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ISSN 0250 3638

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DIFFERENCE EQUATIONS

by

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PREPRINT SERIES IN MATHEMATICS

No.25/1985

BUCURESTI

Med 19218

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May 1983

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ASYMPTOTIC EXPANSIONS

FOR COUPLED DIFFERENTIAL AND DIFFERENCE EQUATIONS

WITH APPLICATIONS TO A PROBLEM IN HYDRAULICS

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1. Introduction

A simple problem in hydraulics if considered by the method of characteristics leads to coupled differential and difference equations. Such equations may be solved by using a step-by-step procedure. If the step-size is small this is inconvenient and does not allow a simple discussion of the role of parameters involved. It is why asymptotic expansions seem to be a better way to compute.

We shall describe such expansions and discuss them in connection with the study of nonstationary motion of the water in a system consisting of a reservoir, pressure tunnel, throttled surge tank and valve.

2. The model problem

A hydraulic system consisting of reservoir, pressure tunnel, throttled surge tank and valve is considered. The nonstationary motion of the water in the system is studied corresponding to the flow suddenly passing from zero to some constant value. The mathematical model after the linearization is

$$\frac{\partial H}{\partial t} + \frac{c^2}{gf} \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial H}{\partial x} + \frac{1}{gf} \frac{\partial Q}{\partial t} = 0$$

$$H(0, t) = 0, \quad Q(0, t) = \hat{q} + F \frac{dz(t)}{dt}$$

$$H(l, t) = z(t) + R'_0 \frac{dz(t)}{dt}$$

with zero initial conditions.

We consider a nondimensional form by setting

$$y(\xi) = \frac{1}{l} z\left(\frac{\xi l}{|\hat{q}|}\right), \quad h(\xi, \zeta) = \frac{1}{l} H\left(l\xi, \frac{\xi l f}{|\hat{q}|}\right)$$

$$q(\xi, \zeta) = \frac{1}{|\hat{q}|} Q\left(l\xi, \frac{\xi l f}{|\hat{q}|}\right)$$

We get the equations

$$\frac{\partial h}{\partial \zeta} + \frac{c^2}{gl} \frac{\partial q}{\partial \xi} = 0, \quad \frac{\partial h}{\partial \xi} + \frac{|\hat{q}|^2}{glf^2} \frac{\partial q}{\partial \zeta} = 0$$

$$h(0, \zeta) = 0, \quad q(1, \zeta) = \frac{\hat{q}}{|\hat{q}|} + \frac{F}{f} y'(\zeta)$$

$$h(1, \zeta) = y(\zeta) + \frac{R'_0 |\hat{q}|}{lf} y'(\zeta)$$

Introduce Riemann invariants $q = u - v$, $h = \frac{c|\hat{q}|}{glf}(u + v)$.

Then

$$\frac{\partial u}{\partial \xi} + \frac{|\hat{q}|}{cf} \frac{\partial u}{\partial \zeta} = 0, \quad \frac{\partial v}{\partial \xi} - \frac{|\hat{q}|}{cf} \frac{\partial v}{\partial \zeta} = 0$$

$$u(0, \zeta) + v(0, \zeta) = 0, \quad u(1, \zeta) - v(1, \zeta) = \frac{\hat{q}}{|\hat{q}|} + \frac{F}{f} y'(\zeta)$$

$$\frac{c|\hat{q}|}{glf} [u(1, \zeta) + v(1, \zeta)] = y(\zeta) + \frac{R'_0 |\hat{q}|}{lf} y'(\zeta)$$

It follows that

$$u(\xi, \zeta) = u\left(\xi - \frac{cf}{|\hat{q}|} \zeta, 0\right) \quad \text{for} \quad \zeta \leq \frac{|\hat{q}|}{cf} \xi$$

$$u(\xi, \zeta) = u\left(0, \zeta - \frac{|\hat{q}|}{cf} \xi\right) \quad \text{for} \quad \zeta \geq \frac{|\hat{q}|}{cf} \xi$$

$$v(\xi, \zeta) = v\left(\xi + \frac{cf}{|\hat{q}|} \zeta, 0\right) \quad \text{for} \quad \zeta \leq \frac{(1-\varepsilon)|\hat{q}|}{cf}$$

$$v(\xi, \zeta) = v\left(1, \zeta - \frac{(1-\varepsilon)|\hat{q}|}{cf}\right) \quad \text{for} \quad \zeta > \frac{(1-\varepsilon)|\hat{q}|}{cf}$$

and from here

$$u(1, z) = 0 \text{ for } 0 \leq z \leq \frac{|\hat{q}|}{cf}, \quad v(0, z) = 0, \quad 0 \leq z \leq \frac{|\hat{q}|}{cf}$$

$$u(1, z) = u(0, z - \frac{|\hat{q}|}{cf}) \text{ for } z > \frac{|\hat{q}|}{cf}$$

$$v(0, z) = v(1, z - \frac{|\hat{q}|}{cf}) \text{ for } z > \frac{|\hat{q}|}{cf}$$

Denote $v(1, z) = w(z)$ and use the boundary conditions to get finally

$$\frac{F}{f} y'(\bar{z}) + w(\bar{z}) + w(\bar{z} - \frac{2|\hat{q}|}{cf}) + \frac{\hat{q}}{|\hat{q}|} = 0$$

$$\frac{R'_0 |\hat{q}|}{gfl} y'(\bar{z}) + y(\bar{z}) = \frac{c|\hat{q}|}{gfl} [w(\bar{z}) - w(\bar{z} - \frac{2|\hat{q}|}{cf})]$$

$$y(0) = 0, \quad w(\bar{z}) = 0 \text{ for } \bar{z} \leq 0.$$

Introduce the fundamental parameters

$$\varepsilon = \frac{2|\hat{q}|}{cf}, \quad k = \frac{c|\hat{q}|}{gfl}, \quad \gamma = \frac{2f^2 g R'_0}{|\hat{q}| F}$$

and write the model in the standard form

$$y'(\bar{z}) = -\frac{f}{kF} y(\bar{z}) - \frac{2f}{F} w(\bar{z} - \varepsilon) - \frac{f}{F} \frac{\hat{q}}{|\hat{q}|}$$

$$w(\bar{z}) = \frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} w(\bar{z} - \varepsilon) + \frac{1}{k} \frac{1}{1 + \frac{\gamma\varepsilon}{4}} y(\bar{z}) - \frac{\frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

Remark that $\frac{\gamma\varepsilon}{4} = \frac{fgR'_0}{cF}$. Remark also that

$$u(\xi, z) = -w(z - \frac{\hat{q}}{cf} (1 + \xi)) \quad , \quad v(\xi, z) = w(z - \frac{\hat{q}}{cf} (1 - \xi))$$

$$h(\xi, z) = \frac{c|\hat{q}|}{gfl} [w(z - \frac{|\hat{q}|}{cf} (1 - \xi)) - w(z - \frac{|\hat{q}|}{cf} (1 + \xi))]$$

hence our model allows the study of the level of water in the surge tank and of the pressure in the pressure tunnel.

The above model will be used for large values of the coefficient γ of discharge ; for smaller values up to the order of $\frac{1}{\sqrt{\varepsilon}}$

we set

$$y(\bar{z}, \varepsilon) = \sqrt{\varepsilon} \hat{y}(\frac{\bar{z}}{\sqrt{\varepsilon}}, \varepsilon) \quad , \quad w(\bar{z}, \varepsilon) = \hat{w}(\frac{\bar{z}}{\sqrt{\varepsilon}}, \varepsilon)$$

and we get for \hat{y}, \hat{w} the equations

$$\hat{y}'(\sigma) = -\sqrt{\varepsilon} \frac{\frac{f}{kF}}{1 + \frac{\gamma\varepsilon}{4}} \hat{y}(\sigma) - \frac{\frac{2f}{F}}{1 + \frac{\gamma\varepsilon}{4}} \hat{w}(\sigma - \sqrt{\varepsilon}) - \frac{\frac{f}{F}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$\hat{w}(\sigma) = \frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \hat{w}(\sigma - \sqrt{\varepsilon}) + \frac{\sqrt{\varepsilon}}{k(1 + \frac{\gamma\varepsilon}{4})} \hat{y}(\sigma) - \frac{\frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

We shall denote

$$A = -\frac{\frac{f}{kF}}{1 + \frac{\gamma\varepsilon}{4}}, \quad B = -\frac{\frac{2f}{F}}{1 + \frac{\gamma\varepsilon}{4}}, \quad f_1 = -\frac{\frac{f}{F}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$G = \frac{1}{k(1 + \frac{\gamma\varepsilon}{4})}, \quad H = -\frac{\frac{\gamma\sqrt{\varepsilon}}{2}}{1 + \frac{\gamma\varepsilon}{4}}, \quad f_2 = -\frac{\frac{\gamma\sqrt{\varepsilon}}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

and the model can be written

$$\hat{y}'(\sigma) = \sqrt{\varepsilon} A \hat{y}(\sigma) + B \hat{w}(\sigma - \sqrt{\varepsilon}) + f_1$$

$$\hat{w}(\sigma) = \sqrt{\varepsilon} G \hat{y}(\sigma) + (1 + \sqrt{\varepsilon} H) \hat{w}(\sigma - \sqrt{\varepsilon}) + \sqrt{\varepsilon} f_2$$

It corresponds to a critical case.

3. The noncritical case (Large coefficient of discharge)

The model corresponds to the general problem

$$\dot{x}(t) = Ax(t) + Bz(t - \varepsilon) + f_1(t) \quad x(t_0) = x_0$$

$$z(t) = Gx(t) + Hz(t - \varepsilon) + f_2(t) \quad z(t) = \varphi(t), \quad -\varepsilon \leq t \leq 0$$

It is assumed that H has the spectrum uniformly inside the unit circle. In our specific situation t is replaced by σ , x by y and z by w ; H corresponds to the number $\frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}}$ and an inequality of the form

$$-\rho \leq \frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \leq \rho \quad \text{means} \quad \frac{1 - \rho}{1 + \rho} \leq \frac{\gamma\varepsilon}{4} \leq \frac{1 + \rho}{1 - \rho}$$

and these inequalities describe the range of values for the coefficient of discharge considered in the asymptotic expansions we shall establish.

The problem is a special case of a situation considered elsewhere [1]. The first approximation is defined by

$$\tilde{x}'(t) = \tilde{A}x + \tilde{f}(t) \quad , \quad \tilde{x}(t_0) = x_0 \quad , \quad \tilde{z}(t) = G\tilde{x}(t) + H\tilde{z}(t-\varepsilon) + f_2(t)$$

$$\tilde{z}(t) = \varphi(t) \quad -\varepsilon \leq t \leq t_0 \quad ,$$

$$\tilde{A} = A + B(I-H)^{-1}G \quad , \quad \tilde{f}(t) = f_1(t) + B(I-H)^{-1}f_2(t)$$

In our specific situation

$$A = -\frac{kF}{1 + \frac{\gamma\varepsilon}{4}} \quad , \quad B = -\frac{\frac{2f}{F}}{1 + \frac{\gamma\varepsilon}{4}} \quad , \quad f_1 = -\frac{\frac{f}{F}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$G = \frac{\frac{1}{k}}{1 + \frac{\gamma\varepsilon}{4}} \quad , \quad H = \frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \quad , \quad f_2 = -\frac{\frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

hence $\tilde{y}'(\tau) = -\frac{4f}{kF\gamma\varepsilon} \tilde{y}(\tau) \quad , \quad \tilde{y}(0) = 0 \quad \text{and} \quad \tilde{y}(\tau) = 0.$

$$\text{Then} \quad \tilde{w}(\tau) = \frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \tilde{w}(\tau - \varepsilon) - \frac{\frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$\tilde{w}(\tau) = 0 \quad \tau < 0$$

For the solution of this difference equation we have

$$\tilde{w}(\tau) = - \left[\sum_{j=0}^{[\frac{\tau}{\varepsilon}]} \left(\frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \right)^j \right] \frac{\frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \frac{\hat{q}}{|\hat{q}|} = - \frac{\hat{q}}{2|\hat{q}|} \left[1 - \left(\frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \right)^{[\frac{\tau}{\varepsilon}] + 1} \right]$$

hence

$$\tilde{h}(\tau, \tau) = \frac{c\hat{q}}{2g\ell f} \left(\frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \right)^{[\frac{\tau}{\varepsilon} - \frac{1+\varepsilon}{2}]} \left\{ 1 - \left(\frac{1 - \frac{\gamma\varepsilon}{4}}{1 + \frac{\gamma\varepsilon}{4}} \right)^{[\frac{\tau}{\varepsilon} - \frac{1+\varepsilon}{2}] - [\frac{\tau}{\varepsilon} - \frac{1+\varepsilon}{2}]} \right\}$$

If we return to the initial variables

$$\tilde{H}(x, t) = \frac{c\hat{q}}{2g\ell} \left(\frac{cF - g\ell R'_c}{cF + g\ell R'_c} \right)^{[\frac{tc-x}{2\ell} - \frac{1}{2}]} \left\{ 1 - \left(\frac{cF - g\ell R'_c}{cF + g\ell R'_c} \right)^{[\frac{tc+x}{2\ell} - \frac{1}{2}] - [\frac{tc-x}{2\ell} - \frac{1}{2}]} \right\}$$

For $\frac{\gamma\varepsilon}{4} < 1$ that is $R'_c < \frac{cF}{g\ell}$ the pressure is monotonically decreasing while for $R'_c > \frac{cF}{g\ell}$ it is oscillatory and

for $R'_c \rightarrow \infty$ gives the usual water hammer in the pressure tunnel. According to the general theory in [1] the expansion is uniformly valid for $t \geq 0$ and the error is of order ε .

4. The critical case
(small coefficient of discharge)

To study the critical case we shall use the same notations as in the model problem.

$$y'(\sigma) = \sqrt{\epsilon} Ay(\sigma) + Bw(\sigma - \sqrt{\epsilon}) + f_1(\sigma)$$

$$w(\sigma) = \sqrt{\epsilon} Gy(\sigma) + (I + \sqrt{\epsilon} H)w(\sigma - \sqrt{\epsilon}) + \sqrt{\epsilon} f_2(\sigma)$$

$$y(0) = y_0, \quad w(\sigma) = \varphi(\sigma), \quad -\sqrt{\epsilon} \leq \sigma < 0.$$

The reduced model is

$$y'(\sigma) = By(\sigma) + f_1(\sigma), \quad w'(\sigma) = Gy(\sigma) + Hw(\sigma) + f_2(\sigma)$$

$$y(0) = y_0, \quad w(0) = \varphi(0).$$

Denote by \tilde{y} the first component of the solution, and by \tilde{w} the second one; define \tilde{w} from the difference equation

$$w(\sigma) = \sqrt{\epsilon} G\tilde{y}(\sigma) + (I + \sqrt{\epsilon} H)\tilde{w}(\sigma - \sqrt{\epsilon}) + \sqrt{\epsilon} f_2(\sigma)$$

$$w(\sigma) = \varphi(\sigma) \quad \sigma < 0$$

$$\text{Write } y(\sigma, \epsilon) = \tilde{y}(\sigma) + \sqrt{\epsilon} \eta(\sigma, \epsilon)$$

$$w(\sigma, \epsilon) = \tilde{w}(\sigma, \epsilon) + \sqrt{\epsilon} \mathcal{J}(\sigma, \epsilon)$$

To validate the expansion we have to prove that η and \mathcal{J} are bounded. From the equations for y and w we get equations for η and \mathcal{J} .

$$\eta'(\sigma, \epsilon) = \sqrt{\epsilon} A \eta(\sigma, \epsilon) + B \mathcal{J}(\sigma - \sqrt{\epsilon}, \epsilon) + A \tilde{y}(\sigma) + \frac{1}{\sqrt{\epsilon}} B [\tilde{w}(\sigma - \sqrt{\epsilon}, \epsilon) - \tilde{w}(\sigma)]$$

$$\mathcal{J}(\sigma, \epsilon) = \sqrt{\epsilon} G \eta(\sigma, \epsilon) + (I + \sqrt{\epsilon} H) \mathcal{J}(\sigma - \sqrt{\epsilon}, \epsilon)$$

Our main assumption, which is satisfied in the model problem, is that $|(I + \sqrt{\epsilon} H)^k|$ is bounded (with respect to k).

