ISSN 0250 3638

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Boris SINGER

PREPRINT SERIES IN MATHEMATICS
No.83/1982

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December 1982

*) Institute of Mathematics, Str.Academiei nr.14, Bucharest, Romania.

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Boris Singer

We can model a stochastic phenomenon through a process with finite set of times and states, the model being more adequate as we take into consideration more moments of time and observe finer networks of states. We shall use the notion of projective limit to define the process approximated by such a sequence of finite processes.

In §1 the notion of projective limit of a sequence of measurable topological spaces is also extended over probabilities and transition probabilities in order to define the projective limit of a sequence of processes. In §2 it is shown that, under certain circumstances, the projective limit of a sequence of finite Markov processes is also a Markov process whose transition system is obtained as a limit of the sequence of transition systems of finite processes. In §4, it is shown that rather general processes (non-homogeneous in time, with states in a compact metric space), can be rediscovered by means of the projective limit of finite processes and, at the same time, a device of constructing this sequence of processes is given.

1. Projective Limit

We shall define the projective limit of a countable system of measurable topological spaces, endowed with probabilities, or transition probabilities.

These notions are used in this work only in the case of projective limits of finite processes.

1.1. Definition. Let $\sigma = (S_n)_N$ be a sequence of sets and $\varphi = (\varphi^n_m)$ a family of applications, so that, for any $m < n < p$, $\varphi^n_m : S_n \to S_m$

and $\varphi^n_m \circ \varphi^p_n = \varphi^p_m$. The doublet (σ,φ) is called a projective system of sets (or, in short, a system).

Let S be a set and $\psi=(\psi_n)_N$ a family of applications $\psi_n : S \longrightarrow S_n$, $n\in N$. (S,ψ) is called a projective limit of the system of sets (σ,φ) if and only if for any set S' and family of applications $(\psi'_n)_N$, with $\psi'_n : S' \longrightarrow S_n$, $\varphi^n_m\circ\psi'_n=\psi'_m$, there is $\psi : S' \longrightarrow S$ so that, for any $n<m$ in N, the following diagram is commutative:

$$\begin{tikzcd} S_m & & S_n \arrow[ll, "\varphi^n_m"'] \\ & S \arrow[ul, "\psi_m"] \arrow[ur, "\psi_n"'] & \\ & S' \arrow[u, "\psi"] \arrow[uul, "\psi'_m"] \arrow[uur, "\psi'_n"'] & \end{tikzcd}$$

Let $S=\{ x\in\prod_N S_n \ / \ pr_m x=\varphi^n_m(pr_n x),\ m<n \}$, where $pr_k : \prod_N S_i \longrightarrow S_k$ is the k-canonical projection. Let $\psi_n = pr_n/S$, $n\in N$ and $\psi=(\psi_n)_N$. The doublet (S,ψ) is a projective limit of the system (σ,φ). We shall note φ_n the application φ^{n+1}_n, $n\in N$.

We have, for any $m<n$, $\varphi^n_m=\varphi_m\circ\varphi_{m+1}\circ\dots\circ\varphi_{n-1}$ and $S=\{x\in\prod_N S_k \ / \ pr_n(x)=\varphi_n(pr_{n+1}(x))\}$.

1.2. <u>Definition</u>. Let $\sigma=(S_n)_N$, $\varphi=(\varphi^n_m)$, (σ,φ) be a projective system having the projective limit (S,ψ).

1. We suppose that, for any $m\in N$, the Borel algebra $\mathcal{S}_m$ is given on S_m and for $m<n$, $\varphi^n_m : S_n \longrightarrow S_m$ is measurable. Let $\mathcal{S}=\mathcal{B}(\psi_n \ / n\in N)$ be the minimum Borel algebra on S, for which all applications ψ_n are measurable. $((S,\mathcal{S}),\psi)$ is called a projective limit of the system of measurable spaces (σ,φ).

2. We suppose that, for any $m\in N$, τ_m is a topology on S_m and the application $\varphi^n_m : S_n \longrightarrow S_m$ is continuous, $m<n$. Let $\tau=top(\psi_n \ / \ n\in N)$ be the minimum topology on S for which all applications ψ_n are continuous. The doublet $((S,\tau),\psi)$ is called the projective limit of the system of topological spaces (σ,φ).

3. We suppose that, for any $m\in N$, S_m is endowed with the topology τ_m and the σ-algebra $\mathcal{S}_m=\mathcal{B}(\tau_m)$ and p_m is a probability

on $(S_m, \mathcal{S}_m)$ and the applications $\psi^n_m : S_n \longrightarrow S_m$ are measurable, $m<n$. Let $((S,\mathcal{S}),\psi)$ be a projective limit of the system of measurable spaces (σ,φ) and p a probability on $(S,\mathcal{S})$. p is called the projective limit of the sequence $(p_n)_N$ if, for any $m\in N$, $p_n\circ\psi^{n^{-1}}_m \xrightarrow[n\to\infty]{} p\circ\psi^{-1}_m$ in the space of probabilities on (S_m,τ_m) ($p=\varprojlim p_n$).

4. Let (σ,φ), τ_m, $\mathcal{S}_m$, $((S,\mathcal{S}),\psi)$ as in 3. For any $n\in N$, let Q_n be a transition probability from $(S_n,\mathcal{S}_n)$ to $(S_n,\mathcal{S}_n)$ and Q a transition probability from $(S,\mathcal{S})$ to $(S,\mathcal{S})$. Q is called the projective limit of the sequence (Q_n) if, for any $x\in S$, the sequence $(Q_n(\psi_n(x),\cdot))_N$ has the projective limit $Q(x,\cdot)$. ($Q=\varprojlim Q_n$).

1.3. Remarks. 1. The set $\bigcup_{n\in N}\psi^{-1}_n(\mathcal{S}_n)$ is a family of generators for $\mathcal{S}$ closed to finite intersection.

2. The projective limit of a sequence of probabilities or of a sequence of transition probabilities, if there are any, is unique.

1.4. In the following, we shall preserve the notations introduced in 1.1. - 1.3. for S_n, S, τ_n, τ, $\mathcal{S}_n$, $\mathcal{S}$, p_n, p, Q_n, Q. We keep on supposing that for any $n\in N$, S_n is finite and $\mathcal{S}_n=\tau_n=\mathcal{P}(S_n)$. Obviously $\psi^n_m: S_n\longrightarrow S_m$ is continuous.

Remarks. 1. $S\neq\emptyset$

2. $\tau = \text{top}(\psi^{-1}_n(s) / n\in N,\ s\in S_n) = \{\emptyset, \bigcup_{k\in N}\psi^{-1}_{n_k}(s_k)/s_k\in S_{n_k}, n_1\le n_2\le\ldots\}$

$\mathcal{S} = \mathcal{B}(\{\psi^{-1}_n(s)/n\in N, s\in S_n\}) = \mathcal{B}(\tau)$.

Proof. 1. We consider, for $m\in N$ and $s\in S_m$,

$n(s)=\min\{n\in N / n>m, \psi^{n^{-1}}_m(s)=\emptyset\}$, $\min\emptyset=\infty$. It is obvious that $\max\{n(s) / s\in S_0\}=\infty$. We choose s_0 with the property $n(s_0)=\infty$. Repeating this reasoning we may find a sequence $(s_k)_N$ so that $n(s_k)=\infty$ and $\psi_k(s_{k+1})=s_k$. It results that $(s_k)_N\in S$.

2. Let $\mathcal{A}=\{\bigcup_{i\in N}\psi^{-1}_{n_i}(s_i) / n_i\le n_{i+1}, s_i\in S_{n_i}, i\in N\}$. We have $\tau\supset\mathcal{A}\supset\{\psi^{-1}_n(s) / n\in N, s\in S_n\}$. Because for any $m<n$, $s\in S_m$, $s'\in S_n$

$\Psi^{-1}_m(s)$ and $\Psi^{-1}_n(s')$ are disjoint or include each other, it results that $\mathcal{A}$ is closed to arbitrary union and finite intersection. We have $\mathcal{S}=\mathcal{B}(\Psi^{-1}_n(s)\ /\ n\in N,\ s\in S_n)\subset\mathcal{B}(\tau)=\mathcal{B}(\mathcal{A})\subset\mathcal{S}$. Therefore $\mathcal{S}=\mathcal{B}(\tau)$

1.5. Remarks. 1. For any $n\in N$, $s\in S_n$, $\Psi^{-1}_n(s)$ is closed and open in (S,τ).

2. (S,τ) is a metrisable space with the metric $d(\cdot,\cdot)=\sum_{i\in N} 1/2^i\,\delta_{\psi_i(\cdot),\psi_i(\cdot)}$ and is induced on S by the topology-product on $\prod_{i\in N} S_i$.

