ in the sense that $G(\varphi) = \{\sigma \in \mathrm{Aut}(M) \mid \varphi \circ \sigma = \varphi\}$ acts ergodically on M and the modular automorphism group σ_t^{φ} of M associated with φ is a periodic flow. A penetrating study of such algebras was made by Takesaki [11].

Let T>0 be the period of σ_t^{φ}. Put $\rho = e^{-2\pi/T}$, $0 < \rho < 1$.

Set $M_n = \{x \in M \mid \sigma_t^{\varphi}(x) = \rho^{int} x\}$, $n \in \mathbb{Z}$.

For each $n \in \mathbb{Z}$, we consider the integration:

$$\mathcal{E}_n(x) = \frac{1}{T}\int_0^T \rho^{-inT} \sigma_t^{\varphi}(x)\, dt \quad , \quad x \in M.$$

Then

$$\mathcal{E}_n(M) = M_n \quad , n \in \mathbb{Z} .$$

$$\mathcal{E}_n \circ \mathcal{E}_m = \delta_{nm} \mathcal{E}_n \qquad m, n \in \mathbb{Z} .$$

$$\mathcal{E}_n(a x b) = a \mathcal{E}_n(x) b \qquad a, b \in M_o \ , x \in M .$$

$$M_n M_m = M_{n+m} \qquad n, m \in \mathbb{Z} .$$

$$M_n^* = M_{-n} \qquad n \in \mathbb{Z} .$$

Collect some results from [11] in the following:

4.1. Theorem (i) The subspace M_1 of M contains an isometry u such that for $n \geq 1$, $M_n = M_o U^n$ and $M_{-n} = U^{*n} M_o$.

(ii) In the pre-Hilbert space structure induced by the state φ , M is decomposed into an orthogonal direct sum as follows:

$$M = \cdots \oplus U^{*n} M_o \oplus \cdots \oplus U^* M_o \oplus M_o \oplus M_o U \oplus \cdots \oplus M_o U^n \oplus \cdots$$

(iii) M_o is of type II_1

(iv) M is of type III .

Let B denote the C^*-subalgebra of M generated by M_o and u. Obviously B is σ_t^φ-invariant, $t \in \mathbb{R}$.

Moreover since the mapping $t \longmapsto \sigma_t^\varphi(x)$ is norm-continuous for every $x \in M_n$, $n \in \mathbb{Z}$, it follows that the mapping $t \longmapsto \sigma_t^\varphi(x)$ is normcontinuous for every $x \in B$. Therefore, we can consider the C^*-dynamical system $(B, \sigma_t^\varphi, \mathbb{R})$. By Lemma 2.1 and Theorem 4.1 (i) it follows that $Sp(\sigma^\varphi)$ is isomorphic with $\mathbb{Z}$.

As in §2 let $\mathcal{A}(\mathbb{Z}, \sigma^\varphi)$ denote the algebra of all elements of B with non negative spectrum.

4.2. Proposition. $\mathcal{A}(\mathbb{Z}, \sigma^\varphi)$ is a maximal subalgebra of B if and only if M_o is a factor.

Proof. Suppose M_o is a factor. We follow the proof of

Theorem 3.2. Let $\mathcal{B} \subset B$ be such that $\mathcal{A}(\mathbb{Z},\sigma^{\varphi}) \subsetneq \mathcal{B}$. Then there exist $b_o \in \mathcal{B}$ and $n \in \mathbb{N}$ such that $\mathcal{E}_{-n}(b_o) \neq 0$. We may suppose $n=1$. Let $K = \{x \in M_o \mid (\exists)\ b \in \mathcal{B},\ \mathcal{E}_{-1}(b) = U^* x\}$. K is a linear subspace of M_o. If we put $e = uv^*$, it can be shown easily that the mapping $Ad(u)(x) = u\,x\,v^*$ is an isomorphism of M_o onto eM_oe.

We claim that eKe is a two-sided ideal of eM_oe. Indeed if $b \in \mathcal{B}$ is such that $\mathcal{E}_{-1}(b) = x \in eKe$ and $a \in eM_oe$, then

$$\begin{aligned}(U^*aU)U^*x &= U^*aex\\ &= U^*eaex\\ &= U^*ax.\end{aligned}$$

Therefore, since $u^*au \in M_o \subset \mathcal{A}(\mathbb{Z},\sigma^{\varphi}) \subset \mathcal{B}$, we have

$$\mathcal{E}_{-1}(U^*aU \cdot b) = U^*a\,x \quad \text{so} \quad ax \in eKe.$$

Similarly

$$\mathcal{E}_{-1}(ba) = U^*xa \quad , \text{so} \quad xa \in eKe.$$

Since M_o is a factor it follows that eM_oe is a factor. Therefore eM_oe is simple. Hence $eKe = eM_oe$. Then, there exists $b_o \in \mathcal{B}$ such that $\mathcal{E}_{-1}(b_o) = u^*e = u^*$.

By definition of B, there exist two "polynomials" $p,q \in \mathcal{A}(\mathbb{Z},\sigma^{\varphi})$ and $h \in B$ such that

$$b_ou = 1 + pu + u^*q^* + h \quad , \qquad \|h\| < 1/2 .$$

The rest of the proof of the "if" part is the same as that of Theorem 3.2. Now, suppose that $\mathcal{A}(\mathbb{Z},\sigma^{\varphi})$ is maximal in B.

If M_o is not a factor, let $\tilde{\theta}$ be the automorphisms of the center Z_o of M_o defined as follows:

$$uzu^* = \tilde{\theta}(z)e \qquad \text{(see [11 Lemma 1.20]).}$$

There are the following possibilities:

I. $\tilde{\theta}$ is not ergodic.

In this case, there exists a projection $q \in Z_o$ such that $\tilde{\theta}(q)=q$. Then q belongs to the center Z of M. Therefore, the algebra $\mathcal{B}$ generated by $\mathcal{A}(\mathbb{Z},\sigma^{\varphi})$ and qB is such that $\mathcal{A}(\mathbb{Z},\sigma^{\varphi}) \subsetneqq \mathcal{B} \subsetneqq B$, a contradiction. Hence M_o is a factor.

II. $\tilde{\theta}$ is ergodic.

In this case there exists $q \in Z_o$ such that $q\,\tilde{\theta}(q)=0$. Then it can be easily verified that the algebra $\mathcal{B}$ generated by $\mathcal{A}(\mathbb{Z},\sigma^{\varphi})$ and the set $\{ U^{*}\tilde{\theta}(q)y \mid y \in M_o \}$ is such that $\mathcal{A}(\mathbb{Z},\sigma^{\varphi}) \subsetneqq \mathcal{B} \subsetneqq B$. This contradiction shows that M_o is a factor.

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