We write

$$\check{w}(\sigma) - \sqrt{\epsilon} G \check{y}(\sigma) - (I + \sqrt{\epsilon} H) \check{w}(\sigma - \sqrt{\epsilon}) - \sqrt{\epsilon} f_2(\sigma) =$$

$$= \check{w}(\sigma) - \check{w}(\sigma - \sqrt{\epsilon}) - \sqrt{\epsilon} G \check{y}(\sigma) - \sqrt{\epsilon} H \check{w}(\sigma - \sqrt{\epsilon}) - \sqrt{\epsilon} f_2(\sigma) =$$

$$= \int_{\sigma - \sqrt{\epsilon}}^{\sigma} [G \check{y}(s) + H \check{w}(s) + f_2(s)] ds - \sqrt{\epsilon} G \check{y}(\sigma) - \sqrt{\epsilon} H \check{w}(\sigma - \sqrt{\epsilon}) - \sqrt{\epsilon} f_2(\sigma) =$$

$$= \int_{\sigma - \sqrt{\varepsilon}}^{\sigma} \{ G[\tilde{y}(s) - \check{y}(s)] + H[\check{w}(s) - \check{w}(\sigma - \sqrt{\varepsilon})] + f_2(s) - f_2(\sigma) \} ds = \mathcal{O}(\varepsilon)$$

since all functions involved are Lipschitz.

It follows that

$$\check{w}(\sigma) = \sqrt{\varepsilon} G \check{y}(\sigma) + (I + \sqrt{\varepsilon} H) \check{w}(\sigma - \sqrt{\varepsilon}) + \sqrt{\varepsilon} f_2(\sigma) + \mathcal{O}(\varepsilon)$$

hence

$$\check{w}(\sigma) - \tilde{w}(\sigma, \varepsilon) = (I + \sqrt{\varepsilon} H) (\check{w}(\sigma - \sqrt{\varepsilon}) - \tilde{w}(\sigma - \sqrt{\varepsilon}, \varepsilon)) + \mathcal{O}(\varepsilon)$$

On the first interval we have

$$\check{w}(\sigma - \sqrt{\varepsilon}) = \varphi(0) + \int_0^{\sigma - \sqrt{\varepsilon}} [G \check{y}(s) + H \check{w}(s) + f_2(s)] ds$$

$$\tilde{w}(\sigma - \sqrt{\varepsilon}) = \varphi(\sigma - \sqrt{\varepsilon})$$

hence on this interval $|\check{w}(\sigma - \sqrt{\varepsilon}) - \tilde{w}(\sigma - \sqrt{\varepsilon}, \varepsilon)| \leq c \sqrt{\varepsilon}$ if the initial function is Lipschitz.

For the difference equation

$$z(\sigma) = (I + \sqrt{\varepsilon} H) z(\sigma - \sqrt{\varepsilon}) + f(\sigma)$$

the solution is written as

$$z(\sigma) = (I + \sqrt{\varepsilon} H)^l z(\sigma - l\sqrt{\varepsilon}) + \sum_{k=0}^{l-1} (I + \sqrt{\varepsilon} H)^k f(\sigma - k\sqrt{\varepsilon})$$

for $(l-1)\sqrt{\varepsilon} < \sigma \leq l\sqrt{\varepsilon}$ and if $|z(\sigma - l\sqrt{\varepsilon})| \leq c \sqrt{\varepsilon}$, $|f(\sigma - k\sqrt{\varepsilon})| \leq c \varepsilon$

we have the estimate

$$|z(\sigma)| \leq |(I + \sqrt{\varepsilon} H)^l| c \sqrt{\varepsilon} + \sum_{k=0}^{l-1} |(I + \sqrt{\varepsilon} H)^k| c \varepsilon$$

Under our assumption the spectral radius of the matrix $I + \sqrt{\varepsilon} H$ is less than $1 + \mathcal{O}(\sqrt{\varepsilon})$; it follows that

$$\sum_{k=0}^{l-1} |(I + \sqrt{\varepsilon} H)^k| \leq \sum_{k=0}^{l-1} (1 + c \sqrt{\varepsilon})^k \leq \frac{(1 + c \sqrt{\varepsilon})^l}{c \sqrt{\varepsilon}} < \frac{e^{c l \sqrt{\varepsilon}}}{c \sqrt{\varepsilon}} \leq \frac{e^{c \sigma}}{c \sqrt{\varepsilon}}$$

and we deduce the estimate $|z(\sigma)| \leq c \sqrt{\varepsilon}$. We use now for η

the representation formula

$$\eta(\sigma, \varepsilon) = \int_0^{\sigma} X^*(\sigma - s) [A \check{y}(s) + \frac{1}{\sqrt{\varepsilon}} B (\check{w}(\sigma - \sqrt{\varepsilon}, \varepsilon) - \check{w}(s))] ds$$

where X is defined from the system

$$X'(\sigma) = \sqrt{\varepsilon} A^* X(\sigma) + G^* Y(\sigma)$$

$$Y(\sigma) = [I + \sqrt{\varepsilon} H] Y(\sigma - \sqrt{\varepsilon}) + \sqrt{\varepsilon} B^* X(\sigma - \sqrt{\varepsilon})$$

$$X(\sigma) = 0 \text{ for } \sigma < 0, X(0) = I, Y(\sigma) = 0 \text{ for } \sigma < 0.$$

As in [1] we use a representation formula for Y replace it in the variation of constants formula for X and write

$$X(\sigma) = e^{\sqrt{\epsilon} A^* \sigma} + \sum_{k=1}^{\lfloor \frac{t}{\sqrt{\epsilon}} \rfloor} \sqrt{\epsilon} \int_{(k-1)\sqrt{\epsilon}}^t e^{\sqrt{\epsilon} A^*(\sigma-s)} G^* (I + \sqrt{\epsilon} H^*)^{k-1} B^* X(s-k\sqrt{\epsilon}) ds$$

From here, using again the estimate for the spectral radius of the matrix $I + \sqrt{\epsilon} H$ we get an estimate

$$|X(\sigma)| \leq M_1 + M_2 \int_0^\sigma |X(s)| ds$$

hence X is bounded by a constant that depend upon the interval considered. We deduce from here that η is bounded and then \int results also bounded. Remark that a similar critical case has been considered in [2].

Returning to the model problem, the reduced first approximation equations will be

$$y'(\sigma) = -\frac{2f}{1 + \frac{J\epsilon}{4}} y(\sigma) - \frac{f}{1 + \frac{J\epsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$w'(\sigma) = \frac{1}{k(1 + \frac{J\epsilon}{4})} y(\sigma) - \frac{\frac{J\sqrt{\epsilon}}{2}}{1 + \frac{J\epsilon}{4}} w(\sigma) - \frac{\frac{J\sqrt{\epsilon}}{4}}{1 + \frac{J\epsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

$$y(0) = 0, \quad w(0) = 0.$$

Taking into account that \int is of order $\frac{1}{\sqrt{\epsilon}}$ we have $\frac{J\epsilon}{4}$ of order $\sqrt{\epsilon}$ and we may write the reduced model as

$$y'(\sigma) = -\frac{2f}{F} w(\sigma) - \frac{f}{F} \frac{\hat{q}}{|\hat{q}|}$$

$$w'(\sigma) = \frac{1}{k} y(\sigma) - \frac{J\sqrt{\epsilon}}{2} w(\sigma) - \frac{J\sqrt{\epsilon}}{4} \frac{\hat{q}}{|\hat{q}|}$$

We get for y the equation

$$y''(\sigma) + \frac{J\sqrt{\epsilon}}{2} y'(\sigma) + \frac{2f}{kF} y(\sigma) = 0, \quad y(0) = 0, \quad y'(0) = -\frac{f}{F} \frac{\hat{q}}{|\hat{q}|}$$

If \hat{y} is a solution for this equation then y will be a solution for

$$y''(\tau, \epsilon) + \frac{J}{2} y'(\tau, \epsilon) + \frac{2f}{kF} y(\tau\epsilon) = 0$$

(recall that $y(\tau, \epsilon) = \sqrt{\epsilon} \hat{y}(\tau, \epsilon)$)

Returning to the dimensional variables we get

$$\ddot{z} + \frac{J|\hat{q}|}{2fl} \dot{z} + \frac{gl}{Fl} z = 0 \quad \text{that is} \quad \ddot{z} + \frac{R_c f g}{Fl} \dot{z} + \frac{fg}{Fl} z = 0$$

and this is the usual equation for mass ascillation in the

surge tank. We have thus proved that this equation is valid as a first approximation for coefficient of discharge of order $\frac{1}{\sqrt{\epsilon}}$.

Remark that the same equation is obtained if we neglect spatial variation of the flow, integrate the equation

$$\frac{\partial H}{\partial x} + \frac{1}{g\beta} \frac{\partial Q}{\partial t} = 0 \text{ from } 0 \text{ to } l \text{ to get}$$

$$l \frac{dQ}{dt} + fg [H(l,t) - H(0,t)] = 0$$

and consider ~~the~~ the boundary conditions

$$l \frac{dQ}{dt} + fg \left[z + R_0 \frac{dz}{dt} \right] = 0 \quad Q = \hat{q} + F \frac{dz}{dt}$$

which leads for z to the same equation as above. Hence we

have proved that for J of order $\frac{1}{\sqrt{\epsilon}}$ we may neglect spatial variation of the flow.

Recall we had

$$h(1, \zeta, \epsilon) = k [w(\zeta, \epsilon) - w(\zeta - \sqrt{\epsilon}, \epsilon)] = k \left[\hat{w}\left(\frac{\zeta}{\sqrt{\epsilon}}\right) - \hat{w}\left(\frac{\zeta}{\sqrt{\epsilon}} - \sqrt{\epsilon}\right) \right]$$

We proved that $\hat{w}(\sigma) = \tilde{w}(\sigma) + O(\sqrt{\epsilon})$ and also $\tilde{w}(\sigma) = \check{w}(\sigma) + O(\sqrt{\epsilon})$

We may now approximate the pressure

$$h(1, \zeta, \epsilon) \approx k \left[\tilde{w}\left(\frac{\zeta}{\sqrt{\epsilon}}\right) - \tilde{w}\left(\frac{\zeta}{\sqrt{\epsilon}} - \sqrt{\epsilon}\right) \right] \text{ where } \tilde{w} \text{ is computed}$$

from

$$w(\sigma) = \frac{1 - \frac{J\epsilon}{4}}{1 + \frac{J\epsilon}{4}} w(\sigma - \sqrt{\epsilon}) + \frac{\sqrt{\epsilon}}{k(1 + \frac{J\epsilon}{4})} \hat{y}(\sigma) - \frac{\frac{J\epsilon}{4}}{1 + \frac{J\epsilon}{4}} \frac{\hat{q}}{|\hat{q}|}$$

The pressure is thus approximated by

$$h(1, \zeta, \epsilon) \approx \frac{1}{1 + \frac{J\epsilon}{4}} y(\zeta, \epsilon) - \frac{\frac{J\epsilon}{4}}{1 + \frac{J\epsilon}{4}} k \frac{\hat{q}}{|\hat{q}|} - \frac{k \frac{J\epsilon}{2}}{1 + \frac{J\epsilon}{4}} \hat{w}\left(\frac{\zeta}{\sqrt{\epsilon}} - \sqrt{\epsilon}\right)$$

Since the error is here of order $\sqrt{\epsilon}$ and J of order of $\frac{1}{\sqrt{\epsilon}}$

we may take $h(1, \zeta, \epsilon) \approx y(\zeta, \epsilon)$ where we recall that y is

defined from $y'' + \frac{J}{2} y' + \frac{2f}{kF} y = 0$, $y(0) = 0$, $y'(0) = -\frac{f}{F} \frac{\hat{q}}{|\hat{q}|}$

5. A special critical case

We consider now without a direct application to our model problem the special critical case

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}[x_2(t) - x_2(t-\varepsilon)] + f_1(t) \\ x_2(t) &= A_{21}x_1(t) - (I + \varepsilon A_{22})x_2(t-\varepsilon) + f_2(t) \\ x_1(0) &= x_1^0, \quad x_2(t) = \varphi(t), \quad -\varepsilon \leq t < 0. \end{aligned}$$

Assume that A_{22} is Hurwitz; this assumption will imply that the spectral radius of $I + \varepsilon A_{22}$ is less than $1 - O(\varepsilon)$ and it will follow that $|(I + A_{22})^{[\frac{t}{\varepsilon}]}| \leq M$ for $0 \leq t \leq T$.