3. (S,d) is the compact metric space because S is closed on $\prod_{i\in N} S_i$.

1.6. Remarks. 1. The sequence of probabilities $(p_n)_N$ admits a projective limit if and only if, for any $m\in N$ and $s\in S_m$, the sequence $(p_n\circ\Psi^{n^{-1}}_m(s))_N$ is convergent. In this case $\lim p_n\circ\Psi^{n^{-1}}_m(s)=p\circ\Psi^{-1}_m(s)$ where $p=\overleftarrow{\lim}\, p_n$.

2. The sequence of transition probabilities $(Q_n)_N$ admits a projective limit if and only if, for any $x\in S$, $m\in N$, $y\in S_m$, the sequence $(Q_n(\Psi_n(x),\Psi^{n^{-1}}_m(y)))_N$ is convergent. Let $Q=\overleftarrow{\lim}\, Q_n$. Then $\lim_{n\to\infty} Q_n(\Psi_n(x),\Psi^{n^{-1}}_m(y))$ $=Q(x,\Psi^{-1}_n(y))$.

Proof. 1. We consider the clan $\mathcal{C}=\bigcup_{m\in N}\Psi^{-1}_m(\mathcal{S}_m)$ and family of generators $\mathcal{G}=\{\Psi^{-1}_n(s)\ /\ n\in N,\ s\in S_n\}$. We define the application $p:\mathcal{G}\longrightarrow[0,1]$ through $p(\Psi^{-1}_m(s))=\lim p_n\circ\Psi^{n^{-1}}_m(s)$, for any $m\in N$ and $s\in S_m$. Obviously, p is monotonous and finitely additive on $\mathcal{G}$. It results that p is extended uniquely to an finitely additive application on $\mathcal{C}$. Since $\mathcal{C}$ is a family of compacts in S, with the Caratheodory theorem, it results that p is uniquely extended on $\mathcal{S}=\mathcal{B}(\mathcal{C})$.

2. It results from 1. that for any $x\in S$, there is $Q(x,\cdot)$ a probability on $(S,\mathcal{S})$, which is the projective limit of the sequence $(Q_n(\Psi_n(x),\cdot))_N$. For any $s\in S_m$, $m\in N$, the application $Q(\cdot,\Psi^{-1}_m(s))=\lim Q_n(\Psi_n(\cdot),\Psi^{n^{-1}}_m(s))$ is $\mathcal{S}$-measurable. It results that the set $\{A\in\mathcal{S}\ /\ Q(\cdot,A)$ is $\mathcal{S}$-measurable$\}$ is a Borel algebra and includes $\mathcal{G}$. Therefore, for any A, $Q(\cdot,A)$ is measurable.

1.7. Proposition. Let $(S,\mathcal{S})$ be the projective limit of the system

$(\sigma,\mathcal{Y})$ and $(T_n)_N$ an increasing sequence of finite sets, $T=\bigcup_{n\in N} T_n$

For any $m<n$ in N, we define the applications $\bar{\varphi}^n_m : S_n^{T_n} \longrightarrow S_m^{T_m}$ so that $\bar{\varphi}^n_m((x_t)_{t\in T_n}) = (\varphi^n_m(x_t))_{t\in T_m}$, $(x_t)_{t\in T_n}\in S_n^{T_n}$ and $\bar{\Psi}_n; S^T \longrightarrow S_n^{T_m}$, so $\bar{\Psi}_m((y_t)_T)=(\Psi_m(y_t))_{t\in T_m}$, $(y_t)_{t\in T}\in S^T$.

Then $(S^T,(\bar{\Psi}_n)_N)$ is the projective limit of the system $((S_n^{T_n}),(\bar{\varphi}^n_m))$ and the projective limit topology on S^T is the product of the topologies on S and the projective limit Borel algebra is the product of the Borel algebras on S and, at the same time, the σ-algebra of the topologies on S^T.

Proof. Obviously $\bar{\varphi}^n_m\circ\bar{\Psi}_n=\bar{\Psi}_m$, $m<n$. Let $(S',(\Psi'_n)_N)$ be a projective limit of the system $(S_n^{T_n},(\bar{\varphi}^n_m)_{m<n})$. We define the function $\eta: S' \longrightarrow S^T$ where, for any $x\in S'$, $\eta(x)$ is the only element of the set $\bigcap_{n\in N}\bar{\Psi}_n^{-1}\Psi'_n(x)$. It remains to show that, for any $m\in N$, $\bar{\Psi}_m\circ\eta=\Psi'_m$. Let $x\in S'$. We have $\bar{\Psi}_m\circ\eta(\{x\})=\bar{\Psi}_m(\bigcap_N\bar{\Psi}_n^{-1}\Psi'_n(x))$
$\subset\bigcap_N\bar{\varphi}^n_m\Psi'_n(\{x\})=\bigcap_N\Psi'_m(\{x\})=\{\Psi'_m(x)\}$.

The statements about topology and the limit Borel algebra obtained on S^T result from 1.4.2.

§2. The construction of Markov processes as a projective limit of finite processes

2.1. In the following we shall note T a totally ordered set with the initial element t_o and the increasing sequence of finite sub-sets $(T_n)_N$ with $T=\bigcup_{n\in N}T_n$. We consider any set T_n with the order induced from T and with the initial element t_o. For given $n\in N$ we shall note $\theta=\max T_n$, $T'=T_n-\{\theta\}$, for t other than θ, $\bar{t}=\min\{u\in T_n / t<u\}$ is the succesor of t in T_n and for $t\neq t_o$, $\underline{t}=\max\{u\in T_n/u<t\}$ is the predecessor of t in T_n.

For any $n\in N$, we shall note by $x_n=(x_t^n)_{t\in T_n}$ the process with states S and times T_n, defined on $(S_n^{T_n},\mathcal{Y}_n^{T_n},p_n)$ by the canonic projections $x_t^n; S_n^T\longrightarrow S_n$, $t\in T_n$. If x_n is a Markov process, then it is determined by an initial probability p_{no} on S_n

and a transition system $(Q^n_{tt'})_{t<t'\in T_n}$ from S_n to S_n. We have for any $(s_t)_{T_n}\in S_n^{T_n}$, $p((s_t)_{t\in T_n})=p_{n0}(s_{t_0})\cdot\prod_{t\in T'}Q^n_{t\bar t}(s_t,s_{\bar t})$ and for $t<t'$ in T_n, $Q^n_{tt'}=Q^n_{t\bar t}Q^n_{\bar t\bar{\bar t}}\dots Q^n_{\bar{t'}t'}$.

We shall note by $x=(x_t)_T$ the process of canonic projections defined on the probability field $(S^T,\mathcal{F}^T,p)$. If x is a Markov process, it results that there exists a transition system $(Q_{tt'})_{t<t'}$ corresponding to x. Let $p_0=p\cdot x_{t_0}^{-1}$ be the initial probability of the process.

2.2. <u>Definition</u>. We state that the sequence of processes $(x_n)_N$ has the process x as a projective limit if and only if $p=\lim p_n$ (according to 1.2.3.). We note $x=\varprojlim x_n$.

2.3. <u>Definition</u>. For $n\in N$, let Q_n be the transition probability on S_n and let Q be the transition probability on S. We state that the sequence $(Q_n)_N$ has the uniform projective limit Q if and only if for any $m\in N$, $s\in S_m$, $\lim\limits_{n\to\infty}\sup\limits_{a\in S}|Q(a,\varphi_m^{-1}(s))-Q_n(\psi_n(a),\varphi_m^{n-1}(s))|=0$.