Define \tilde{x}_1 from the reduced model

$$\tilde{x}_1'(t) = A_{11}\tilde{x}_1(t) + f_1(t), \quad \tilde{x}_1(0) = x_1^0$$

and \tilde{x}_2 from the difference equation

$$\tilde{x}_2(t) = A_{21}\tilde{x}_1(t) - (I + \varepsilon A_{22})x_2(t-\varepsilon) + f_2(t)$$

Let $\hat{x}_1(t) = x_1(t) - \tilde{x}_1(t), \quad \hat{x}_2(t) = x_2(t) - \tilde{x}_2(t).$

We have for \hat{x}_1, \hat{x}_2 the equations

$$\begin{aligned} \hat{x}_1'(t) &= A_{11}\hat{x}_1(t) + A_{12}[\hat{x}_2(t) - \hat{x}_2(t-\varepsilon)] + A_{12}[\tilde{x}_2(t) - \tilde{x}_2(t-\varepsilon)] \\ \hat{x}_2(t) &= A_{21}\hat{x}_1(t) - (I + \varepsilon A_{22})\hat{x}_2(t-\varepsilon) \\ \hat{x}_1(0) &= 0, \quad \hat{x}_2(t) = 0, \quad -\varepsilon \leq t < 0 \end{aligned}$$

Replace $\hat{x}_2(t)$ in the first equation by using the second one, and use the representation

$$\hat{x}_2(t) = \sum_{j=1}^{\ell} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(t - (j-1)\varepsilon),$$

$(\ell-1)\varepsilon \leq t < \ell\varepsilon$ in the integral equation

$$\begin{aligned} \hat{x}_1(t) &= \int_0^t (A_{11} + A_{12}A_{21}) \hat{x}_1(s) ds + A_{12} \int_0^t [\tilde{x}_2(s) - \tilde{x}_2(s-\varepsilon)] ds - \\ &\quad - 2A_{12} \int_0^{t-\varepsilon} (I + \frac{\varepsilon}{2} A_{22}) \hat{x}_2(s) ds \end{aligned}$$

We have

$$\int_0^{t-\varepsilon} \hat{x}_2(s) ds = \sum_{\ell=1}^{[\frac{t}{\varepsilon}]-1} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} \hat{x}_2(s) ds + \int_{[\frac{t}{\varepsilon}]\varepsilon - \varepsilon}^{t-\varepsilon} \hat{x}_2(s) ds =$$

$$\begin{aligned}
 &= \sum_{l=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_{t-\epsilon}^{\lfloor \frac{t}{\epsilon} \rfloor \epsilon} \sum_{j=1}^l \frac{(-1)^{j-1}}{(I + \epsilon A_{22})^{j-1}} A_{21} \hat{x}_1(s - (j-1)\epsilon) ds + \\
 &+ \int_{\lfloor \frac{t}{\epsilon} \rfloor \epsilon - \epsilon}^{\lfloor \frac{t}{\epsilon} \rfloor \epsilon} \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s - (j-1)\epsilon) ds = \\
 &= \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \sum_{l=j}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_{t-\epsilon}^{\lfloor \frac{t}{\epsilon} \rfloor \epsilon} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s - (j-1)\epsilon) ds + \\
 &+ \int_{\lfloor \frac{t}{\epsilon} \rfloor \epsilon - \epsilon}^{\lfloor \frac{t}{\epsilon} \rfloor \epsilon} \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s - (j-1)\epsilon) ds \\
 &= \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_{(j-1)\epsilon}^{t-\epsilon} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s - (j-1)\epsilon) ds + \\
 &+ (-1)^{\lfloor \frac{t}{\epsilon} \rfloor - 1} (I + \epsilon A_{22})^{\lfloor \frac{t}{\epsilon} \rfloor - 1} A_{21} \int_{\lfloor \frac{t}{\epsilon} \rfloor \epsilon - \epsilon}^{t-\epsilon} \hat{x}_1(s - \lfloor \frac{t}{\epsilon} \rfloor \epsilon + \epsilon) ds = \\
 &= \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_0^{t-j\epsilon} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s) ds + (-1)^{\lfloor \frac{t}{\epsilon} \rfloor - 1} (I + \epsilon A_{22})^{\lfloor \frac{t}{\epsilon} \rfloor - 1} A_{21} \int_0^{t-j\epsilon} \hat{x}_1(s) ds \\
 &+ \epsilon A_{22} \int_0^{t-j\epsilon} \hat{x}_1(s) ds = \sum_{j=1}^{\lfloor \frac{t}{\epsilon} \rfloor} \int_0^{t-j\epsilon} (-1)^{j-1} (I + \epsilon A_{22})^{j-1} A_{21} \hat{x}_1(s) ds
 \end{aligned}$$

Denote $z(t) = \tilde{x}_2(t) - \tilde{x}_2(t-\epsilon)$; we get an equation of the form

$$z(t) = -(I + \epsilon A_{22})z(t-\epsilon) + A_{21}[\tilde{x}_1(t) - \tilde{x}_1(t-\epsilon)] + f_2(t) - f_2(t-\epsilon)$$

for $t \geq \epsilon$

$$z(t) = A_{21} \tilde{x}_1(t) + f_2(t) - 2(I + \frac{\epsilon}{2} A_{22})\psi(t-\epsilon) \quad \text{for } 0 \leq t < \epsilon.$$

With corresponding notations it follows that

$$z(t) = (-1)^l (I + \epsilon A_{22})^l \psi(t - l\epsilon) + \sum_{j=1}^l (-1)^{j-1} (I + \epsilon A_{22})^{j-1} \phi(t - (j-1)\epsilon)$$

for $l\epsilon < t \leq (l+1)\epsilon$.

We deduce that

$$\begin{aligned}
 \int_0^t z(s) ds &= \sum_{l=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_{l\epsilon}^{(l+1)\epsilon} z(s) ds + \int_{\lfloor \frac{t}{\epsilon} \rfloor \epsilon}^t z(s) ds + \int_0^{\epsilon} \psi(s) ds = \\
 &= \int_0^{\epsilon} \psi(s) ds + \sum_{l=1}^{\lfloor \frac{t}{\epsilon} \rfloor - 1} \int_{l\epsilon}^{(l+1)\epsilon} (-1)^l (I + \epsilon A_{22})^l \psi(s - l\epsilon) ds + \\
 &+ \int_{\lfloor \frac{t}{\epsilon} \rfloor \epsilon}^t (-1)^{\lfloor \frac{t}{\epsilon} \rfloor} (I + \epsilon A_{22})^{\lfloor \frac{t}{\epsilon} \rfloor} \psi(s - \lfloor \frac{t}{\epsilon} \rfloor \epsilon) ds +
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{\ell=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} \int_{\ell\varepsilon}^{(\ell+1)\varepsilon} \sum_{j=1}^{\ell} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(s - (j-1)\varepsilon) ds + \\
& + \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} (-1)^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} (I + \varepsilon A_{22})^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} \int_{\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon}^t \phi(s - (j-1)\varepsilon) ds = \\
& = \int_0^t \sum_{\ell=0}^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} (-1)^\ell (I + \varepsilon A_{22})^\ell \psi(s) ds + \int_{\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon}^t (-1)^{\lfloor \frac{t}{\varepsilon} \rfloor} (I + \varepsilon A_{22})^{\lfloor \frac{t}{\varepsilon} \rfloor} \psi(s - \lfloor \frac{t}{\varepsilon} \rfloor \varepsilon) ds + \\
& + \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_0^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma \\
\text{Remark that } & \sum_{\ell=0}^{\lfloor \frac{t}{\varepsilon} \rfloor} (-1)^\ell (I + \varepsilon A_{22})^\ell = \{ I + (-1)^{\lfloor \frac{t}{\varepsilon} \rfloor + 1} (I + \varepsilon A_{22})^{\lfloor \frac{t}{\varepsilon} \rfloor + 1} \} (2I + \varepsilon A_{22})^{-1},
\end{aligned}$$

and we may estimate the terms containing ψ .

We have next

$$\begin{aligned}
& \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} A_{12} \int_0^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma = A_{12} \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \left(\sum_{\ell=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - j} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma \right. \\
& \left. + A_{22} \int_0^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma \right) = \\
& = A_{12} \sum_{\ell=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - \ell} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma + \\
& + A_{12} \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_{\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon - j\varepsilon}^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} \phi(\sigma) d\sigma
\end{aligned}$$

Taking into account that

$$\phi(\sigma) = A_{21} [\tilde{x}_1(\sigma) - \tilde{x}_1(\sigma - \varepsilon)] + f_2(\sigma) - f_2(\sigma - \varepsilon)$$

we deduce that the above sum is of order ε .

In the same way

$$\begin{aligned}
& \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_0^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma = \\
& = \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \sum_{\ell=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - j} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma + \\
& + \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_{\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon - j\varepsilon}^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma = \\
& = \sum_{\ell=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor - \ell} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma + \\
& + \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_{\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon - j\varepsilon}^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma
\end{aligned}$$

It follows that

$$\left| \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} \int_0^{t-j\varepsilon} (-1)^{j-1} (I + \varepsilon A_{22})^{j-1} A_{21} \hat{x}_1(\sigma) d\sigma \right| \leq M \int_0^t |\hat{x}_1(\sigma)| d\sigma$$

and finally we get an inequality of the form

$$|\hat{x}_1(t)| \leq \varepsilon L + M \int_0^t |\hat{x}_1(\sigma)| d\sigma$$

and \hat{x}_1 is of order ε on the considered interval $[0, T]$.

The estimate for \hat{x}_2 will have the order which results from the matrix A_{21} .

In fact, we have

$$|\hat{x}_2(t)| \leq c \varepsilon |A_{21}| \sum_{j=1}^{\lfloor \frac{t}{\varepsilon} \rfloor} |I + \varepsilon A_{22}|^{j-1} \leq c |A_{21}|$$

References

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ASYMPTOTIC EXPANSIONS FOR COUPLED
DIFFERENCE AND DIFFERENTIAL - DIFFERENCE
EQUATIONS WITH SMALL DELAYS

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1. Introduction

We shall consider a system of the form

$$\begin{aligned} \dot{x}_1(t) &= A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_{11}(t)x_1(t-\varepsilon) + \\ &\quad + B_{12}(t)x_2(t-\varepsilon) \\ x_2(t) &= A_{21}(t)x_1(t) + B_{21}(t)x_1(t-\varepsilon) + B_{22}(t)x_2(t-\varepsilon) \end{aligned} \quad (1)$$

with $x_1(t) = \varphi_1(t)$, $x_2(t) = \varphi_2(t)$ for $s-\varepsilon \leq t < s$, $x_1(s) = x_1^0$

The solution may be obtained directly using a step by step procedure. But if the delay $\varepsilon > 0$ is small, as Vasilieva remarked more than 20 years ago [1] such procedure will be very tedious; to get the solution on a reasonable interval a large number of steps will be needed and there is no hope to get informations concerning the asymptotic behavior of the solutions. That is why a procedure that might give in a simpler way the main terms of an asymptotic expansion with respect to ε could be useful. Such expansions have been proposed by Vasilieva and her colleagues (see for example [2]). Our results differ mainly in that we are interested in uniform expansions on $[t_0, \infty)$, obtaining thus as a necessary step estimates for the "fundamental matrix solution" from which a stability result of Gradstein-Klimušev-Krasovski type follows.

Special cases of systems with constant coefficients have been considered in connection with a problem in hydraulics and have been published separately [3].

2. Estimates for a difference equation

Consider the difference equation

$$y(t) = B_{22}(t)y(t-\varepsilon)$$

Denote by $C_2(t, s, \varepsilon)$ the fundamental matrix solution defined by the initial conditions $C_2(s, s, \varepsilon) = I$ and $C_2(t, s, \varepsilon) = 0$ for $s - \varepsilon \leq t < s$. Assume that uniformly with respect to t the spectral radius of the matrix $B_{22}(t)$ satisfies $|\sigma(B_{22}(t))| \leq \hat{\rho} < 1$. Assume also that $t \rightarrow B_{22}(t)$ is uniformly Lipschitz and uniformly bounded.

It is known fact that under such assumption

$$|C_2(t, s, \varepsilon)| \leq c \rho^{(t-s)/\varepsilon} \quad \text{for} \quad \hat{\rho} < \rho < 1,$$

where c depends upon ρ and $\sup_t |B_{22}(t)|$ (for a slightly different version see [2] page 262).

Fix now $s < \zeta \leq t$ and write

$$C_2(t, s, \varepsilon) = B_{22}(\zeta)C_2(t-\varepsilon, s, \varepsilon) + [B_{22}(t) - B_{22}(\zeta)]C_2(t-\varepsilon, s, \varepsilon)$$

we have for $s + (l-1)\varepsilon < t \leq s + l\varepsilon$

$$C_2(t, s, \varepsilon) = [B_{22}(\zeta)]^l C_2(t-l\varepsilon, s, \varepsilon) + \sum_{k=0}^{l-1} [B_{22}(\zeta)]^k [B_{22}(t-k\varepsilon) - B_{22}(\zeta)] C_2(t-(k+1)\varepsilon, s, \varepsilon)$$

the formula being directly checked by induction.

For $\zeta = t$ it follows

$$C_2(t, s, \varepsilon) = [B_{22}(t)]^l C_2(t-l\varepsilon, s, \varepsilon) + \sum_{k=0}^{l-1} [B_{22}(t)]^k [B_{22}(t-k\varepsilon) - B_{22}(t)] C_2(t-(k+1)\varepsilon, s, \varepsilon)$$

$$-B_{22}(t)] C_2(t-(k+1)\epsilon, s, \epsilon)$$

and for $z=s$

$$C_2(t, s, \epsilon) = [B_{22}(s)]^\ell C_2(t-\ell\epsilon, s, \epsilon) + \sum_{k=0}^{\ell-1} [B_{22}(s)]^k [B_{22}(t-k\epsilon) - B_{22}(s)] C_2(t-(k+1)\epsilon, s, \epsilon)$$

(recall that $s+(\ell+1)\epsilon < t \leq s+\ell\epsilon$).

For $t=s+\ell\epsilon$ the last formula gives

$$C_2(s+\ell\epsilon, s, \epsilon) = [B_{22}(s)]^\ell + \sum_{k=0}^{\ell-1} [B_{22}(s)]^k [B_{22}(s+(\ell-k)\epsilon) - B_{22}(s)] C_2(s+(\ell-k-1)\epsilon, s, \epsilon)$$

hence

$$|C_2(s+\ell\epsilon, s, \epsilon) [B_{22}(s)]^\ell| \leq \sum_{k=0}^{\ell-1} c \rho^k (\ell-k)\epsilon \rho^{\ell-k-1} = c\epsilon \rho^{\ell-1} \frac{\ell(\ell+1)}{2}$$

Consider now the equation

$$y(t) = B_{22}(t)y(t-\epsilon) + f(t) - \hat{\alpha}(t-t_0)$$

and assume $|f(t)| \leq c e^{-\hat{\alpha}(t-t_0)}$

We look for an estimate for the solution defined for $t \geq t_0$

by the initial function φ given on $[t_0-\epsilon, t_0]$. For

$t_0+(\ell-1)\epsilon < t \leq t_0+\ell\epsilon$ we have

$$y(t) = C_2(t, t-\ell\epsilon, \epsilon) \varphi(t-\ell\epsilon) + \sum_{k=0}^{\ell-1} C_2(t, t-k\epsilon, \epsilon) f(t-k\epsilon)$$

(the formula is checked by induction).

It follows that

$$|y(t)| \leq c \rho^\ell \sup_{[t_0-\epsilon, t_0]} |\varphi| + c \sum_{k=0}^{\ell-1} \rho^k e^{-\hat{\alpha}(t-k\epsilon-t_0)} \leq$$

$$\leq c \rho^{\frac{t-t_0}{\epsilon}} \sup_{[t_0-\epsilon, t_0]} |\varphi| + c e^{-\hat{\alpha}(t-t_0)} \sum_{k=0}^{\ell-1} (\rho e^{\hat{\alpha}\epsilon})^k$$

and for ϵ small enough since $\rho e^{\hat{\alpha}\epsilon} < 1$ we get

$$|y(t)| \leq c \left[\rho^{\frac{t-t_0}{\epsilon}} \sup_{[t_0-\epsilon, t_0]} |\varphi| + e^{-\hat{\alpha}(t-t_0)} \right] \quad \text{or with } \rho = e^{-\alpha}$$

$$|y(t)| \leq c \left[e^{-\frac{\alpha(t-t_0)}{\varepsilon}} \sup_{[t_0-\varepsilon, t_0]} |\varphi| + e^{-\hat{\alpha}(t-t_0)} \right]$$

3. A stability result

To system (1) we associate the reduced one obtained by formally setting $\varepsilon=0$; it is a system of ordinary differential equations of the form $\dot{x}_1 = \tilde{A}(t)x_1$ with

$$\tilde{A}(t) = A_{11}(t) + B_{11}(t) + [A_{12}(t) + B_{12}(t)] [I - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)]$$

Our main assumption will be that the reduced system is exponentially stable and that all coefficient matrices are uniformly bounded and uniformly Lipschitz. We shall prove that under such assumptions (recall that we assumed also that

$$|\sigma(B_{22}(t))| \leq \hat{\rho} < 1) \text{ system (1) is also exponentially stable}$$

(this is what we call a result of Gradstein-Klimušev-Krasovski type by analogy with the usual case of singularly perturbed systems).