2.4. <u>Proposition</u>. For any $n\in N$, let $(Q^n_{tt'})_{t<t'\in T_n}$ be a family of transition probabilities on S. We suppose that, for any $m\in N$, $t<t'\in T_m$, $u\text{-}\lim Q^n_{tt'}=Q_{tt'}$. Then $(Q_{tt'})_{t<t'\in T}$ is a transition system on S. In this situation, we state that the sequence of systems $((Q^n_{tt'})_{t<t'\in T_n})_N$ has the system $(Q_{tt'})_{t<t'\in T}$ as a projective limit.

<u>Proof</u>. Let $s<t<u$ in T_q. It is enough to show that, for any $b\in S$ and any $m\in N$, $m>q$ and any $a\in S_m$, we have:

$$Q_{st}Q_{tu}(b,\psi_m^{-1}(a))=Q_{su}(b,\psi_m^{-1}(a))$$

$$Q_{st}Q_{tu}(b,\psi_m^{-1}(a))=\lim_{m<n\to\infty}\sum_{y'\in S_n}Q^n_{tu}(y',\varphi_m^{n-1}(a))\,Q_{st}(b,\psi_n^{-1}(y'))$$

$$Q_{su}(b,\psi_m^{-1}(a))=\lim_{m<n\to\infty}Q^n_{su}(\psi_n(b),\varphi_m^{n-1}(a))=$$

$$=\lim_{m<n\to\infty}Q^n_{st}Q^n_{tu}(\psi_n(b),\varphi_m^{n-1}(a))=\lim_{m<n\to\infty}\sum_{y'\in S_n}Q^n_{st}(\psi_n(b),y')\,Q^n_{tu}(y',\varphi_m^{n-1}(a))$$

We have $\left|\int Q^n_{tu}(\psi_n(y),\varphi_m^{n-1}(a))\,Q_{st}(b,dy)-Q^n_{su}(\psi_n(b),\varphi_m^{n-1}(a))\right|=\dots$

$$=\left|\sum_{y'\in S_n}Q^n_{tu}(y',\varphi_m^{n-1}(a))\left[Q_{st}(b,\psi_n^{-1}(y'))-Q^n_{st}(\psi_n(b),y')\right]\right|\le a_{np}+b_{np},$$

where $p\in N$, $p<n$ and

$$a_{np} = \Big| \sum_{y' \in S_n} [Q^n_{tu}(y', \varphi^{n^{-1}}_m(a)) - Q^p_{tu}(\varphi^n_p(y'), \varphi^{p^{-1}}_m(a))] \times [Q_{st}(b, \psi_n^{-1}(y')) - Q^n_{st}(\psi_n(b), y')] \Big|$$

$$b_{np} = \Big| \sum_{y'' \in S_p} Q^p_{tu}(y'', \varphi^{p^{-1}}_m(a)) [Q_{st}(b, \psi_p^{-1}(y'')) - Q^n_{st}(\psi_n(b), \varphi^{n^{-1}}_p(y''))] \Big|$$

since $Q^p_{tu}(y', \varphi^{p^{-1}}_m(a))$ are equal for $y' \in \varphi^{n^{-1}}_p(y'')$. We have

$a_{np} \leq 2 \sup\limits_{y' \in S_n} | Q^n_{tu}(y', \varphi^{n^{-1}}_m(a)) - Q^p_{tu}(\varphi^n_p(y'), \varphi^{p^{-1}}_m(a))|$ and $\lim\limits_{p \leq n \to \infty} a_{np} = 0$.

Terefore, for any $\varepsilon > 0$, there is $p_\varepsilon \in N$ so that, for $n > p_\varepsilon$, $|a_{np_\varepsilon}| < \varepsilon$, $\lim\limits_{p_\varepsilon < n \to \infty} b_{np_\varepsilon} = 0$, S_{p_ε} being a finite set.

2.5. <u>Theorem</u>. Let $(x_n)_N$ be a sequence of processes (we preserve the pattern from 2.1. and 2.2.) so that $(p_{no})_N$ has the uniform projective limit p_o and $(Q^n_{tt'})$ has the uniform projective limit $(Q_{tt'})_{t<t' \in T}$. Then the sequence $(x_n)_N$ admits as a projective limit a Markov process x with initial distribution p_o and the transition system $(Q_{tt'})_{t<t' \in T}$.

<u>Proof</u>. Since the family $(Q_{tt'})_{t<t' \in T}$ has the property Chapman-Kolmogorov, it results that there exists a probability p on S^T so that the process $x = (x_t)_T$ should be a Markov process with the initial probability p_o and the family of transition distributions $(Q_{tt'})_{t<t' \in T}$. It remains to show that, according to 1.9., for any $m \in N$ and $s = (s_t)_{T_n} \in S^{T_m}_m$, we have $\lim\limits_{n \to \infty} p_n(\bar\varphi^{n^{-1}}_m(s)) = p \circ \bar\psi^{-1}_m(s)$.

Let $m \in N$ be fixed, arbitrarily. Let $\theta = \max T_m$ and $T' = T_m - \{\theta\}$.

For $n \in N$, we note $A = \{u = (u_t)_{T'} \in S^{T'}_n / \varphi^n_m(u_t) = s_t, \ t \in T'\}$.

We have
$$p_n(\bar\varphi^{n^{-1}}_m(s)) = \sum_{u \in A} Q^n_{\underline{\theta}\theta}(u_{\underline{\theta}}, \varphi^{n^{-1}}_m(s_\theta)) \, p_n((x^n_t)_{t \in T'})^{-1}(u)$$

$$p \circ \bar\psi^{-1}_m(s) = \int\limits_{\prod_{t \in T'} \psi^{-1}_m(s_t)} Q_{\underline{\theta}\theta}(y_{\underline{\theta}}, \psi^{-1}_m(s_\theta)) \, dp((x_t)_{t \in T'})^{-1}((y_t)_{t \in T'})$$

Let $\quad a_n = |p_n \circ \bar\varphi^{n^{-1}}_m(s) - p \circ \bar\psi^{-1}_m(s)| = \Big| \sum\limits_{u \in A} [p_n \circ (x^n_t)^{-1}(u) - p \circ \bar\psi^{-1}_m(u)] \Big| =$

$$= \Big| \sum_{u \in A} [Q^n_{\underline{\theta}\theta}(u_{\underline{\theta}}, \varphi^{n^{-1}}_m(s_\theta)) \, p_n((x^n_t)_{T'})^{-1}(u) - \int\limits_{\prod_{t \in T'} \psi^{-1}_n(u_t)} Q_{\underline{\theta}\theta}(y_{\underline{\theta}}, \psi^{-1}_m(s_\theta)) \, dp_o((x_t)_{T'})^{-1}((y_t)_{T'}) \Big|$$

$\leq d_{nr} + e_{nr}$ where, $m \leq r < n$,

$$d_{nr} = \Big| \sum_{u \in A} [Q^n_{\underline{\theta}\theta}(u_{\underline{\theta}}, \varphi^{n^{-1}}_m(s_\theta)) - Q^r_{\underline{\theta}\theta}(\varphi^n_r(u_\theta), \varphi^{r^{-1}}_m(s_\theta))] \cdot [p_n((x^n_t)_{T'})^{-1}((u_t)_{T'}) - p_o((x_t)_{T'})^{-1}(\prod_{T'} \psi^{-1}_n(u_t))] \Big| \leq$$

$$\leq 2 \sup_{\substack{z\in S\\ s\in S_n}} |Q^n_{\underline{\theta}\theta}(\psi_n(z), \varphi^{n-1}_m(s)) - Q^r_{\underline{\theta}\theta}(\psi_r(z), \varphi^{r-1}_m(s))| \xrightarrow[r,n]{} 0$$

Let $\varepsilon > 0$. There exists $r \in N$ so that, for any $n > r$, we have $d_{nr} < \varepsilon$. Thus, we fix r.

$$e_{nr} = |\sum_{u\in A} Q^r_{\underline{\theta}\theta}(\varphi^n_r(u_{\underline{\theta}}), \varphi^{r-1}_m(s_\theta)) \cdot [p_n \circ (x^n_t)^{-1}_{T'}((u_t)_{T'}) - p_o(x_t)^{-1}_{T'}(\prod_{t\in T'} \psi^{-1}_n(u_t))]|$$

We note $B_v = \{ u=(u_t)_{t\in T'} \in S_n^{T'} / \varphi^{n-1}_n(u_t) = s_t,\ t\in T'-\{\underline{\theta}\},\ \varphi^n_r(u_{\underline{\theta}}) = v\}$

We have $e_{nr} = |\sum_{v\in \varphi^{r-1}_m(s_{\underline{\theta}})} Q^r_{\underline{\theta}\theta}(v, \varphi^{r-1}_m(s_\theta)) \sum_{u\in B_v} [p_n \circ (x^n_t)^{-1}_{T'}((u_t)_{T'}) - p_o(x_t)^{-1}_{T'}(\prod_{t\in T'} \psi^{-1}_n(u_t))]|$

$$\leq \sum_{v\in \varphi^{r-1}_m(s_{\underline{\theta}})} [Q^r_{\underline{\theta}\theta}(v, \varphi^{r-1}_m(s_\theta)) \cdot |\sum_{u\in B_v} [p_n \circ (x^n_t)^{-1}_{T'}((u_t)_{T'}) - p_o(x_t)^{-1}_{T'}(\prod_{t\in T'} \psi^{-1}_n(u_t))]|]$$

r being fixed, it results that the set $\varphi^{r}_{m}(s)$ is finite. It is enough to show that for any $v \in S_r$, $a'_n = |\sum_{u\in B_v} [p_n \circ ((x^n_t)_{T'})^{-1}(u) - p_o((x_t)_{T'})^{-1}(\prod_{t\in T'} \psi^{-1}_n(u_t))]|$ converges to 0, after n; a'_n is treated similarly to a_n, thus the set of the times considered in T_m being reduced.

Finally, it remains to show that, for fixed p, $p > m$, u fixed in S_p, $\lim_{n\to\infty} | p_{no} \circ \varphi^{n-1}_p(u)) - p_o(\psi^{-1}_p(u))| = 0$, which is obvious.

2.6. <u>Proposition</u>. Let $(Q_{tt'})_{t<t'\in T}$ be the uniform projective limit of the sequence of systems $((Q^n_{tt'})_{t<t'\in T_n})_N$. Let V be the vectorial space of the real functions defined on S generated by the set $\{\chi_{\psi^{-1}_n(s)} / n\in N, s\in S_n\}$ and let C be the set of the continuous real functions defined on S. We have

1) V is uniformly dense in C and

2) whatever $f \in C$ and $t < t'$ in T, the real application defined on S, $y \longmapsto Q_{tt'}(y,f)$ is continuous.

§3. The transformation of the space of states

3.1. <u>Definition</u>. Let (S,τ) be a topological space and $\sim$ an equivalence relation on S. Let $S' = S/\sim$ and $\pi: S \longrightarrow S'$ be the canonical surjection. We note $\mathcal{S} = \mathcal{B}(\tau)$, $\tau' = \{ D' / D' \subset S', \pi^{-1}(D') \in \tau\}$, $\mathcal{S}' = \mathcal{B}(\tau')$, $\mathcal{S}'' = \{ A' / A' \subset S', \pi^{-1}(A') \in \mathcal{S}\}$.

3.2. Lemma. Let the application $f : A \longrightarrow B$ and $\mathcal{A}=f^{-1}(\mathcal{P}(B))$. We consider the Borel algebras $\mathcal{T}_1 = \mathcal{B}(\{D/\ D\subset B\ ,\ f^{-1}(D)\in \mathcal{A}\})$, $\mathcal{T}_2 = \{D/\ D\subset B\ ,\ f^{-1}(D)\in\mathcal{B}(\mathcal{A})\}$. Then $\mathcal{T}_1=\mathcal{T}_2$.

Proof. Obviously $\mathcal{T}_1\subset\mathcal{T}_2$. $f^{-1}(\mathcal{T}_1)$ is a Borel algebra and $\mathcal{A}\subset f^{-1}(\mathcal{T}_1)$. Then $\mathcal{B}(\mathcal{A})\subset f^{-1}(\mathcal{T}_1)$ and $\mathcal{T}_2=\{D/\ f^{-1}(D)\in f^{-1}(\mathcal{T}_1)\}$. It remains to show that $\{D/\ f^{-1}(D)\in f^{-1}(\mathcal{T}_1)\}\subset\mathcal{T}_1$. Let $D\subset B$ and $C\in\mathcal{T}_1$ so that $f^{-1}(D)=f^{-1}(C)$. It results that $f^{-1}(D\Delta C)=\phi$. It is easy to show that $\mathcal{T}_1\supset\mathcal{P}(B-\mathrm{Im} f)$. Then $D\Delta C = E\in\mathcal{T}_1$. It results that $D = C\Delta E\in\mathcal{T}_1$

3.3. Remarks. 1) τ' is a topology and $\mathcal{S}''$ is a σ-algebra;
2) τ' is the maximum topology for which π is continuous;
3) $\mathcal{S}'\subset\mathcal{S}''$. If $\tau\subset\pi^{-1}(\mathcal{T}(S))$ then $\mathcal{S}'=\mathcal{S}''$.

3.4. Definition. Let $x=(x_t)_{t\in T}$ be a Markov process defined on the probability field (Ω, K, p) with the set of times T totally ordered and with the initial element t_o, having the set of states $(S,\mathcal{S})$ and the family of transition probabilities $(Q_{tt'})_{t<t'}$. Let $p_o=p\circ x_{t_o}^{-1}$ be the initial probability of the process x. We define on S the following equivalence relation for any $u,v\in S$, $u\sim v$ if and only if $Q_{tt'}(u,\ )=Q_{tt'}(v,\ )$, whatever $t<t'$ in T. For $S',\pi,\mathcal{S}'$ we preserve the notations introduced in 3.1. For any $t<t'$, we define the function $Q'_{tt'}: S'\times\mathcal{S}'\longrightarrow[0,1]$ by $Q'_{tt'}(u',A')=Q_{tt'}(u,\pi^{-1}(A'))$, where $u\in u'\in S'$ and $A'\in\mathcal{S}'$. Obviously, the definition of $Q_{tt'}$ does not depend on the choice of u in u'. We note for any $t\in T$, $x'_t=\pi\circ x_t$, $x'=\pi\circ x=(\pi\circ x_t)_{t\in T}$.

Lemma. Under the circumstances of the definition 3.4. let $f:S'\longrightarrow R$ be $\mathcal{S}'$-measurable, positive or bounded. Then, whatever $s<t$, $u'\in S'$, $u\in u'$, $Q'_{st}(u',f) = Q_{st}(u,f\circ\pi)$.

Proof. It is enough to proof this lemma for $f=\chi_{B'}$, $B'\in\mathcal{S}'$, because $\{\chi_{B'}/B'\in\mathcal{S}'\}$ is cloded to product and, obviously, $\mathcal{S}'=\mathcal{B}(\chi_{B'}/B'\in\mathcal{S}')$. We have, for any $s<t$, $u\in u'\in S'$, $Q'_{st}(u',\chi_{B'})=Q'_{st}(u',B')=Q_{st}(u,\pi^{-1}(B'))$ $=Q_{st}(u,\chi_{B'}\circ\pi)$.

3.5. Proposition. 1) The family of applications $(Q'_{tt'})_{t<t'}$, defined in 3.4. is a transition system from $(S',\mathcal{S}')$ to $(S',\mathcal{S}')$;

2) The process $x'=\pi\circ x$ is a Markov process with the transition system $(Q'_{tt'})_{t<t'}$ and the initial probability $p'_0=p_0\circ\pi^{-1}$.

Proof. 1) For any $t<t'$ in T, $Q'_{tt'}=Q_{tt'}(\pi(\cdot),\pi^{-1}(\cdot))$ is a probability in the second variable and measurable in the first. Let $s<t<v$ in T, $u\in u'\in S'$, $A'\in\mathcal{S}'$ we have $Q'_{st}Q'_{tv}(u',A')=Q'_{st}(u',Q'_{tv}(\cdot,A'))=$ $=Q_{st}(u,Q_{tv}(\cdot,\pi^{-1}(A')))=Q_{sv}(u,\pi^{-1}(A'))=Q'_{sv}(u',A')$.