Consider an arbitrary solution of (1). From the difference

$$x_2(t) = C_2(t, t-l\varepsilon, \varepsilon) \varphi_2(t-l\varepsilon) + \sum_{k=0}^{l-1} G_2(t, t-k\varepsilon, \varepsilon) \cdot$$

$$[A_{21}(t-k\varepsilon)x_1(t-k\varepsilon) + B_{21}(t-k\varepsilon)x_1(t-(k+1)\varepsilon)]$$

From the first equation we may write, by using the fundamental matrix $\tilde{C}(t, s)$ associated to \tilde{A}

$$x_1(t) = \tilde{C}(t, s)x_1(s) + \int_s^t \tilde{C}(t, \sigma) [(A_{11}(\sigma) - \tilde{A}(\sigma))x_1(\sigma) + B_{11}(\sigma)x_1(\sigma-\varepsilon) + A_{12}(\sigma)x_2(\sigma) + B_{12}(\sigma)x_2(\sigma-\varepsilon)] d\sigma.$$

We shall replace here x_2 by using the above representation formula to get an integral equation for x_1

$$\begin{aligned}
 x_1(t) = & \tilde{C}(t,s)x_1(s) + \int_{\Delta}^t \tilde{C}(t,\sigma) \left[(A_{11}(\sigma) - \tilde{A}(\sigma))x_1(\sigma) + \right. \\
 & \left. + B_{11}(\sigma)x_1(\sigma - \varepsilon) \right] d\sigma + \\
 & \sum_{\ell=1}^{\left[\frac{t-s}{\varepsilon} \right]} \int_{\Delta + (\ell-1)\varepsilon}^{\Delta + \ell\varepsilon} \tilde{C}(t,\sigma) \left\{ A_{12}(\sigma) \left[C_2(\sigma, \sigma - \ell\varepsilon, \varepsilon) \varphi_2(\sigma - \ell\varepsilon) + \right. \right. \\
 & \left. \left. + \sum_{k=0}^{\ell-1} C_2(\sigma, \sigma - k\varepsilon, \varepsilon) (A_{21}(\sigma - k\varepsilon)x_1(\sigma - k\varepsilon) + \right. \right. \\
 & \left. \left. + B_{21}(\sigma - k\varepsilon)x_1(\sigma - (k+1)\varepsilon)) \right] + B_{12}(\sigma) \left[C_2(\sigma - \varepsilon, \sigma - (\ell+1)\varepsilon, \varepsilon) \varphi_2(\sigma - (\ell+1)\varepsilon) + \right. \right. \\
 & \left. \left. + \sum_{k=0}^{\ell-1} C_2(\sigma - \varepsilon, \sigma - (k+1)\varepsilon, \varepsilon) (A_{21}(\sigma - (k+1)\varepsilon)x_1(\sigma - (k+1)\varepsilon) + \right. \right. \\
 & \left. \left. + B_{21}(\sigma - (k+1)\varepsilon)x_1(\sigma - (k+1)\varepsilon)) \right] \right\} d\sigma + \\
 & \int_{\Delta + \left[\frac{t-s}{\varepsilon} \right] \varepsilon}^t \tilde{C}(t,\sigma) \left\{ A_{12}(\sigma) \left[C_2(\sigma, \sigma - \left(\left[\frac{t-s}{\varepsilon} \right] + 1 \right) \varepsilon, \varepsilon) \varphi_2(\sigma - \left(\left[\frac{t-s}{\varepsilon} \right] + 1 \right) \varepsilon) + \right. \right. \\
 & \left. \left. + \sum_{k=0}^{\left[\frac{t-s}{\varepsilon} \right]} C_2(\sigma, \sigma - k\varepsilon, \varepsilon) (A_{21}(\sigma - k\varepsilon)x_1(\sigma - k\varepsilon) + \right. \right. \\
 & \left. \left. + B_{21}(\sigma - k\varepsilon)x_1(\sigma - (k+1)\varepsilon)) \right] + B_{12}(\sigma) \left[C_2(\sigma - \varepsilon, \sigma - \left(\left[\frac{t-s}{\varepsilon} \right] + 2 \right) \varepsilon, \varepsilon) \varphi_2(\sigma - \left(\left[\frac{t-s}{\varepsilon} \right] + 2 \right) \varepsilon) + \right. \right. \\
 & \left. \left. + \sum_{k=0}^{\left[\frac{t-s}{\varepsilon} \right]} C_2(\sigma - \varepsilon, \sigma - (k+1)\varepsilon, \varepsilon) (A_{21}(\sigma - (k+1)\varepsilon)x_1(\sigma - (k+1)\varepsilon) + \right. \right. \\
 & \left. \left. + B_{21}(\sigma - (k+1)\varepsilon)x_1(\sigma - (k+2)\varepsilon)) \right] \right\} d\sigma
 \end{aligned}$$

Here we have assumed $t > s + \varepsilon$; for $s \leq t \leq s + \varepsilon$ the sum is missing.

The representation formula for x_2 works only if $s < t$, hence in using this formula for $\ell = 1$ we consider it conventionally, the impossible summation $\sum_{k=0}^{-1}$ being deleted, which gives the same result as if we took $x_1(t) = 0$ for $t < s - \varepsilon$.

To get from this integral equation an estimate for x_1 we shall perform a series of transformations.

A. Consider the terms

$$\begin{aligned}
 & \sum_{\ell=1}^{\left[\frac{t-s}{\varepsilon} \right]} \int_{\Delta + (\ell-1)\varepsilon}^{\Delta + \ell\varepsilon} \tilde{C}(t,\sigma) \left\{ A_{12}(\sigma) \sum_{k=1}^{\ell} C_2(\sigma, \sigma - (k-1)\varepsilon, \varepsilon) (A_{21}(\sigma - (k-1)\varepsilon)x_1(\sigma - (k-1)\varepsilon) + \right. \\
 & \left. + B_{21}(\sigma - (k-1)\varepsilon)x_1(\sigma - k\varepsilon)) + B_{12}(\sigma) \sum_{k=1}^{\ell} C_2(\sigma - \varepsilon, \sigma - k\varepsilon, \varepsilon) (A_{21}(\sigma - k\varepsilon)x_1(\sigma - k\varepsilon) + \right. \\
 & \left. + B_{21}(\sigma - k\varepsilon)x_1(\sigma - (k+1)\varepsilon)) \right\} d\sigma
 \end{aligned}$$

and perform the permutation of the order of summation to get

$$\sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor} \int_{\Delta+(k-1)\epsilon}^{\Delta+k\epsilon} \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - (k-1)\epsilon, \epsilon) (A_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - (k-1)\epsilon) +$$

$$+ B_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - k\epsilon)) + B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - k\epsilon, \epsilon) (A_{21}(\sigma - k\epsilon) x_1(\sigma - k\epsilon) +$$

$$+ B_{21}(\sigma - k\epsilon) x_1(\sigma - (k+1)\epsilon)) \} d\sigma = \sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor} \int_{\Delta+(k-1)\epsilon}^{\Delta+k\epsilon} \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - (k-1)\epsilon, \epsilon) \cdot$$

$$\cdot (A_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - (k-1)\epsilon) + B_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - k\epsilon)) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - k\epsilon, \epsilon) (A_{21}(\sigma - k\epsilon) x_1(\sigma - k\epsilon) + B_{21}(\sigma - k\epsilon) x_1(\sigma - (k+1)\epsilon)) \} d\sigma$$

If we add now the terms not containing the initial function

φ_2 in the last integral we obtain the integral equation

$$x_1(t) = \tilde{C}(t, s) x_1(s) + \int_s^t \tilde{C}(t, \sigma) [(A_{11}(\sigma) - \tilde{A}(\sigma)) x_1(\sigma) + B_{11}(\sigma) x_1(\sigma - \epsilon)] d\sigma +$$

$$+ \sum_{\ell=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor} \int_{\Delta+(\ell-1)\epsilon}^{\Delta+\ell\epsilon} \tilde{C}(t, \sigma) [A_{12}(\sigma) C_2(\sigma, \sigma - \ell\epsilon, \epsilon) \varphi_2(\sigma - \ell\epsilon) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - (\ell+1)\epsilon, \epsilon) \varphi_2(\sigma - (\ell+1)\epsilon)] d\sigma +$$

$$+ \int_{s+\lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon}^t \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon, \epsilon) \varphi_2(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 2)\epsilon, \epsilon) \varphi_2(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 2)\epsilon) \} d\sigma +$$

$$+ \sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor} \int_{\Delta+(k-1)\epsilon}^t \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - (k-1)\epsilon, \epsilon) (A_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - (k-1)\epsilon) +$$

$$+ B_{21}(\sigma - (k-1)\epsilon) x_1(\sigma - k\epsilon)) + B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - k\epsilon, \epsilon) (A_{21}(\sigma - k\epsilon) x_1(\sigma - k\epsilon) +$$

$$+ B_{21}(\sigma - k\epsilon) x_1(\sigma - (k+1)\epsilon)) \} d\sigma + \int_{\Delta+\lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon}^t \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - \lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon, \epsilon) \cdot$$

$$(A_{21}(\sigma - \lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon) x_1(\sigma - \lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon) + B_{21}(\sigma - \lfloor \frac{t-\Delta}{\epsilon} \rfloor \epsilon) x_1(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon)) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \epsilon, \sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon, \epsilon) (A_{21}(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon) x_1(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon) +$$

$$+ B_{21}(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1)\epsilon) x_1(\sigma - (\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 2)\epsilon)) \} d\sigma$$

Performing now a change of variables in the integrals we get

$$x_1(t) = \tilde{C}(t, s) x_1(s) + \int_s^t \tilde{C}(t, \sigma) [(A_{11}(\sigma) - \tilde{A}(\sigma)) x_1(\sigma) + B_{11}(\sigma) x_1(\sigma - \epsilon)] d\sigma +$$

$$+ \sum_{\ell=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor} \int_{\Delta+(\ell-1)\epsilon}^{\Delta+\ell\epsilon} \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - \ell\epsilon, \epsilon) \varphi_2(\sigma - \ell\epsilon) +$$

$$\begin{aligned}
 & + B_{12}(\sigma) C_2(\sigma - \varepsilon, \sigma - (l+1)\varepsilon, \varepsilon) \varphi_2(\sigma - (l+1)\varepsilon) \} d\sigma + \\
 & + \int_{\Delta + [\frac{t-\Delta}{\varepsilon}]\varepsilon}^t \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - ([\frac{t-\Delta}{\varepsilon}] + 1)\varepsilon, \varepsilon) \varphi_2(\sigma - ([\frac{t-\Delta}{\varepsilon}] + 1)\varepsilon) + \\
 & + B_{12}(\sigma) C_2(\sigma - \varepsilon, \sigma - ([\frac{t-\Delta}{\varepsilon}] + 2)\varepsilon, \varepsilon) \varphi_2(\sigma - ([\frac{t-\Delta}{\varepsilon}] + 2)\varepsilon) \} d\sigma + \\
 & + \sum_{k=1}^{[\frac{t-\Delta}{\varepsilon}] + 1} \int_{\Delta - \varepsilon}^{\Delta} \tilde{C}(t, \bar{z} + k\varepsilon) [A_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) + \\
 & + B_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z})] \varphi_1(\bar{z}) d\bar{z} + \\
 & + \sum_{k=1}^{[\frac{t-\Delta}{\varepsilon}] + 1} \int_{\Delta}^{\Delta} \tilde{C}(t, \bar{z} + (k+1)\varepsilon) B_{12}(\bar{z} + (k+1)\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) \varphi(\bar{z}) d\bar{z} + \\
 & + \sum_{k=1}^{[\frac{t-\Delta}{\varepsilon}] + 1} \left\{ \int_{\Delta}^{\Delta - \varepsilon} \tilde{C}(t, \bar{z} + (k-1)\varepsilon) A_{12}(\bar{z} + (k-1)\varepsilon) C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z}) x_1(\bar{z}) d\bar{z} + \right. \\
 & + \int_{\Delta} \tilde{C}(t, \bar{z} + k\varepsilon) [A_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) + \\
 & + B_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z})] x_1(\bar{z}) d\bar{z} + \\
 & \left. + \int_{\Delta} \tilde{C}(t, \bar{z} + (k+1)\varepsilon) B_{12}(\bar{z} + (k+1)\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) x_1(\bar{z}) d\bar{z} \right\}
 \end{aligned}$$

B. We consider the last sum and rewrite it as

$$\begin{aligned}
 & \sum_{k=1}^{[\frac{t-\Delta}{\varepsilon}] + 1} \left\{ \int_{\Delta}^{t - (k-1)\varepsilon} \tilde{C}(t, \bar{z}) A_{12}(\bar{z}) C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z}) x_1(\bar{z}) d\bar{z} + \right. \\
 & + \int_{\Delta}^{t - k\varepsilon} \tilde{C}(t, \bar{z}) [A_{12}(\bar{z}) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z}) + B_{12}(\bar{z}) C_2(\bar{z} + (k+1)\varepsilon, \bar{z}, \varepsilon) \cdot \\
 & \cdot A_{21}(\bar{z})] x_1(\bar{z}) d\bar{z} + \int_{\Delta}^{t - (k+1)\varepsilon} \tilde{C}(t, \bar{z}) B_{12}(\bar{z}) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z}) x_1(\bar{z}) d\bar{z} \} + \\
 & + \sum_{k=1}^{[\frac{t-\Delta}{\varepsilon}] + 1} \left\{ \int_{\Delta}^{t - (k-1)\varepsilon} [\tilde{C}(t, \bar{z} + (k-1)\varepsilon) A_{12}(\bar{z} + (k-1)\varepsilon) \cdot \right. \\
 & \cdot \tilde{C}(t, \bar{z}) A_{12}(\bar{z})] C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z}) x_1(\bar{z}) d\bar{z} + \\
 & + \int_{\Delta}^{t - k\varepsilon} \left\{ \tilde{C}(t, \bar{z} + k\varepsilon) [A_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) + B_{12}(\bar{z} + k\varepsilon) C_2(\bar{z} + (k+1)\varepsilon, \bar{z}, \varepsilon) \cdot \right. \\
 & \cdot A_{21}(\bar{z})] \cdot \tilde{C}(t, \bar{z}) [A_{12}(\bar{z}) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z}) + \\
 & + B_{12}(\bar{z}) C_2(\bar{z} + (k-1)\varepsilon, \bar{z}, \varepsilon) A_{21}(\bar{z})] \} x_1(\bar{z}) d\bar{z} + \\
 & \left. + \int_{\Delta}^{t - (k+1)\varepsilon} [\tilde{C}(t, \bar{z} + (k+1)\varepsilon) B_{12}(\bar{z} + (k+1)\varepsilon) C_2(\bar{z} + k\varepsilon, \bar{z} + \varepsilon, \varepsilon) B_{21}(\bar{z} + \varepsilon) \cdot \right.
 \end{aligned}$$

$$-\tilde{C}(t, \tau) B_{12}(\tau) C_2(\tau+k\varepsilon, \tau+\varepsilon, \varepsilon) B_{21}(\tau)] x_1(\tau) d\tau \}$$