2) Let $f: S'\longrightarrow R$ bounded and measurable and let $\mathcal{F}_t=\mathcal{B}(x_s/s\le t)$, $\mathcal{F}'_t=\mathcal{B}(x'_s/s\le t)$, $t\in T$. Let $s<t$. We have $\mathcal{F}_t>\mathcal{F}'_t$.

$$E(f\circ x'_t/\mathcal{F}'_s)=E(E(f\circ\pi\circ x_t/\mathcal{F}_s)/\mathcal{F}'_s)=E(Q_{st}(x_s,f\circ\pi)/\mathcal{F}'_s)=$$
$$=E(Q'_{st}(\pi\circ x_s,f)/\mathcal{F}'_s)=Q'_{st}(x'_s,f).$$

3.6. Proposition. Let $((S,\tau),(\psi_n)_N)$ be the projective limit of topological spaces of the projective system $((S_n,\tau_n)_N,(\psi^n_m)_{m<n})$ (according to 1.2.). We suppose that, for any $n\in N$, the topology is given by a metric d_n and we have :

1) $\limsup\limits_{\substack{m,n\to\infty\\ m<n}}\ \sup\limits_{b,c\in\psi^{n\,-1}_m(\psi_n(a))} d_n(b,c)=0$, for any $a\in S$;

2) there exists $\delta: S\times S\longrightarrow R_+$ so that $\delta(a,b)=\lim d_n(\psi_n(a),\psi_n(b))$, $a,b\in S$.

We define on S the following equivalence relation $a\approx b$ if and only if $\delta(a,b)=0$. Let $S''=S/\sim$. the function $d': S''\times S''\to R$ so that, for any $a\in a'\in S''$ and $b\in b'\in S''$, $d'(a',b')=\delta(a,b)$ is a metric on S''. The canonical surjection $\pi: S\longrightarrow S/_{\approx}=S''$ is continuous.

Proof. Let $a',b'\in S'$ and $a,a_1\in a'$, $b,b_1\in b'$,

$$|\delta(a_1,b_1)-\delta(a,b)|=\lim_{n\to\infty}|d_n(\psi_n(a_1),\psi_n(b_1))-d_n(\psi_n(a),\psi_n(b))|\le$$
$$\le\lim_{n\to\infty}[d_n(\psi_n(a),\psi_n(a_1))+d_n(\psi_n(b),\psi_n(b_1))]=0$$

Therefore d' is well defined. Obviously, d' is a metric. Let $(a_n)_N$ be a sequence of elements in S, with $\lim\limits_{n\to\infty}a_n=a$. Let $\varepsilon>0$. There exists $n_\varepsilon \in N$ so that, for $n > n_\varepsilon$, $\sup_{b,c \in \varphi_{n_\varepsilon}^{n-1}(\psi_{n_\varepsilon}(a))} d_n(b,c) < \varepsilon$.

There exists $m \in N$ so that, if $p > m$, then $a_p \in \psi_{n_\varepsilon}^{-1}(a)$. Therefore for any $p > m$, $n > n_\varepsilon$, $d_n(\psi_n(a_p), \psi_n(a)) < \varepsilon$. It results that $\lim d'(a_p, a) = 0$.

§4. The approximation of a stocastic process with finite processess

A Markov process being given, we want to find a sequence of finite processes with whose projective limit we can rediscover the given process.

4.1. Let the Markov process $y = (y_t)_{t\in T}$, defined on the probability field $(\Omega, \mathcal{K}, q)$ with states in the compact metric space (E,d) and the totally ordered, countable set of times T with the initial element t_0. Let $\mathcal{E}$ be the Borel algebra of the topology on E, $q_0 = p \circ y_{t_0}^{-1}$ the initial probability of the process y and $(R_{tt'})_{t<t'}$ a transition system for y.

Let $(S_n)_N$ be an increasing sequence of finite and measurable partitions of E so that :

(4.1.1.) $\lim\limits_{n\to\infty} \max\limits_{u\in S_n} \delta(u) = 0$, where, for $A \in \mathcal{E}$, $\delta(A) = \sup\limits_{a,b\in A} d(a,b)$ is the diameter of the set A;

(4.1.2.) For any $n \in N$, $u \in S_n$, $t<t'$ in T, the application $R_{tt'}(\cdot, u) : E \longrightarrow R$ is continuous.

The notations in §2 will be used further on.

We define the family of applications $(\varphi_m^n)_{m<n}$ so that for any $n, m \in N$, $\varphi_m^n : S_n \longrightarrow S_m$, $\varphi_m^n(u) = v$ if and only if $u \subset v$, whatever $u \in S_n$ and $v \in S_m$.

Let $(S, (\psi_n)_N)$ be the projective limit of the system $((S_n)_N, (\varphi_m^n)_{m<n})$, τ the projective limit topology and $\mathcal{S}$ the projective limit Borel algebra (according to 1.2.).

<u>Remark</u>. If the increasing sequence of partitions $(S_n)_N$ satifies (4.1.1.) and (4.1.2.) then $\mathcal{B}(\bigcup\limits_{n\in N} S_n) \subset \mathcal{E}$

Proof. Obviously, $\mathcal{B}(\bigcup_N S_n) \subset \mathcal{E}$. Let K be compact in E.

Let $A_n = \{ a \in S \ / \ d(a,K) > \frac{1}{n} \}$ and $B_n = \bigcup_{\substack{n \in N \\ u \in S_n \\ u \cap A_n \neq \phi,\ \delta(u) < \frac{1}{n}}} u \in \mathcal{B}(\bigcup_N S_n)$, B_n being the countable union of elements in $\mathcal{E}$. $CK = \bigcup_{n \in N} A_n \in \mathcal{B}(\bigcup_N S_n)$

$\mathcal{B}(\{ K \ / \ K$ compact in $S \}) = \mathcal{E} = \mathcal{B}(\bigcup_{n \in N} S_n)$.

4.2. __Definition.__ Let $\varphi : E \longrightarrow S$ the application defined for any $a \in E$ by $\varphi(a) = (u_n)_{n \in N}$, where $a \in u_n$, whatever $n \in N$. Since $(S_n)_N$ is an increasing sequence of partitions, φ is well defined. We shall note $x = (x_t)_{t \in T}$ the process $x = \varphi \circ y = (\varphi \circ y_t)_{t \in T}$. For any $n \in N$, we choose an application $g_n : S_n \longrightarrow S$, where $g_n(u) \in u$, whatever $u \in S_n$.

4.3. __Remark.__ φ is injective and measurable.

Proof. Since the Borel algebra is separate, it results that, whatever $a, b \in E$, there exists $n \in N$ and $u, v \in S_n$, $u \neq v$, so that $a \in u$, $b \in v$; therefore $\Psi_n(\varphi(a)) \neq \Psi_n(\varphi(b))$. For any $m \in N$, $u \in S_m$, $\varphi^{-1}(\Psi_m^{-1}(u)) = u \in \mathcal{E}$. Because $\mathcal{S} = \mathcal{B}(\bigcup_N \Psi_n^{-1}(\mathcal{S}_n))$ it results that φ is measurable.

4.4. __Lemma.__ For any $a \in S$, $t < t'$ in T and $A \in \mathcal{S}$ there exists $\lim_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), \varphi^{-1}(A))$.

Proof. Let $t < t'$ in T. The sequence $(g_n(\Psi_n(a)))_N$ is Cauchy (from 4.1.1.), therefore convergent in E. Let $b = \lim_{n \to \infty} g_n(\Psi_n(a)) \in E$. Let $A = \Psi_m^{-1}(u)$, $m \in N$, $u \in S_m$, $\varphi^{-1}(A) = u$.

$\lim_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), u) = R_{tt'}(b, u)$ (din 4.1.3.). Whatever $a \in S$, the set $\{ A \ / $ there exists $\lim_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), \varphi^{-1}(A)) \}$ forms a Borel algebra and includes $\bigcup_N \Psi_n^{-1}(\mathcal{S}_n)$.

4.5. __Definition.__ For any $t < t'$ in T, we define the function $Q_{tt'} : S \times \mathcal{S} \to [0,1]$ by $Q_{tt'}(a, A) = \lim_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), \varphi^{-1}(A))$ for any $a \in S$ and $A \in \mathcal{S}$.

4.6. __Remark.__ For any $a \in E$, whatever $A \in \mathcal{S}$, $Q_{tt'}(\varphi(a), A) = R_{tt'}(a, \varphi^{-1}(A))$.

Proof. For any $a\in E$, $A\in\bigcup_N S_n$, $Q_{tt'}(\varphi(a),A)=$

$=\lim_{n\to\infty} R_{tt'}(g_n(\psi_n(\varphi(a))),\varphi^{-1}(A)) = R_{tt'}(a,A)$ since

$\lim_{n\to\infty} g_n(\psi_n(\varphi(a)))=a$. The set $\mathcal{M}=\{A\in\mathcal{S}/\ Q_{tt'}(\varphi(a),A)=R_{tt'}(a,\varphi^{-1}(A))\}$ is closed to disjoint countable union and to increasing limit and includes $\bigcup_{n\in N} S_n$; it results that $\mathcal{M}$ is a Borel algebra.

4.7. Proposition. Whatever $b\in S$, there exists $a=\lim_{n\to\infty} g_n(\psi_n(b))\in E$ so that, for any $t<t'$ in T, $Q_{tt'}(b,\cdot) = Q_{tt'}(\varphi(a),\cdot)$.

Proof. Let $b\in S$. It is enough to show that there exists $a\in S$ so that, whatever $t<t'$, $m\in N$, $A\in S_m$, $Q_{tt'}(b,\psi_m^{-1}(A))=Q_{tt'}(\varphi(a),\psi_m^{-1}(A))$

We have $Q_{tt'}(b,\psi_m^{-1}(A))=\lim_{n\to\infty} R_{tt'}(g_n(\psi_n(b)),A)=R_{tt'}(a,A)=$

$=Q_{tt'}(\varphi(a),\psi_m^{-1}(A))$, where $a=\lim_{n\to\infty} g_n(\psi_n(b))$.

4.8. Proposition. The family $(Q_{tt'})_{t<t'}$ is a transition system on $(S,\mathcal{S})$.

Proof. Let $t<t'$ in T. For any $A\in\mathcal{S}$, $\lim_{n\to\infty} R_{tt'}(g_n(\psi_n(\cdot)),\varphi^{-1}(A))=$

$=Q_{tt'}(\cdot,A)$ is a limit of measurable applications, therefore $Q_{tt'}(\cdot,A)$ is measurable. Obviously, $Q_{tt'}(a,\cdot)$ is a probability, $a\in S$.

The Chapman-Kolmogorov property remains to be checked.

Let $t<t'<t''$ in T and $a\in S$. It is enough to chec

$Q_{tt'}Q_{t't''}(a,\psi_m^{-1}(b))=Q_{tt''}(a,\psi_m^{-1}(b))$, for any $m\in N$ and $b\in S_m$.

$$Q_{tt'}Q_{t't''}(a,\psi_m^{-1}(b)) = \int Q_{t't''}(z,\psi_m^{-1}(b))\,Q_{tt'}(a,dz) =$$

$$=\lim_{n\to\infty}\int R_{t't''}(g_n(\psi_n(z)),b)\,Q_{tt'}(a,dz) =$$

$$=\lim_{n\to\infty}\sum_{z\in S_n} R_{t't''}(g_n(z),b)\,Q_{tt'}(a,\psi_n^{-1}(z))=$$

$$=\lim_{n\to\infty}\lim_{k\to\infty}\sum_{z\in S_n} R_{t't''}(g_n(z),b)\,R_{tt'}(g_k(\psi_k(a)),z)$$

$$Q_{tt''}(a,\psi_m^{-1}(b)) = \lim_{n\to\infty} R_{tt''}(g_n(\psi_n(a)),b)$$

Let $a_{nk}=\left|\sum_{z\in S_n} R_{t't''}(g_n(z),b)\,R_{tt'}(g_k(\psi_k(a)),z)-R_{tt''}(g_n(\psi_n(a)),b)\right|$

$\leq b_{nk}+c_{nk}$, where

$$b_{nk} = |R_{tt''}(g_k(\psi_k(a)), b) - R_{tt''}(g_n(\psi_n(a)), b)| \xrightarrow[\substack{n,k\to\infty \\ n<k}]{} 0 \quad \text{and}$$

$$c_{nk} = \left|\sum_{z\in S_n} R_{t't''}(g_n(z), b)\cdot R_{tt'}(g_k(\psi_k(a)), z) - R_{tt''}(g_k(\psi_k(a)), b)\right| =$$

$$= \left|\sum_{z\in S_n} \int_z [R_{t't''}(g_n(z), b) - R_{t't''}(w, b)]\, R_{tt'}(g_k(\psi_k(a)), dw)\right| \leq$$

$$\leq \sup_{\substack{w,w'\in E \\ \psi_n(\varphi(w)) = \psi_n(\varphi(w'))}} |R_{t't''}(w', b) - R_{t't''}(w, b)| \xrightarrow[n\to\infty]{} 0$$

since $R_{tt'}(\cdot, b)$ is continuous, therefore uniformly continuous, (E,d) being compact.

4.9. Lemma. 1) $\varphi^{-1}(\mathcal{S}) = \mathcal{E}$.

2) We note, for t in T, $\mathcal{F}_t = \mathcal{B}(y_s, s\leq t)$. Then $\mathcal{F}_t = \mathcal{B}(x_s, s\leq t)$

Also, $\mathcal{B}(y_t) = \mathcal{B}(x_t)$ for any $t\in T$.

3) For any $a \in E$ and measurable and limited $f : S \longrightarrow R$, we have $R_{st}(a, f\circ\varphi) = Q_{st}(\varphi(a), f)$.

Proof. 1) φ is measurable therefore $\varphi^{-1}(\mathcal{S}) \subset \mathcal{E}$.

$$\mathcal{E} = \mathcal{B}(\bigcup_{n\in N} S_n) \text{ and } \varphi^{-1}(\mathcal{S}) \supset \bigcup_N S_n . \text{ So } \varphi^{-1}(\mathcal{S}) \supset \mathcal{E}.$$

2) Obvious.

3) Let $a \in E$, $m \in N$, $b \in S_m$. $Q_{st}(\varphi(a), \chi_{\psi_m^{-1}(b)}) = Q_{st}(\varphi(a), \psi_m^{-1}(b)) =$
$= \lim_{n\to\infty} R_{st}(g_n(\psi(a)), b) = R_{st}(a,b) = R_{st}(a, \chi_{\psi_m^{-1}(b)}\circ\varphi)$. The set of functions $\{\chi_{\psi_m^{-1}(b)} / m\in N,\ b \in S_m\}$ is closed to product. It results that for any measurable and limited $f : S \longrightarrow R$, $R_{st}(a, f\circ\varphi) = Q_{st}(\varphi(a), f)$.

4.10. Proposition. The process $x = (x_t)_{t\in T} = (\varphi\circ y_t)_{t\in T}$ defined on $(\Omega, \mathcal{K}, q)$ with states $(S, \mathcal{S})$ is a Markov process with the initial distribution $p_o = q_o\circ\varphi^{-1}$ and a transition system $(Q_{tt'})_{t<t'}$.

Proof. Let $s<t$ in T and $f : S \longrightarrow R$ be measurable, bounded.

$$E(f\circ(\varphi\circ y_t)/\mathcal{F}_s) = E((f\circ\varphi)\circ y_t/\mathcal{F}_s) = R_{st}(y_s, f\circ\varphi) =$$

$$= Q_{st}(\varphi\circ y_s, f).$$

Therefore $\varphi\circ y$ is a Markov process and has the transition system $(Q_{tt'})_{t<t'}$ and $q\circ(\varphi\circ y_{t_0})^{-1} = q_o\circ\varphi^{-1} = p_o$.

4.11. Definition. We consider on S the equivalence relation $\sim$ defined by means of the transition system $(Q_{tt'})_{t<t'}$, according to 3.3. We shall preserve the notations introduced in §3 for S', π, x', τ', φ'. We note $\alpha : E \longrightarrow S'$ the application $\alpha = \pi \circ \varphi$. $x' = \alpha \circ y = (\alpha \circ y_t)_{t \in T}$.

4.12. Definition. For $n \in N$, we define on S_n the metric d_n so that $d_n(a,b) = d(g_n(a), g_n(b))$, $a, b \in S_n$.

Remark. The sequence of metric spaces $(S_n, d_n)_{n \in N}$ satisfies (3.6.1. and (3.6.2.). According to 3.6., we shall find the equivalence relation $\approx$ on S, $S'' = S/_{\approx}$ and metric d' on S''.