Rewrite the first sum as

$$\begin{aligned} & \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_s^t \tilde{C}(t, \tau) [A_{12}(\tau) C_2(\tau+(k-1)\varepsilon, \tau, \varepsilon) A_{21}(\tau) + A_{12}(\tau) C_2(\tau+k\varepsilon, \tau+\varepsilon, \varepsilon) B_{21}(\tau) + \\ & + B_{12}(\tau) C_2(\tau+(k-1)\varepsilon, \tau, \varepsilon) A_{21}(\tau) + B_{12}(\tau) C_2(\tau+k\varepsilon, \tau+\varepsilon, \varepsilon) B_{21}(\tau)] x_1(\tau) d\tau + \\ & + \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_{t-(k-1)\varepsilon}^t \tilde{C}(t, \tau) A_{12}(\tau) C_2(\tau+(k-1)\varepsilon, \tau, \varepsilon) A_{21}(\tau) x_1(\tau) d\tau + \\ & + \int_{t-k\varepsilon}^t \tilde{C}(t, \tau) [A_{12}(\tau) C_2(\tau+k\varepsilon, \tau+\varepsilon, \varepsilon) B_{21}(\tau) + B_{12}(\tau) C_2(\tau+(k-1)\varepsilon, \tau, \varepsilon) \cdot \\ & \cdot A_{21}(\tau)] x_1(\tau) d\tau + \int_{t-(k+1)\varepsilon}^t \tilde{C}(t, \tau) B_{12}(\tau) C_2(\tau+k\varepsilon, \tau+\varepsilon, \varepsilon) B_{21}(\tau) x_1(\tau) d\tau \} \end{aligned}$$

Consider again only the first sum and write it by replacing

C_2 with the formula

$$C_2(\tau+l\varepsilon, \tau, \varepsilon) = [B_{22}(\tau)]^l + \Gamma_2(\tau, l, \varepsilon)$$

$$|\Gamma_2(\tau, l, \varepsilon)| \leq c\varepsilon^{\ell-1} \frac{\ell(\ell+1)}{2} ;$$

we get

$$\begin{aligned} & \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_s^t \tilde{C}(t, \tau) \{ A_{12}(\tau) [B_{22}(\tau)]^{k-1} A_{21}(\tau) + A_{12}(\tau) [B_{22}(\tau)]^{k-1} \cdot \\ & \cdot B_{21}(\tau) + B_{12}(\tau) [B_{22}(\tau)]^{k-1} A_{21}(\tau) + B_{12}(\tau) [B_{22}(\tau+\varepsilon)]^{k-1} B_{21}(\tau) \} x_1(\tau) d\tau + \\ & + \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_s^t \tilde{C}(t, \tau) [A_{12}(\tau) \Gamma_2(\tau, k-1, \varepsilon) A_{21}(\tau) + A_{12}(\tau) \Gamma_2(\tau+\varepsilon, k-1, \varepsilon) B_{21}(\tau) + \\ & + B_{12}(\tau) \Gamma_2(\tau, k-1, \varepsilon) A_{21}(\tau) + B_{12}(\tau) \Gamma_2(\tau+\varepsilon, k-1, \varepsilon) B_{21}(\tau)] x_1(\tau) d\tau \end{aligned}$$

Consider again only the first sum and write it

$$\begin{aligned} & \int_s^t \tilde{C}(t, \tau) \{ [A_{12}(\tau) + B_{12}(\tau)] [I - B_{22}(\tau)]^{-1} A_{21}(\tau) + \\ & + [A_{12}(\tau) + B_{12}(\tau)] [I - B_{22}(\tau+\varepsilon)]^{-1} B_{21}(\tau) \} x_1(\tau) - \\ & - \sum_{k=\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 2}^{\infty} \int_s^t \tilde{C}(t, \tau) \{ [A_{12}(\tau) + B_{12}(\tau)] [B_{22}(\tau)]^{k-1} A_{21}(\tau) + \\ & + [A_{12}(\tau) + B_{12}(\tau)] [B_{22}(\tau+\varepsilon)]^{k-1} B_{21}(\tau) \} x_1(\tau) d\tau \end{aligned}$$

Write finally the first integral as

$$\int_{\Delta}^t \tilde{C}(t, \tau) [A_{12}(\tau) + B_{12}(\tau)] [I - B_{22}(\tau)]^{-1} [A_{21}(\tau) + B_{21}(\tau)] x_1(\tau) d\tau +$$

$$+ \int_{\Delta}^t \tilde{C}(t, \tau) [A_{12}(\tau) + B_{12}(\tau)] \left\{ [I - B_{22}(\tau + \varepsilon)]^{-1} - [I - B_{22}(\tau)]^{-1} \right\} \cdot$$

$$B_{21}(\tau) x_1(\tau) d\tau$$

C. We have

$$\int_{\Delta}^t \tilde{C}(t, \sigma) [(A_{11}(\sigma) - \tilde{A}(\sigma)) x_1(\sigma) + B_{11}(\sigma) x_1(\sigma - \varepsilon)] d\sigma =$$

$$= \int_{\Delta}^{\Delta + \varepsilon} \tilde{C}(t, \sigma) B_{11}(\sigma) \varphi_1(\sigma - \varepsilon) d\sigma + \int_{\Delta}^{t - \varepsilon} [\tilde{C}(t, \sigma + \varepsilon) B_{11}(\sigma + \varepsilon) - \tilde{C}(t, \sigma) B_{11}(\sigma)] x_1(\sigma) d\sigma +$$

$$+ \int_{\Delta}^t \tilde{C}(t, \sigma) [A_{11}(\sigma) + B_{11}(\sigma) - \tilde{A}(\sigma)] x_1(\sigma) d\sigma - \int_{t - \varepsilon}^t \tilde{C}(t, \sigma) B_{11}(\sigma) x_1(\sigma) d\sigma =$$

$$= - \int_{\Delta}^t \tilde{C}(t, \sigma) [A_{12}(\sigma) + B_{12}(\sigma)] [I - B_{22}(\sigma)]^{-1} [A_{21}(\sigma) + B_{21}(\sigma)] x_1(\sigma) d\sigma +$$

$$+ \int_{\Delta}^{t - \varepsilon} [\tilde{C}(t, \sigma + \varepsilon) B_{11}(\sigma + \varepsilon) - \tilde{C}(t, \sigma) B_{11}(\sigma)] x_1(\sigma) d\sigma - \int_{t - \varepsilon}^t \tilde{C}(t, \sigma) B_{11}(\sigma) x_1(\sigma) d\sigma$$

$$+ \int_{\Delta - \varepsilon}^{\Delta} \tilde{C}(t, \sigma + \varepsilon) B_{11}(\sigma + \varepsilon) \varphi_1(\sigma) d\sigma$$

We see now that the integral equation for x_1 may be written

$$x_1(t) = \tilde{C}(t, s) x_1(s) + \int_{\Delta - \varepsilon}^{\Delta} \tilde{C}(t, \sigma + \varepsilon) B_{11}(\sigma + \varepsilon) \varphi_1(\sigma) d\sigma +$$

$$+ \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_{\Delta - \varepsilon}^{\Delta} \tilde{C}(t, \tau + k\varepsilon) [A_{12}(\tau + k\varepsilon) C_2(\tau + k\varepsilon, \tau + \varepsilon, \varepsilon) B_{21}(\tau + \varepsilon) +$$

$$+ B_{12}(\tau + k\varepsilon) C_2(\tau + (k-1)\varepsilon, \tau, \varepsilon) A_{21}(\tau)] \varphi_1(\tau) d\tau +$$

$$+ \sum_{k=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1} \int_{\Delta - \varepsilon}^{\Delta} \tilde{C}(t, \tau + (k+1)\varepsilon) B_{12}(\tau + (k+1)\varepsilon) C_2(\tau + k\varepsilon, \tau + \varepsilon, \varepsilon) B_{21}(\tau + \varepsilon) \cdot$$

$$\varphi_1(\tau) d\tau + \sum_{l=1}^{\lfloor \frac{t-\Delta}{\varepsilon} \rfloor} \int_{\Delta + (l-1)\varepsilon}^{\Delta + l\varepsilon} \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - l\varepsilon, \varepsilon) \varphi_2(\sigma - l\varepsilon) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \varepsilon, \sigma - (l+1)\varepsilon, \varepsilon) \varphi_2(\sigma - (l+1)\varepsilon) \} d\sigma +$$

$$+ \int_{\Delta + \lfloor \frac{t-\Delta}{\varepsilon} \rfloor \varepsilon}^t \tilde{C}(t, \sigma) \{ A_{12}(\sigma) C_2(\sigma, \sigma - \lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1, \varepsilon) \varphi_2(\sigma - (\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 1)\varepsilon) +$$

$$+ B_{12}(\sigma) C_2(\sigma - \varepsilon, \sigma - (\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 2)\varepsilon, \varepsilon) \varphi_2(\sigma - (\lfloor \frac{t-\Delta}{\varepsilon} \rfloor + 2)\varepsilon) \} d\sigma -$$

$$- \int_{t - \varepsilon}^t \tilde{C}(t, \sigma) B_{11}(\sigma) x_1(\sigma) d\sigma + \int_{\Delta}^{t - \varepsilon} [\tilde{C}(t, \sigma + \varepsilon) B_{11}(\sigma + \varepsilon) - \tilde{C}(t, \sigma) B_{11}(\sigma)] x_1(\sigma) d\sigma +$$

$$\begin{aligned}
 & + \int_0^t \tilde{C}(t, \tau) [A_{12}(\tau) + B_{12}(\tau)] \left\{ [I - B_{22}(\tau + \epsilon)]^{-1} - [I - B_{22}(\tau)]^{-1} \right\} B_{21}(\tau) \cdot \\
 & \cdot x_1(\tau) d\tau - \sum_{k=\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 2}^{\infty} \int_0^t \tilde{C}(t, \tau) \left\{ A_{12}(\tau) [B_{22}(\tau)]^{k-1} A_{21}(\tau) + \right. \\
 & + B_{12}(\tau) [B_{22}(\tau)]^{k-1} A_{21}(\tau) + [A_{12}(\tau) + B_{12}(\tau)] [B_{22}(\tau + \theta)]^{k-1} B_{21}(\tau) \left. \right\} x_1(\tau) d\tau + \\
 & + \sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1} \int_0^t \tilde{C}(t, \tau) \left\{ [A_{12}(\tau) + B_{12}(\tau)] \Gamma_2(\tau, (k-1)\epsilon) A_{21}(\tau) + [A_{12}(\tau) + \right. \\
 & + B_{12}(\tau)] \Gamma_2(\tau + \epsilon, (k-1)\epsilon) B_{21}(\tau) \left. \right\} x_1(\tau) d\tau + \sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1} \left\{ \int_{t-(k-1)\epsilon}^t \tilde{C}(t, \tau) A_{12}(\tau) \cdot \right. \\
 & C_2(\tau + (k-1)\epsilon, \tau, \epsilon) A_{21}(\tau) x_1(\tau) d\tau + \int_{t-k\epsilon}^t \tilde{C}(t, \tau) [A_{12}(\tau) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) B_{21}(\tau) + \\
 & + B_{12}(\tau) C_2(\tau + (k-1)\epsilon, \tau, \epsilon) A_{21}(\tau)] x_1(\tau) d\tau + \int_{t-(k+1)\epsilon}^t \tilde{C}(t, \tau) B_{12}(\tau) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) \cdot \\
 & B_{21}(\tau) x_1(\tau) d\tau \left. \right\} + \sum_{k=1}^{\lfloor \frac{t-\Delta}{\epsilon} \rfloor + 1} \left\{ \int_0^{t-(k-1)\epsilon} [\tilde{C}(t, \tau + (k-1)\epsilon) A_{12}(\tau + (k-1)\epsilon) - \right. \\
 & - \tilde{C}(t, \tau) A_{12}(\tau)] C_2(\tau + (k-1)\epsilon, \tau, \epsilon) A_{21}(\tau) x_1(\tau) d\tau + \\
 & + \int_0^{t-k\epsilon} \left\{ \tilde{C}(t, \tau + k\epsilon) [A_{12}(\tau + k\epsilon) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) B_{21}(\tau + \epsilon) + \right. \\
 & + B_{12}(\tau + k\epsilon) C_2(\tau + (k-1)\epsilon, \tau, \epsilon) A_{21}(\tau)] - \tilde{C}(t, \tau) [A_{12}(\tau) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) B_{21}(\tau) + \\
 & + B_{12}(\tau) C_2(\tau + (k-1)\epsilon, \tau, \epsilon) A_{21}(\tau)] \left. \right\} x_1(\tau) d\tau + \int_0^t [\tilde{C}(t, \tau + (k+1)\epsilon) \cdot \\
 & \cdot B_{12}(\tau + (k+1)\epsilon) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) B_{21}(\tau + \epsilon) - \tilde{C}(t, \tau) B_{12}(\tau) C_2(\tau + k\epsilon, \tau + \epsilon, \epsilon) \cdot \\
 & \cdot B_{21}(\tau)] x_1(\tau) d\tau \left. \right\}
 \end{aligned}$$

We are now ready for obtaining the estimates.

D. For the terms containing the initial function φ_1 we get the estimate:

$$\begin{aligned}
 & c \sup_{[s-\varepsilon, s]} |\varphi_1| \left\{ \int_{\Lambda-\varepsilon}^{\Lambda} e^{-\tilde{\alpha}(t-\sigma-\varepsilon)} d\sigma + \sum_{k=1}^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \left[\int_{\Lambda-\varepsilon}^{\Lambda} e^{-\tilde{\alpha}(t-\tau-k\varepsilon)} d\tau \right] \rho^{k-1} + \right. \\
 & \left. + \int_{\Lambda-\varepsilon}^{\Lambda} e^{-\tilde{\alpha}(t-\tau-(k+1)\varepsilon)} d\tau \right] \rho^{k-1} \Big\} \leq c \sup |\varphi_1| \left\{ e^{-\tilde{\alpha}(t-s-\varepsilon)} e^{-\tilde{\alpha}(t-s)} + \right. \\
 & \left. + \sum_{k=1}^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \rho^{k-1} \left[e^{-\tilde{\alpha}(t-k\varepsilon-s)} e^{-\tilde{\alpha}(t-(k-1)\varepsilon-s)} + e^{-\tilde{\alpha}(t-(k+1)\varepsilon-s)} e^{-\tilde{\alpha}(t-k\varepsilon-s)} \right] \right\} \\
 & = c \sup |\varphi_1| e^{-\tilde{\alpha}(t-s)} \left\{ e^{\tilde{\alpha}\varepsilon} - 1 + \sum_{k=1}^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \rho^{k-1} (e^{\tilde{\alpha}(k-1)\varepsilon} e^{\tilde{\alpha}(k-1)\varepsilon}) \right\} \\
 & \leq c \sup |\varphi_1| e^{-\tilde{\alpha}(t-\Lambda)} e^{\tilde{\alpha}\varepsilon} \left\{ e^{\tilde{\alpha}\varepsilon} - 1 + \sum_{k=1}^{\infty} (\rho e^{\tilde{\alpha}\varepsilon})^{k-1} (e^{2\tilde{\alpha}\varepsilon} - 1) \right\} \leq c \varepsilon e^{-\tilde{\alpha}(t-\Lambda)} \sup |\varphi_1|
 \end{aligned}$$

For the terms containing the initial function φ_2 we get the estimate

$$\begin{aligned}
 & c \sup |\varphi_2| \left[\sum_{l=1}^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \int_{\Lambda+l\varepsilon}^{\Lambda+l\varepsilon} e^{-\tilde{\alpha}(t-\sigma)} d\sigma \rho^l + \int_{\Lambda+\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor \varepsilon}^t e^{-\tilde{\alpha}(t-\sigma)} d\sigma \rho^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \right] \leq \\
 & \leq c \sup |\varphi_2| \left[\sum_{l=1}^{\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor} \int_{\Lambda+l\varepsilon}^{\Lambda+l\varepsilon} e^{-\tilde{\alpha}(t-\sigma)} \rho^{\frac{\sigma-\Lambda}{\varepsilon}} d\sigma + \int_{\Lambda+\lfloor \frac{t-\Lambda}{\varepsilon} \rfloor \varepsilon}^t e^{-\tilde{\alpha}(t-\sigma)} \rho^{\frac{\sigma-\Lambda}{\varepsilon}} d\sigma \right] = \\
 & = c \sup |\varphi_2| \left[\int_{\Lambda}^t e^{-\tilde{\alpha}(t-\sigma)} e^{-\alpha \frac{(\sigma-\Lambda)}{\varepsilon}} d\sigma \right] \leq c \varepsilon e^{-\tilde{\alpha}(t-\Lambda)} \sup |\varphi_2|
 \end{aligned}$$

for $\varepsilon > 0$ small enough.