4,13. Proposition. We suppose in the following that the process $(y_t)_{t \in T}$ satisfies the condition ;

(4.13.1) For any $a, b \in E$, there exists $t < t'$ in T, so that $R_{tt'}(a,\cdot) \neq R_{tt'}(b,\cdot)$.

Then we shall have 1) The relations $\sim$ and $\approx$ coincide (therefore $S' = S''$).

2) For any $a, b \in E$, we have $d(a,b) = d'(\alpha(a), \alpha(b))$.

Proof. Let $a, b \in S$. We suppose that $\lim d_n(\Psi_n(a), \Psi_n(b)) = 0$.(that is $a \approx b$). Let $t < t'$ in T, $m \in N$, $c \in S_m$. We have $Q_{tt'}(a, \Psi_m^{-1}(c)) = \lim\limits_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), c) = \lim\limits_{n \to \infty} R_{tt'}(g_n(\Psi_n(b)), c) = Q_{tt'}(b, \Psi_m^{-1}(c))$.
It results that $a \sim b$.

Reciprocally, we suppose $a \sim b$. Let $a' = \lim g_n(\Psi_n(a)) \in E$. and $b' = \lim g_n(\Psi_n(b)) \in E$. We suppose that $d(a',b') \neq 0$. Therefore there exists $t < t'$ in T, $n \in N$, $c \in S_n$ so that $R_{tt'}(a',c) \neq R_{tt'}(b',c)$. We have $Q_{tt'}(a, \Psi_n^{-1}(c)) = \lim\limits_{n \to \infty} R_{tt'}(g_n(\Psi_n(a)), c) = R_{tt'}(a',c) \neq$ $\neq Q_{tt'}(b, \Psi_n^{-1}(c))$. Therefore $\pi(a) \neq \pi(b)$. It is absurd. It results that $d(a',b') = \lim\limits_{n \to \infty} d(g_n(\Psi_n(a)), g_n(\Psi_n(b))) = 0$ and $a \approx b$.

2) Let $a, b \in E$. We have $d'(\alpha(a), \alpha(b)) = \lim d(g_n(\Psi_n(\varphi(a))), g_n(\Psi_n(\varphi(b)))) =$ $= d(a,b)$.

4.14. Theorem. 1) α is an isometry between (E,d) and (S',d') and the topology τ' is just the topology of the d' metric.

2) For any $a\in E$, $A\in \mathcal{E}$, t, t' in T, we have
$R_{tt'}(a,A) = Q'_{tt'}(\alpha(a),\alpha(A))$ and $\rho\circ x_{t_0}^{-1} = p_0\circ\pi^{-1}$.

An application which satisfies 4.14.1. and 4.14.2. will be called isomorphism between processes x and y.

Proof. 1) From 4.13., it results that α is an injection. Let $a'\in S'$ and $a\in\pi^{-1}(a)$. The sequence $(g_n(\psi_n(a)))_N$ is convergent in E. Let $b=\lim\limits_{n\to\infty} g_n(\psi_n(a))$. It remains to show that $a\sim\varphi(b)$. Let $t<t'$ in T, $n\in N$, $c\in S_n$. We have $Q_{tt'}(\varphi(b),\psi_n^{-1}(c))=R_{tt'}(b,c)=$
$=\lim\limits_{n\to\infty} R_{tt'}(g_n(\psi_n(a)),c)=Q_{tt'}(a,\psi_n^{-1}(c))$. It results that $a'=\pi(a)=$
$=\pi(\varphi(b))=\alpha(b)$. Let us note with τ'' the topology of the metric d' on S'. We shall show that $\tau''=\tau'$. Let $a'\in S'$, $\varepsilon>0$, $a\in\pi^{-1}(a')$. We shall note $B'_\varepsilon(a')=\{b'\in S' \,/\, d'(a',b')<\varepsilon\}$. $\pi^{-1}(B'(a'))=$
$=\{b\in S \,/\, \lim\limits_{n\to\infty} d_n(\psi_n(b),\psi_n(a))<\varepsilon\}$. We shall show that $\pi^{-1}(B'_\varepsilon(a'))\in\tau$ and it will follow that $B'_\varepsilon(a')\in\tau'$. Let $b\in\pi^{-1}(B'_\varepsilon(a'))$ and $\lim\limits_{n\to\infty} d_n(\psi_n(b),\psi_n(a))<\varepsilon$. There exists $k\in N$, $r>0$ so that $\rho(\psi_k(b))<\varepsilon-r$. It follows that, for any $c\in\psi_k^{-1}(\psi_k(b))$,
$\lim\limits_{n\to\infty} d_n(\psi_n(c),\psi_n(a))\leq \lim\limits_{n\to\infty} d_n(\psi_n(c),\psi_n(b))+\lim\limits_{n\to\infty} d_n(\psi_n(b),\psi_n(a))<\varepsilon$.

Therefore $\psi_k^{-1}(\psi_k(b))\subset\pi^{-1}(B'_\varepsilon(a'))$.

That is $\pi^{-1}(B'_\varepsilon(a'))\in\tau$.

In order to show that $\tau'\subset\tau''$ we shall proof that α is τ'-continuous.

Let F' be closed in τ'. We suppose by reductio ad absurdum that $\alpha^{-1}(F')$ is not closed in E. Therefore, there exists $a\in E-\alpha^{-1}(F')$ and $(a_n)_N$ a sequence in $\alpha^{-1}(F')$ so that $\lim\limits_{n\to\infty} a_n=a$. Since (S,τ) is compact, there exists a convergent sub-sequence $(b_n)_N$ of $(\varphi(a_n))_N$, $\lim\limits_{n\to\infty} b_n=b$. Since $\pi^{-1}(F')$ is closed, $b\in\pi^{-1}(F')$. Let $t<t'$ in T, $m\in N$, $c\in S_m$. We have $\lim\limits_{n\to\infty} Q_{tt'}(b_n,\psi_m^{-1}(c))=\lim\limits_{n\to\infty} R_{tt'}(a_n,c)=$

$R_{tt'}(a,c)=Q_{tt'}(\varphi(a),\psi_m^{-1}(c))$, $\lim_{n\to\infty} Q_{tt'}(b_n,\psi_m^{-1}(c))=Q_{tt'}(b,\psi_m^{-1}(c))$, since, for any $\varepsilon>0$, there exists n_ε so that , if $n,k>n_\varepsilon$, $\psi_k(b_n)\subset\psi_{n_\varepsilon}(b)$ and if $d,d'\in\psi_{n_\varepsilon}(b)$, then $|R_{tt'}(d,c)-R_{tt'}(d',c)|<\varepsilon$. We have $|Q_{tt'}(b_n,\psi_m^{-1}(c))-Q_{tt'}(b,\psi_m^{-1}(c))|=\lim_{n\to\infty} R_{tt'}(g_k(\psi_k(b_n)),c)-R_{tt'}(g_k(\psi_k(b),c))|<\varepsilon$. It results that, for any $t<t'$ in T, $m\in N$, $c\in S_m$, we have $Q_{tt'}(\varphi(a),\psi_m^{-1}(c))=Q_{tt'}(b,\psi_m^{-1}(c))$, ,that is $\alpha(a)=\pi(\varphi(a))=\pi(b)\in F'$. This contradicts the hypothesis of reductio ad absurdum. Therefore, α is continuous. Let G be open in $\mathcal{E}'$ and a' a certain element in G. $\alpha^{-1}(G)$ is open in E, so, there exists $\varepsilon>0$, so that $B_\varepsilon(\alpha^{-1}(a'))\subset\alpha^{-1}(G)$. From 4.13. it follows that $B'_\varepsilon(a')\subset G$. That is $G\in\mathcal{T}''$.

2) Let $t<t'$ in T, $a\in E$, $A\in\mathcal{E}$. We have $Q_{tt'}(\alpha(a),\alpha(A))=$ $=Q_{tt'}(\varphi(a),\pi^{-1}(\alpha(A)))=R_{tt'}(a,\varphi^{-1}(\pi^{-1}(\alpha(A))))=R_{tt'}(a,A)$.

4.15. Let $(T_n)_N$ be an increasing sequence of finite sub-sets of T with $T=\bigcup_{n\in N} T_n$ and $t_o\in T_o$. (Obviously, T must be countable). For established n, we shall use the notations introduced in 2.1. for $\bar{t}$, $\underline{t}$, θ, T' ($\theta=\max T_n$, $T'=T_n-\{\theta\}$, $\bar{t}$ is the successor of t in T_n).

For any $n\in N$ and $k_o\in N$, we define the probability p^k_{on} on S_k by $p^k_{on}(A)=q_o(A)$, whatever $A\in S_k$ and the transition system on S_k $(Q^{nk}_{tt'})_{t<t'\in T_n}$, where, for $t\in T'_n$, $Q^{nk}_{t\bar{t}}(u,v)=R_{t\bar{t}}(g_k(u),v)$, whatever $u,v\in S_k$. Obviously, $Q^{nk}_{tt'}=Q^{nk}_{t\bar{t}}Q^{nk}_{\bar{t}\bar{\bar{t}}}\dots Q^{nk}_{\underline{t}'t'}$, whatever $t<t'$ in T_n .

We note p^k_n the probability on $S_k^{T_n}$ for which the Markov process $x^k_n=(x^{n,k}_t)_{t\in T_n}$ of the projections of $S_k^{T_n}$ has the initial probability p_{on} and the transition system $(Q^{nk}_{tt'})_{t<t'\in T_n}$.

Choosing a strictly incresing sequence of natural numbers $(k_n)_N$, for any $n\in N$, we shall note $\varphi^{k_n}_{k_m}$ the function $\varphi^{k_n}_{k_m}:S_{k_n}\to S_{k_m}$ defined that $\varphi^{k_n}_{k_m}(u)=v$ if and only if $u\subset v$.

4.16. <u>Remark</u>. For any subsequence $(k_n)_N$ of the natural numbers sequence, the system $((S_{k_n}),(\varphi^{k_n}_{k_m}))$ has $(S,(\Psi_{k_n})_{n\in N})$ as its projective limit. The sequence of probabilities $(p^{k_n}_{on})_{n\in N}$ has as a projective limit the probability $p_o=q_o\circ\varphi^{-1}$.

4.17. Lemma. There exists $(k_n)_N$ a sequence of strictly increasing natural numbers so that, the transition system on S, $(Q_{tt'})_{t<t'\in T}$ is the uniform projective limit of the system $((Q^{nk_n}_{tt'})_{t<t'\in T_n})_{n\in N}$ (see 2.4.).

Proof. Let $(\alpha_n)_{n\in N}$ be a real strictly decreasing to 0 sequence (eventually $\alpha_n=\frac{1}{n}$, $n\in N$). We shall build the sequence $(k_n)_{n\in N}$ by induction so that, for any $n\in N$ and whatever $m<n$, $b\in S_{k_m}$ and $t<t'$ in T_m, we have

$$\sup_{a\in S}|Q^{nk_n}_{tt'}(\Psi_{k_n}(a),\varphi^{k_n^{-1}}_{k_m}(b))-Q_{tt'}(a,\Psi^{-1}_{k_m}(b))|<\alpha_n$$

We suppose that we found $k_1,k_2,\dots,k_{n-1}$, with this property. We must show that, fixing $n, t<t'$ in T_n and $b\in S_{k_m}$, where $m<n$

We have $\lim\limits_{k\to\infty}\sup\limits_{a\in S}|Q^{nk}_{tt'}(\Psi_k(a),\varphi^{k^{-1}}_{k_m}(b))-Q_{tt'}(a,\Psi^{-1}_{k_m}(b))|=0$

For any $a\in S$, we have $|Q^{nk}_{tt'}(\Psi_k(a),\varphi^{k^{-1}}_{k_m}(b))-Q_{tt'}(a,\Psi^{-1}_{k_m}(b))|=$

$$=|Q^{nk}_{tt'}(\Psi_k(a),\varphi^{k^{-1}}_{k_m}(b))-\lim_{i\to\infty}R_{tt'}(g_i(\Psi_i(a)),b)|\leq$$

$$\leq|Q^{nk}_{tt'}(\Psi_k(a),\varphi^{k^{-1}}_{k_m}(b))-R_{tt'}(g_k(\Psi_k(a)),b)|+$$

$$+\lim_{i\to\infty}|R_{tt'}(g_k(\Psi_k(a)),b)-R_{tt'}(g_i(\Psi_i(a)),b)|$$

Since conditions (4.1.1.) and (4.1.2.) are fulfilled and since $R_{tt'}(\cdot,b)$ is uniformly continuous, (E being compact), it follows that there exists $r_1\in N$, $r_1>k_{n-1}$ so that, for any $i,j>r_1$, we have

$$|R_{tt'}(g_j(\Psi_j(a)),b)-R_{tt'}(g_i(\Psi_i(a)),b)|<\alpha_n/2$$

$$|Q^{n,k}_{tt'}(\Psi_k(a),\varphi^{k^{-1}}_{k_m}(b))-R_{tt'}(g_k(\Psi_k(a)),b)|=$$

$$= \left|\sum_{u\in S_k} Q^{nk}_{\bar t t'}(u, \varphi^{k-1}_{k_m}(b))\cdot R_{t\bar t}(g_k(\psi_k(a)), u) - \int R_{\bar t t'}(v,b) R_{t\bar t}(g_k(\psi_k(a)), dv)\right| =$$

$$= \left|\int [Q^{nk}_{\bar t t'}(\psi_k(\varphi(v)), \varphi^{k-1}_{k_m}(b)) - R_{\bar t t'}(v,b)]\, R_{t\bar t}(g_k(\psi_k(a)), dv)\right| \le$$

$$\le \sup_{v\in E} |Q^{nk}_{\bar t t'}(\psi_k(\varphi(v)), \varphi^{k-1}_{k_m}(b)) - R_{\bar t t'}(v,b)| \le$$

$$\le \sup_{v\in E} |Q^{nk}_{\bar t t'}(\psi_k(\varphi(v)), \varphi^{k-1}_{k_m}(b)) - R_{\bar t t'}(g_k(\psi_k(\varphi(v))), b)| +$$

$$+ \sup_{v\in E} |R_{\bar t t'}(g_k(\psi_k(\varphi(v))), b) - R_{\bar t t'}(v,b)|$$

There exists $r_2 \in N$ so that, for any $i > r_2$

$$|R_{\bar t t'}(g_k(\psi_i(\varphi(v))), b) - R_{\bar t t'}(v,b)| < \frac{\alpha_n}{4}$$

$$\sup_{v\in E} |Q^{nk}_{\bar t t'}(\psi_k(\varphi(v)), \varphi^{k-1}_{k_m}(b)) - R_{\bar t t'}(g_k(\psi_k(\varphi(v))), b)| =$$

$$= \sup_{v\in E} \left|\int [Q^{nk}_{\bar t t'}(\psi_k(\varphi(w)), b) - R_{\bar t t'}(w,b)]\, R_{t\bar t}(g_k(\psi_k(\varphi(v))), dw)\right| \le$$

$$\le \sup_{w\in E} |Q^{nk}_{\bar t t'}(\psi_k(\varphi(w)), b) - R_{\bar t t'}(w,b)|$$

and this reasoning is repeated a finite number of times (depending on the number of elements of the set T_n). Thus, we find $r_1, r_2, \dots, r_c$

We choose $k_n = \max(r_1, r_2, \dots, r_c)$.

4.18. It remains only to formulate the following

<u>Theorem</u>. Let $y=(y_t)_{t\in T}$ be a Markov process defined on $(\Omega, \mathcal{K}, q)$ with states in the compact metric space (E,d) and with countable set of times T with the initial element t_o, totally ordered. Let $(T_n)_{n\in N}$ be an increasing sequence of measurable partitions of S checking conditions (4.1.1.) and (4.1.2.).

Then, there exists a sub-sequence $(k_n)_{n\in N}$ of N, so that the sequence of Markov processes $(x_n^{k_n})_{n\in N}$ (where, for any $n \in N$, $x_n^{k_n}$ is defined according to 4.15.) has as its projective limit the process x (from 4.10), and the process $x' = \pi\circ x$ may be defined with the process y (according to 4.14.) if y also fulfils the hypothesys (4.13.1.).