E. We use next the estimate

$$|\tilde{C}(t, \sigma+\gamma) - \tilde{C}(t, \sigma)| \leq \gamma c e^{\tilde{\alpha}\gamma} e^{-\tilde{\alpha}(t-\sigma)}$$

to get

$$\left| \int_{\Lambda}^{t-\varepsilon} \tilde{C}(t, \sigma+\varepsilon) B_{11}(\sigma+\varepsilon) - \tilde{C}(t, \sigma) B_{11}(\sigma) \right] x_1(\sigma) d\sigma \Big| \leq c \varepsilon \int_{\Lambda}^{t-\varepsilon} e^{-\tilde{\alpha}(t-\sigma)} |x_1(\sigma)| d\sigma$$

Keep also the estimates

$$\left| \int_{\Lambda}^t \tilde{C}(t, \sigma) B_{11}(\sigma) x_1(\sigma) d\sigma \right| \leq c \int_{\Lambda}^t e^{-\tilde{\alpha}(t-\sigma)} |x_1(\sigma)| d\sigma$$

$$\begin{aligned}
 & \left| \int_{\Lambda}^{t-\varepsilon} \tilde{C}(t, \tau) [A_{12}(\tau) + B_{12}(\tau)] \left\{ [I - B_{22}(\tau+\varepsilon)]^{-1} - [I - B_{22}(\tau)]^{-1} \right\} B_{21}(\tau) \cdot \right. \\
 & \left. x_1(\tau) d\tau \right| \leq c \varepsilon \int_{\Lambda}^{t-\varepsilon} e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau
 \end{aligned}$$

We estimate next

$$\begin{aligned}
 & \left| \sum_{k=\lfloor \frac{t-\varepsilon}{\varepsilon} \rfloor+2}^{\infty} \int_{\Lambda}^t \tilde{C}(t, \tau) \left\{ [A_{12}(\tau) + B_{12}(\tau)] [B_{22}(\tau)]^{k-1} A_{21}(\tau) + \right. \right. \\
 & \left. \left. + [B_{22}(\tau+\varepsilon)]^{k-1} B_{21}(\tau) \right\} x_1(\tau) d\tau \right| \leq c \rho^{\lfloor \frac{t-\varepsilon}{\varepsilon} \rfloor+1} \int_{\Lambda}^t e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau \leq \\
 & \leq c \int_{\Lambda}^t e^{-\tilde{\alpha}(t-\tau)} e^{-\frac{\tilde{\alpha}(t-\tau)}{\varepsilon}} |x_1(\tau)| d\tau
 \end{aligned}$$

For Λ and ε small enough we have $\rho e^{\tilde{\alpha}\varepsilon} < 1$ and $e^{-\frac{\tilde{\alpha}(t-\tau)}{\varepsilon}} < 1$.

By using the estimate for Π_2 the next sum is estimated by

$$c\varepsilon \sum_{k=1}^{\lfloor \frac{t}{\varepsilon} \rfloor + 1} \int_s^t e^{-\tilde{\alpha}(t-\tau)} k(k-1) \rho^{k-2} |x_1(\tau)| d\tau \leq c\varepsilon \int_s^t e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau$$

For the next sum we get the estimate

$$c \sum_{k=1}^{\lfloor \frac{t}{\varepsilon} \rfloor + 1} \int_{t-k\varepsilon}^t e^{-\tilde{\alpha}(t-\tau)} \rho^{k-1} |x_1(\tau)| d\tau$$

For the first terms in the last sum we have

$$\left| \int_s^{t-(k-1)\varepsilon} [\tilde{C}(t, \tau + (k-1)\varepsilon) A_{12}(\tau + (k-1)\varepsilon) - \tilde{C}(t, \tau) A_{12}(\tau)] C_2(\tau + (k-1)\varepsilon, \tau, \varepsilon) A_{21}(\tau) x_1(\tau) d\tau \right| \leq$$

$$\leq c\varepsilon(k-1) e^{\tilde{\alpha}(k-1)\varepsilon} \rho^{k-1} \int_s^{t-(k-1)\varepsilon} e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau$$

and for the other terms we obtain similar estimates.

F. We may finally write the inequality

$$|x_1(t)| \leq c e^{-\tilde{\alpha}(t-s)} |x_1(s)| + c\varepsilon e^{-\tilde{\alpha}(t-s)} [\sup |\varphi_1| + \sup |\varphi_2|] +$$

$$+ c \int_{t-\varepsilon}^t e^{-\tilde{\alpha}(t-\sigma)} |x_1(\sigma)| d\sigma + c\varepsilon \int_s^t e^{-\tilde{\alpha}(t-\sigma)} |x_1(\sigma)| d\sigma +$$

$$+ c \int_s^t e^{-\tilde{\alpha}(t-\tau)} e^{-\frac{\alpha(t-\tau)}{\varepsilon}} |x_1(\tau)| d\tau + c \sum_{k=1}^{\lfloor \frac{t-s}{\varepsilon} \rfloor} \rho^{k-1} \int_{t-k\varepsilon}^t e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau +$$

$$+ \varepsilon(k-1) (e^{\tilde{\alpha}\varepsilon} \rho)^{k-1} \int_s^{t-k\varepsilon} e^{-\tilde{\alpha}(t-\tau)} |x_1(\tau)| d\tau \}.$$

Let $\mu(T, \varepsilon) = \max_{s \leq t \leq T} e^{\frac{\tilde{\alpha}}{2}t} |x_1(t)|$. We have the inequality

$$e^{\frac{\tilde{\alpha}}{2}t} |x_1(t)| \leq c e^{\frac{\tilde{\alpha}}{2}s} [|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|)] +$$

$$\begin{aligned}
 & + c\mu(T, \varepsilon) e^{-\frac{\alpha}{2}t} \int_0^t e^{\frac{\alpha}{2}\sigma} d\sigma + \varepsilon c\mu(T, \varepsilon) e^{-\frac{\alpha}{2}t} \int_s^t e^{\frac{\alpha}{2}\sigma} d\sigma + \\
 & + c\mu(T, \varepsilon) \int_s^t e^{-\frac{\alpha}{2}(t-\zeta)} e^{-\frac{\alpha}{\varepsilon}(t-\zeta)} d\zeta + \\
 & + c \sum_{k=1}^{\lfloor \frac{t-\varepsilon}{\varepsilon} \rfloor + 1} \left\{ \rho^{k-1} e^{-\frac{\alpha}{2}t} \int_{t-k\varepsilon}^t e^{\frac{\alpha}{2}\zeta} d\zeta + \varepsilon(k-1) \tilde{\rho}^{k-1} e^{-\frac{\alpha}{2}t} \int_s^{t-k\varepsilon} e^{\frac{\alpha}{2}\zeta} d\zeta \right\} \mu(T, \varepsilon)
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mu(T, \varepsilon) & \leq c e^{-\frac{\alpha}{2}s} \left[|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) \right] + \\
 & + c\mu(T, \varepsilon) (1 - e^{-\frac{\alpha}{2}\varepsilon}) + c\varepsilon\mu(T, \varepsilon) + c \sum_{k=1}^{\lfloor \frac{t-\varepsilon}{\varepsilon} \rfloor + 1} \left\{ \rho^{k-1} (1 - e^{-\frac{\alpha}{2}k\varepsilon}) + \right. \\
 & \left. + \varepsilon(k-1) \tilde{\rho}^{k-1} e^{-\frac{\alpha}{2}k\varepsilon} \right\} \mu(T, \varepsilon)
 \end{aligned}$$

We see that $\mu(T, \varepsilon) \leq c e^{-\frac{\alpha}{2}s} \left[|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) \right]$ and,

uniformly for $s \leq t$,

$$|x_1(t)| \leq c e^{-\frac{\alpha}{2}(t-s)} \left[|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) \right]$$

If we replace this estimate into the formula for x_2 we finally get

$$|x_2(t)| \leq c \rho^{\lfloor \frac{t-s}{\varepsilon} \rfloor} \sup |\varphi_2| + c e^{-\frac{\alpha}{2}(t-s)} \left[|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) \right]$$

Let us state the final result as

THEOREM 1. Consider the system (1) and assume:

a) All matrices in the coefficients are uniformly bounded and uniformly Lipschitz on (a, ∞) .

b) The spectral radius of $B_{22}(t)$ satisfies

$$|\sigma(B_{22}(t))| \leq \hat{\rho} \leq 1.$$

c) The system $\dot{x}_1 = \tilde{A}(t)x_1$, where

$$\tilde{A}(t) = A_{11}(t) + B_{11}(t) + [A_{12}(t) + B_{12}(t)] [I - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)]$$

is exponentially stable, that is its fundamental matrix satisfies $|\tilde{C}(t, s)| \leq c e^{-\tilde{\alpha}(t-s)}$

Then, for $\varepsilon > 0$ small enough all solutions of the system satisfy the estimates, uniformly for $s \leq t$,

$$|x_1(t)| \leq c e^{-\frac{\alpha}{2}(t-s)} [|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|)]$$

$$|x_2(t)| \leq c \int_{\frac{t-s}{\varepsilon}}^{\frac{t-s}{\varepsilon}} \sup |\varphi_2| + c e^{-\frac{\alpha}{2}(t-s)} [|x_1(s)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|)]$$

with $\hat{\beta} < \beta < 1$, hence system (1) is exponentially stable.

REMARKS. 1. From theorem 1 we deduce that a matrix-solution of system (1) with $X_1(s,s)=I$, $X_1(t,s)=0$, $X_2(t,s)=0$ $t < s$, satisfies the estimates

$$|X_1(t,s)| \leq c e^{-\hat{\alpha}(t-s)}, \quad |X_2(t,s)| \leq c e^{-\hat{\alpha}(t-s)}$$

2. Consider a standard neutral-type system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-\varepsilon) + B_1(t)\dot{x}(t-\varepsilon)$$

Denote $x_2(t) = \dot{x}(t)$, $x_1(t) = x(t)$ to get a system of the form (1) with $A_{11} = 0$, $A_{12} = I$, $B_{11} = 0$, $B_{12} = 0$, $A_{21} = A_0$, $B_{21} = A_1$, $B_{22} = B_1$

The main assumptions are $\sigma(B_1(t)) \leq \hat{\beta} < 1$, and

$\tilde{A}(t) = [I - B_1(t)]^{-1} (A_0(t) + A_1(t))$ defines an exponentially stable system.

Under these assumptions theorem 1 gives the exponential stability with corresponding estimates for $|x(t)|$ and $|\dot{x}(t)|$.

4. An adjoint system and the variation of constants formula

A. Consider the adjoint system

$$\begin{aligned} \dot{z}_1(s) + z_1(s)A_{11}(s) + z_1(s+\varepsilon)B_{11}(s+\varepsilon) + z_2(s)A_{21}(s) + \\ + z_2(s+\varepsilon)B_{21}(s+\varepsilon) = 0 \end{aligned} \quad (2)$$

$$z_2(s) = z_1(s)A_{12}(s) + z_1(s+\varepsilon)B_{12}(s+\varepsilon) + z_2(s+\varepsilon)B_{22}(s+\varepsilon)$$

Multiply the first equation by $Y_1(s,t)$, the second by $Y_2(s,t)$ integrate from t to T , add and integrate by parts to get:

$$\begin{aligned}
 & z_1(T)Y_1(T,t) - z_1(t)Y_1(t,t) + \int_t^T z_1(s) \left[-Y_1'(s,t) + A_{11}(s)Y(s,t) + \right. \\
 & \left. + B_{11}(s)Y_1(s-\varepsilon,t) + A_{12}(s)Y_2(s,t) + B_{12}(s)Y_2(s-\varepsilon,t) \right] ds + \\
 & + \int_t^T z_2(s) \left[A_{21}(s)Y_1(s,t) + B_{21}(s)Y_1(s-\varepsilon,t) - Y_2(s,t) + B_{22}(s)Y_2(s-\varepsilon,t) \right] ds + \\
 & + \int_{T+\varepsilon}^T z_1(s) B_{11}(s) Y_1(s-\varepsilon,t) ds + \int_{T+\varepsilon}^T z_2(s) B_{21}(s) Y_1(s-\varepsilon,t) ds + \\
 & + \int_T^{T+\varepsilon} z_1(s) B_{12}(s) Y_2(s-\varepsilon,t) ds + \int_T^{T+\varepsilon} z_2(s) B_{22}(s) Y_2(s-\varepsilon,t) ds = 0
 \end{aligned}$$

Choose Y_1, Y_2 defined by

$$Y_1'(s,t) = A_{11}(s)Y_1(s,t) + B_{11}(s)Y_1(s-\varepsilon,t) + A_{12}(s)Y_2(s,t) +$$

$$+ B_{12}(s)Y_2(s-\varepsilon,t)$$

$$Y_2(s,t) = A_{21}(s)Y_1(s,t) + B_{21}(s)Y_1(s-\varepsilon,t) + B_{22}(s)Y_2(s-\varepsilon,t)$$

$$Y_1(t,t) = I, \quad Y_1(s,t) = 0 \text{ for } s < t, \quad Y_2(s,t) = 0 \text{ for } s < t.$$

Then we get the representation

$$\begin{aligned}
 z_1(t) &= z_1(T)Y_1(T,t) + \int_T^{T+\varepsilon} z_1(s) \left[B_{11}(s)Y_1(s-\varepsilon,t) + B_{12}(s)Y_2(s-\varepsilon,t) \right] ds + \\
 & + \int_T^T z_2(s) \left[B_{21}(s)Y_1(s-\varepsilon,t) + B_{22}(s)Y_2(s-\varepsilon,t) \right] ds
 \end{aligned}$$

From the system of equations and the initial conditions we see that $Y_1(s,t) = X_1(s,t)$, $Y_2(s,t) = X_2(s,t)$ (see remark 1) and under the assumptions of the theorem 1 we have the estimates $|Y_1(T,t)| \leq c e^{-\hat{\alpha}(T-t)}$, $|Y_1(s-\varepsilon,t)| + |Y_2(s-\varepsilon,t)| \leq c e^{-\hat{\alpha}(s-\varepsilon-t)}$.

If the "initial" functions for z_1 and z_2 are denoted by ψ_1 , ψ_2 respectively, we deduce

$$|z_1(t)| \leq c e^{-\hat{\alpha}(T-t)} \left[|z_1(T)| + c (\sup |\psi_1| + \sup |\psi_2|) \int_T^{T+\varepsilon} e^{-\hat{\alpha}(s-\varepsilon-t)} ds \right]$$

hence

$$|z_1(t)| \leq c e^{-\hat{\alpha}(T-t)} \left[|z_1(T)| + \varepsilon (\sup |\psi_1| + \sup |\psi_2|) \right]$$

for $\varepsilon > 0$ small enough.

The second equation in (2) gives then

$$z_2(s) = \Psi_2(s+l\varepsilon)C_2(s+l\varepsilon, s, \varepsilon) + \sum_{k=0}^{l-1} [z_1(s+k\varepsilon)A_{12}(s+k\varepsilon) + z_1(s+(k+1)\varepsilon)A_{12}(s+(k+1)\varepsilon)]C_2(s+k\varepsilon, s, \varepsilon)$$

and we deduce the estimate

$$|z_2(t)| \leq c \rho^{\frac{T-t}{\varepsilon}} \sup |\Psi_2| + c e^{-\hat{\lambda}(T-t)} (|z_1(T)| + \varepsilon (\sup |\Psi_1| + \sup |\Psi_2|))$$

For the matrix solution with $\Psi_1=0$, $\Psi_2=0$, $Z_1(T)=I$ we

get the usual exponential estimates

$$|Z_1(t, T)| \leq c e^{-\hat{\lambda}(T-t)}, \quad |Z_2(t, T)| \leq c e^{-\hat{\lambda}(T-t)}, \quad t \leq T.$$

For this solution we shall have $Z_1(t, T) = Y_1(T, t) = X_1(T, t)$.

B. Consider now the affine system

$$\begin{aligned} \dot{x}_1(t) &= A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_{11}(t)x_1(t-\varepsilon) + \\ &\quad + B_{12}(t)x_2(t-\varepsilon) + f_1(t) \\ x_2(t) &= A_{21}(t)x_1(t) + B_{21}(t)x_1(t-\varepsilon) + \\ &\quad + B_{22}(t)x_2(t-\varepsilon) + f_2(t) \end{aligned} \quad (3)$$

under the same assumptions as in theorem 1 and f_1, f_2 continuous.

Write the equations in the variable s , multiply the first equation by $Z_1(s, t)$, the second one by $Z_2(s, t)$ integrate from t_0 to t , add, integrate by parts, use the "initial" conditions for Z_1 and Z_2 .

We get the variations of constants formula

$$\begin{aligned} x_1(t) &= Z_1(t_0, t)x_1(t_0) + \int_{t_0-\varepsilon}^t [Z_1(s+\varepsilon, t)B_{11}(s+\varepsilon) + Z_2(s+\varepsilon, t)B_{21}(s+\varepsilon)]\varphi_1(s)ds + \\ &\quad + \int_{t_0-\varepsilon}^t [Z_1(s+\varepsilon, t)B_{12}(s+\varepsilon) + Z_2(s+\varepsilon, t)B_{22}(s+\varepsilon)]\varphi_2(s)ds + \\ &\quad + \int_{t_0}^t [Z_1(s, t)f_1(s) + Z_2(s, t)f_2(s)]ds \end{aligned}$$

$$\begin{aligned} x_2(t) &= C_2(t, t-l\varepsilon, \varepsilon)\varphi_2(t-l\varepsilon) + \\ &\quad + \sum_{k=0}^{l-1} C_2(t, t-k\varepsilon, \varepsilon) [A_{21}(t-k\varepsilon)x_1(t-k\varepsilon) + B_{21}(t-k\varepsilon)x_1(t-(k-1)\varepsilon)] + \\ &\quad + \sum_{k=0}^{l-1} C_2(t, t-k\varepsilon, \varepsilon)f_2(t-k\varepsilon), \quad \text{for } t_0 + (l-1)\varepsilon < t \leq t_0 + l\varepsilon \end{aligned}$$

Remarks. 1. The presence of the term $A_{12}(t)x_2(t)$ in system (1) may seem artificial since we may replace $x_2(t)$ by using the second equation. It is just the consideration of the adjoint system and the corresponding variations of constants formula that suggested the form we adopted for (1).

2. Under the assumptions in the Theorem 1 the formulae above give for the solutions of the affine system the estimates

$$|x_1(t)| \leq c e^{-\hat{\alpha}(t-s)} [|x_1(t_0)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) + c(\sup |f_1| + \sup |f_2|)]$$

$$|x_2(t)| \leq c \rho^{\lfloor \frac{t-t_0}{\varepsilon} \rfloor} \sup |\varphi_2| + c e^{-\hat{\alpha}(t-s)} [|x_1(t_0)| + \varepsilon (\sup |\varphi_1| + \sup |\varphi_2|) + c(\sup |f_1| + \sup |f_2|)]$$

5. Asymptotic expansions

Consider again system (1) under the assumptions of theorem 1, define \tilde{x}_1 as the solution of $\dot{x}_1 = \tilde{A}(t)x_1$, $\tilde{x}_1(t_0) = x_1(t_0)$ for $t \geq t_0$, $\tilde{x}_1(t) = \varphi_1(t)$ for $t_0 - \varepsilon \leq t < t_0$, and \tilde{x}_2 from the difference equation

$$x_2(t) = A_{21}(t)\tilde{x}_1(t) + B_{21}(t)\tilde{x}_1(t-\varepsilon) + B_{22}(t)x_2(t)$$

$$\tilde{x}_2(t) = \varphi_2(t) \text{ for } t_0 - \varepsilon \leq t < t_0.$$

We shall prove that \tilde{x}_1, \tilde{x}_2 represent the main part of the asymptotic expansions of x_1, x_2 with respect to ε ; more precisely we write

$$x_1(t, \varepsilon) = \tilde{x}_1(t) + \varepsilon \tilde{f}_1(t, \varepsilon), \quad x_2(t, \varepsilon) = \tilde{x}_2(t, \varepsilon) + \varepsilon \tilde{f}_2(t, \varepsilon)$$

and prove that \tilde{f}_1 and \tilde{f}_2 are uniformly bounded, and more-

-over we obtain for them exponential estimates.

We get for ξ_1 and ξ_2 the equations

$$\begin{aligned} \dot{\xi}_1(t, \varepsilon) = & A_{11}(t)\xi_1(t, \varepsilon) + A_{12}(t)\xi_2(t, \varepsilon) + B_{11}(t)\xi_1(t-\varepsilon, \varepsilon) + \\ & + B_{12}(t)\xi_2(t-\varepsilon, \varepsilon) + \frac{1}{\varepsilon} B_{11}(t)[\tilde{x}_1(t-\varepsilon) - \tilde{x}_1(t)] + \\ & + \frac{1}{\varepsilon} A_{12}(t)[\tilde{x}_2(t, \varepsilon) - \hat{x}_2(t)] + \frac{1}{\varepsilon} B_{12}(t)[\tilde{x}_2(t-\varepsilon, \varepsilon) - \hat{x}_2(t)] \end{aligned}$$

$$\dot{\xi}_2(t, \varepsilon) = A_{21}(t)\xi_1(t, \varepsilon) + B_{21}(t)\xi_1(t-\varepsilon, \varepsilon) + B_{22}(t)\xi_2(t-\varepsilon, \varepsilon)$$

with zero initial conditions.

Here we have set $\hat{x}_2(t) = [I - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)] \tilde{x}_1(t)$.

By using the variation of constants formula we may write

$$\begin{aligned} \xi_1(t, \varepsilon) = & \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) [B_{11}(s)(\tilde{x}_1(s-\varepsilon) - \tilde{x}_1(s)) + A_{12}(s)(\tilde{x}_2(s, \varepsilon) - \\ & - \hat{x}_2(s)) + B_{12}(s)(\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s))] ds \end{aligned}$$

We have $\tilde{x}_1(s) = \tilde{C}(s, t_0)x_1(t_0)$; hence

$$\begin{aligned} |\tilde{x}_1(s-\varepsilon) - \tilde{x}_1(s)| & \leq |\tilde{C}(s-\varepsilon, t_0) - \tilde{C}(s, t_0)| |x_1(t_0)| = \\ & = \varepsilon \left| \int_0^1 \tilde{C}'(s-\theta\varepsilon, t_0) d\theta \right| |x_1(t_0)| = \varepsilon \left| \int_0^1 \tilde{A}(s-\theta\varepsilon, t_0) \tilde{C}(s-\theta\varepsilon, t_0) d\theta \right| |x_1(t_0)| \leq \\ & \leq c\varepsilon e^{-\tilde{\alpha}(s-t_0)} |x_1(t_0)| \quad \text{for } s \geq t_0 + \varepsilon \end{aligned}$$

We may then write

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) B_{11}(s) [\tilde{x}_1(s-\varepsilon) - \tilde{x}_1(s)] ds \right| \leq \\ & \leq \frac{1}{\varepsilon} \left| \int_{t_0}^{t_0+\varepsilon} Z_1(s, t) B_{11}(s) [\tilde{x}_1(s-\varepsilon) - \tilde{x}_1(s)] ds \right| + \\ & + \frac{1}{\varepsilon} \left| \int_{t_0}^t Z_1(s, t) B_{11}(s) [\tilde{x}_1(s-\varepsilon) - \tilde{x}_1(s)] ds \right| \leq \frac{c}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} e^{-\tilde{\alpha}(t-s)} ds + \\ & + c \int_{t_0}^t e^{-\tilde{\alpha}(t-s)} e^{-\tilde{\alpha}(s-t_0)} ds |x_1(t_0)| \leq c e^{-\tilde{\alpha}(t-t_0)} \end{aligned}$$

where the constant c depends upon $|x_1(t_0)|$.

We write next

$$\begin{aligned}
 & \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) B_{12}(s) (\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s)) = \\
 & = \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) B_{12}(s) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s)] ds + \\
 & + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} Z_1(s, t) B_{12}(s) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s)] ds = \\
 & = \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} Z_1(s, t) B_{12}(s) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s)] ds + \\
 & + \frac{1}{\varepsilon} \int_{t_0}^{t-\varepsilon} Z_1(s+\varepsilon, t) B_{12}(s+\varepsilon) [\tilde{x}_2(s, \varepsilon) - \hat{x}_2(s)] ds + \\
 & + \frac{1}{\varepsilon} \int_{t_0}^{t-\varepsilon} Z_1(s+\varepsilon, t) B_{12}(s+\varepsilon) [\hat{x}_2(s) - \hat{x}_2(s+\varepsilon)] ds
 \end{aligned}$$

We deduce

$$\begin{aligned}
 & \left| \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) B_{12}(s) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \hat{x}_2(s)] ds \right| \leq c e^{-\hat{\alpha}(t-s)} + \\
 & + \left| \frac{1}{\varepsilon} \int_{t_0}^{t-\varepsilon} Z_1(s+\varepsilon, t) B_{12}(s+\varepsilon) [\tilde{x}_2(s, \varepsilon) - \hat{x}_2(s)] ds \right|
 \end{aligned}$$

We may finally write

$$\begin{aligned}
 & \left| \int_1^{\varepsilon}(t, \varepsilon) \right| \leq c e^{-\hat{\alpha}(t-t_0)} + \left| \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) A_{12}(s) (\tilde{x}_2(s, \varepsilon) - \right. \\
 & \left. - \hat{x}_2(s)) ds \right| + \left| \frac{1}{\varepsilon} \int_{t_0}^{t-\varepsilon} Z_1(s+\varepsilon, t) B_{12}(s+\varepsilon) [\tilde{x}_2(s, \varepsilon) - \hat{x}_2(s)] ds \right|
 \end{aligned}$$

Denote $y_2(t, \varepsilon) = \tilde{x}_2(t, \varepsilon) - \hat{x}_2(t)$. We have $\hat{x}_2(t) - B_{22}(t)\hat{x}_2(t) =$
 $= [A_{21}(t) + B_{21}(t)] \tilde{x}_1(t)$ hence

$$\begin{aligned}
 \hat{x}_2(t) = & A_{21}(t)\tilde{x}_1(t) + B_{21}(t)\tilde{x}_1(t-\varepsilon) + B_{22}(t)\hat{x}_2(t-\varepsilon) + \\
 & + B_{21}(t)[\tilde{x}_1(t) - \tilde{x}_1(t-\varepsilon)] + B_{22}(t)[\hat{x}_2(t) - \hat{x}_2(t-\varepsilon)]
 \end{aligned}$$

We deduce that

$$\begin{aligned}
 y_2(t, \varepsilon) = & B_{22}(t)y_2(t-\varepsilon, \varepsilon) - B_{21}(t)[\tilde{x}_1(t) - \tilde{x}_1(t-\varepsilon)] - \\
 & - B_{22}(t)[\hat{x}_2(t) - \hat{x}_2(t-\varepsilon)]
 \end{aligned}$$

hence, for $t_0 + (l-1)\varepsilon < t \leq t_0 + l\varepsilon$

$$\begin{aligned}
 y_2(t, \varepsilon) = & C_2(t, t-l\varepsilon)y_2(t-l\varepsilon, \varepsilon) + \\
 & + \sum_{k=0}^{l-1} C_2(t, t-k\varepsilon, \varepsilon) \left\{ B_{21}(t-k\varepsilon)[\tilde{x}_1(t-(k+1)\varepsilon) - \right. \\
 & \left. - \tilde{x}_1(t-k\varepsilon)] + B_{22}(t-k\varepsilon)[\hat{x}_2(t-(k+1)\varepsilon) - \hat{x}_2(t-k\varepsilon)] \right\},
 \end{aligned}$$

and we have the estimate

$$|y_2(t, \varepsilon)| \leq c \rho^l \left(\sup_{t_0 + (l-1)\varepsilon \leq t \leq t_0 + l\varepsilon} |x_2(t-l\varepsilon) - \hat{x}_2(t-l\varepsilon)| \right) + \varepsilon c e^{-\tilde{\alpha}(t-t_0)}$$

for $t_0 + (l-1)\varepsilon \leq t \leq t_0 + l\varepsilon$

hence .

$$|y_2(t, \varepsilon)| \leq c e^{-\frac{\alpha(t-t_0)}{\varepsilon}} \left(\sup_{t_0 - \varepsilon \leq t \leq t_0} |\tilde{x}_2(t) - \hat{x}_2(t)| \right) + \varepsilon c e^{-\tilde{\alpha}(t-t_0)} \quad \text{for } t \geq t_0$$

We deduce that

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{t_0}^t Z_1(s, t) A_{12}(s) (\tilde{x}_2(s, \varepsilon) - \hat{x}_2(s)) ds \right| + \\ & + \left| \frac{1}{\varepsilon} \int_{t_0}^{t-\varepsilon} Z_1(s+\varepsilon, t) B_{12}(s+\varepsilon) [\tilde{x}_2(s, \varepsilon) - x_2(s)] ds \right| \leq \\ & \leq \frac{c}{\varepsilon} \int_{t_0}^t \left\{ e^{-\hat{\alpha}(t-s)} \left[e^{-\frac{\alpha(s-t_0)}{\varepsilon}} \left(\sup_{t_0 - \varepsilon \leq \sigma \leq t_0} |\tilde{x}_2(\sigma) - \hat{x}_2(\sigma)| \right) + \right. \right. \\ & \left. \left. + \varepsilon e^{-\tilde{\alpha}(s-t_0)} \right] \right\} ds + \frac{c}{\varepsilon} \int_{t_0}^{t-\varepsilon} \left\{ e^{-\hat{\alpha}(t-s-\varepsilon)} \left[e^{-\frac{\alpha(s-t_0)}{\varepsilon}} \left(\sup_{t_0 - \varepsilon \leq \sigma \leq t_0} |\tilde{x}_2(\sigma) - \hat{x}_2(\sigma)| \right) + \right. \right. \\ & \left. \left. + \varepsilon e^{-\tilde{\alpha}(s-t_0)} \right] \right\} ds \leq c e^{-\tilde{\alpha}(t-t_0)} \end{aligned}$$

With this estimate for \tilde{y}_1 we use the estimate in section 2 to get the corresponding estimate for \tilde{y}_2 and the validity of the asymptotic expansion is established.

Let us state this result as a theorem .

THEOREM 2. Under the same assumptions as in theorem 1, we have for an arbitrary solution of system (1)

$$x_1(t, \varepsilon) = \tilde{x}_1(t) + \varepsilon \tilde{y}_1(t, \varepsilon)$$

$$x_2(t, \varepsilon) = \hat{x}_2(t) + y_2(t, \varepsilon) + \varepsilon \tilde{y}_2(t, \varepsilon)$$

where \tilde{x}_1 and \hat{x}_2 are defined by

$$\tilde{x}_1' = \tilde{A}(t) \tilde{x}_1, \quad \tilde{x}_1(t_0) = x_1(t_0, \varepsilon)$$

$$\hat{x}_2(t) = [I - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)] \tilde{x}_1(t)$$

and $\xi_1, y_2, \hat{\xi}_2$ admit the estimates
 $|\xi_1(t, \varepsilon)| \leq c e^{-\hat{\alpha}(t-t_0)}$, $|\hat{\xi}_2(t, \varepsilon)| \leq c \left[e^{-\hat{\alpha}(t-t_0)} + e^{-\frac{\alpha(t-t_0)}{\varepsilon}} \right]$
 $|y_2(t, \varepsilon)| \leq c e^{-\frac{\alpha(t-t_0)}{\varepsilon}}$. $\sup_{t_0 - \varepsilon \leq t \leq t_0} |x_2(t, \varepsilon) - \hat{x}_2(t)| + \varepsilon c e^{-\tilde{\alpha}(t-t_0)}$

6. Higher order approximations

Let $\tilde{x}_1, \tilde{x}_2, \hat{x}_2$ be as above and write the expansion in the form

$$x_1(t, \varepsilon) = \tilde{x}_1(t) + \sum_{k=1}^{\ell+1} \varepsilon^k \xi_1^k(t, \varepsilon)$$

$$x_2(t, \varepsilon) = \tilde{x}_2(t, \varepsilon) + \sum_{k=1}^{\ell+1} \varepsilon^k \xi_2^k(t, \varepsilon)$$

where $\xi_1^k, \xi_2^k, \hat{\xi}_2^k$ $1 \leq k \leq \ell$, are defined by

$$\begin{aligned} \xi_1^k(t, \varepsilon) = & \tilde{A}(t) \xi_1^k(t, \varepsilon) - \frac{1}{\varepsilon} B_{11}(t) [\xi_1^{k-1}(t, \varepsilon) - \xi_1^{k-1}(t-\varepsilon, \varepsilon)] + \\ & + \frac{1}{\varepsilon} A_{12}(t) [\xi_2^{k-1}(t, \varepsilon) - \hat{\xi}_2^{k-1}(t, \varepsilon)] + \frac{1}{\varepsilon} B_{12}(t) [\xi_2^{k-1}(t-\varepsilon, \varepsilon) - \\ & - \hat{\xi}_2^{k-1}(t, \varepsilon)] \end{aligned}$$

$$\xi_2^k(t, \varepsilon) = A_{21}(t) \xi_1^k(t, \varepsilon) + B_{21}(t) \xi_1^k(t-\varepsilon, \varepsilon) + B_{22}(t) \xi_2^k(t-\varepsilon, \varepsilon)$$

$$\hat{\xi}_2^k(t, \varepsilon) = [I - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)] \xi_1^k(t, \varepsilon)$$

and we have set $\xi_1^0 = \tilde{x}_1$, $\xi_2^0 = \tilde{x}_2$, $\hat{\xi}_2^0 = \hat{x}_2$; finally

$$\begin{aligned} \xi_1^{\ell+1}(t, \varepsilon) = & A_{11}(t) \xi_1^{\ell+1}(t, \varepsilon) + A_{12}(t) \xi_2^{\ell+1}(t, \varepsilon) + B_{11}(t) \xi_1^{\ell+1}(t-\varepsilon, \varepsilon) + \\ & + B_{12}(t) \xi_2^{\ell+1}(t-\varepsilon, \varepsilon) + \frac{1}{\varepsilon} B_{11}(t) [\xi_1^{\ell}(t-\varepsilon, \varepsilon) - \xi_1^{\ell}(t, \varepsilon)] + \\ & + \frac{1}{\varepsilon} A_{12}(t) [\xi_2^{\ell}(t, \varepsilon) - \hat{\xi}_2^{\ell}(t, \varepsilon)] + \frac{1}{\varepsilon} B_{12}(t) [\xi_2^{\ell}(t-\varepsilon, \varepsilon) - \hat{\xi}_2^{\ell}(t, \varepsilon)] \end{aligned}$$

$$\xi_2^{l+1}(t, \varepsilon) = A_{21}(t) \xi_1^{l+1}(t, \varepsilon) + B_{21}(t) \xi_1^{l+1}(t - \varepsilon, \varepsilon) + B_{22}(t) \xi_2^{l+1}(t - \varepsilon, \varepsilon)$$

We have to prove that with $\xi_1^k, \xi_2^k, 1 \leq k \leq l$ as above the equations for ξ_1^{l+1}, ξ_2^{l+1} are the ones we have written, then that $\xi_1^k, \xi_2^k, 1 \leq k \leq l+1$ are bounded.

To check the equations for ξ_1^{l+1}, ξ_2^{l+1} we have to set the expansion into ^{the} initial system (1), replace in the terms not involving ξ_1^{l+1}, ξ_2^{l+1} the coefficient matrix A_{11} by using the formulae for \tilde{A} and for $\hat{\xi}_2^k$ and take into account the equations for $\tilde{x}_1, \tilde{x}_2, \xi_1^k, \xi_2^k, 1 \leq k \leq l$.

Remark also that $\xi_1^k, \xi_2^k, 1 \leq k \leq l+1$ have zero initial conditions.

Assume by induction

$$|\xi_1^{k-1}(t, \varepsilon) - \xi_1^{k-1}(t - \varepsilon, \varepsilon)| \leq c \left[\varepsilon e^{-\hat{\alpha}(t-t_0)} + e^{-\frac{\alpha(t-t_0)}{\varepsilon}} \right]$$

$$|\xi_1^{k-1}(t, \varepsilon)| \leq c e^{-\hat{\alpha}(t-t_0)}$$

$$|\xi_2^{k-1}(t, \varepsilon) - \hat{\xi}_2^{k-1}(t, \varepsilon)| \leq c \left[\varepsilon e^{-\hat{\alpha}(t-t_0)} + e^{-\frac{\alpha(t-t_0)}{\varepsilon}} \right]$$

These assumptions are satisfied by $\tilde{x}_1, \tilde{x}_2, \hat{x}_2$ hence for $k=1$.

Then a direct look at the equations shows that the estimates propagate. What concerns ξ_1^{l+1} and ξ_2^{l+1} the proof of the boundedness proceeds as in the case $l=0$ by using the estimates above.

REMARK. The asymptotic expansions described above does not require more smoothness than assumed in Theorem 1. If more smoothness is available the terms ξ_1^k, ξ_2^k may be decomposed in "interior" terms not depending upon ε and "boundary-layer" terms.

7. Expansions for affine systems

For the affine system we have the representation formula

$$\begin{aligned}
 x_1(t, \varepsilon) = & Z_1(t_0, t)x_1(t_0) + \int_{t_0-\varepsilon}^{t_0} [Z_1(s+\varepsilon, t)B_{11}(s+\varepsilon) + \\
 & + Z_2(s+\varepsilon, t)B_{21}(s+\varepsilon)] \varphi_1(s) + \int_{t_0-\varepsilon}^t [Z_1(s+\varepsilon, t)B_{12}(s+\varepsilon) + \\
 & + Z_2(s+\varepsilon, t)B_{22}(s+\varepsilon)] \varphi_2(s) ds + \\
 & + \int_{t_0}^t [Z_1(s, t)f_1(s) + Z_2(s, t)f_2(s)] ds
 \end{aligned}$$

Recall that $Z_1(s, t) = X_1(t, s)$

$$\begin{aligned}
 Z_2(s, t) = & Z_1(s, t)A_{12}(s) + Z_1(s+\varepsilon, t)B_{12}(s+\varepsilon) + \\
 & + Z_2(s+\varepsilon)B_{22}(s+\varepsilon)
 \end{aligned}$$

Use the asymptotic expansions

$$X_1(t, s) = \tilde{C}(t, s) + \sum_{k=1}^{l+1} \varepsilon^k \tilde{C}^k(t, s)$$

with \tilde{C}^k estimated as above,

define $\hat{Z}_2(s, t) = [\tilde{C}(t, s)A_{21}(s) + \tilde{C}(t, s+\varepsilon)B_{21}(s+\varepsilon)] [I - B_{22}(s+\varepsilon)]^{-1}$

write $Z_2(s, t) = \tilde{Z}_2(s, t) + \sum_{k=1}^{l+1} \varepsilon^k Z_2^k(s, t)$

prove in the usual way that

$$|\tilde{Z}_2(s, t) - \hat{Z}_2(s, t)| \leq c \left(e^{-\frac{\alpha(t-s)}{\varepsilon}} + \varepsilon e^{-\hat{\alpha}(t-s)} \right)$$

and replace in the representation formula.

We have to look closer only at the principal term of the expansion given by

$$\begin{aligned}
 & \tilde{C}(t, t_0)x_1(t_0) + \int_{t_0}^t [\tilde{C}(t, s)f_1(s) + \hat{Z}_2(s, t)f_2(s)] ds = \\
 = & \tilde{C}(t, t_0)x_1(t_0) + \int_{t_0}^t \{ \tilde{C}(t, s)f_1(s) + \tilde{C}(t, s)[A_{21}(s) + \\
 & + B_{21}(s)] [I - B_{22}(s)]^{-1} f_2(s) \} ds + \int_{t_0}^t \{ [\tilde{C}(t, s)A_{21}(s) + \\
 & + \tilde{C}(t, s+\varepsilon)B_{21}(s+\varepsilon)] [I - B_{22}(s+\varepsilon)]^{-1} \tilde{C}(t, s)[A_{21}(s) + \\
 & + B_{21}(s)] [I - B_{22}(s)]^{-1} \} f_2(s) ds
 \end{aligned}$$

We see that here the principal term can be written as

$$\tilde{C}(t, t_0)x_1(t_0) + \int_{t_0}^t \tilde{C}(t, s)\tilde{f}(s)ds$$

with $\tilde{f}(s) = f_1(s) + [A_{21}(s) + B_{21}(s)] [I - B_{22}(s)]^{-1} f_2(s)$

and the principal term is naturally defined by formally setting $\varepsilon = 0$.

8. The nonlinear case.

The system considered had the general form

$$\dot{x}_1(t) = F_1(t, x_1(t), x_2(t), x_1(t-\varepsilon), x_2(t-\varepsilon))$$

$$x_2(t) = F_2(t, x_1(t), x_1(t-\varepsilon), x_2(t-\varepsilon))$$

The reduced model is obtained by setting $\varepsilon = 0$

$$\dot{\tilde{x}}_1(t) = F_1(t, x_1(t), x_2(t), x_1(t), x_2(t))$$

$$x_2(t) = F_2(t, x_1(t), x_1(t), x_2(t))$$

to get \tilde{x}_1 and \hat{x}_2 and

$$\tilde{x}_2(t, \varepsilon) = F_2(t, \tilde{x}_1(t), \tilde{x}_1(t-\varepsilon), \tilde{x}_2(t-\varepsilon))$$

with corresponding initial conditions.

4 -

ASYMPTOTIC EXPANSIONS FOR COUPLED SYSTEMS
OF DIFFERENCE-DIFFERENTIAL AND DIFFERENCE EQUATIONS
A CRITICAL CASE

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1. Introduction

We consider again a coupled system of differential-difference and difference equations with small delays which arises when considering wave propagation with boundary conditions containing differential equations.

As it happens often in applications this system is now critical in the sense that the spectrum of a matrix is close to the stability boundary. To be more precise we consider the system

$$\dot{x}_1(t) = A_{11}(\varepsilon)x_1(t) + A_{12}(\varepsilon)x_2(t) + B_{11}(\varepsilon)x_1(t-\varepsilon) + B_{12}(\varepsilon)x_2(t-\varepsilon)$$

$$x_2(t) = \varepsilon A_{21}(\varepsilon)x_1(t) + \varepsilon B_{21}(\varepsilon)x_1(t-\varepsilon) + [I + \varepsilon B_{22}(\varepsilon)]x_2(t-\varepsilon)$$

The main problem will consist in getting asymptotic expansions and estimates for the fundamental matrices associated to this system, as a basic tool for the study of affine systems.

We consider here the case of constant coefficients; variable coefficients for the first part of the system could be considered with no major changes, but the problem is complicated enough as it stands and we leave the possible extensions for further considerations.

2. Representation formulae and main assumptions

The first component of the solution may be represented in the form:

$$x_1(t) = Z_1(t_0, t)x_1(t_0) + \int_{t_0-\varepsilon}^{t_0} [Z_1(s+\varepsilon, t)B_{11}(\varepsilon) + \varepsilon Z_2(s+\varepsilon, t)B_{21}(\varepsilon)] \varphi_1(s) ds + \int_{t_0-\varepsilon}^{t_0} \{Z_1(s+\varepsilon, t)B_{12}(\varepsilon) + Z_2(s+\varepsilon, t)[I + \varepsilon B_{22}(\varepsilon)]\} \varphi_2(s) ds$$

where φ_1 is the initial function for x_1 and φ_2 the initial function for x_2 ; the matrices Z_1 and Z_2 are the solutions of the problem

$$\dot{Z}_1(s) + Z_1(s)A_{11}(\varepsilon) + Z_1(s+\varepsilon)B_{11}(\varepsilon) + \varepsilon Z_2(s)A_{21}(\varepsilon) + \varepsilon Z_2(s+\varepsilon)B_{21}(\varepsilon) = 0$$

$$Z_2(s) = Z_1(s)A_{12}(\varepsilon) + Z_1(s+\varepsilon)B_{12}(\varepsilon) + Z_2(s+\varepsilon)(I + \varepsilon B_{22}(\varepsilon))$$

$$Z_1(s, t) = 0 \text{ for } s > t, \quad Z_1(t, t) = I, \quad Z_2(s, t) = 0, \quad s \geq t$$

Denote $Z_1(s, t) = \tilde{Z}_1(t-s)$, $Z_2(s, t) = \tilde{Z}_2(t-s)$ (since the system is autonomous) to get

$$\tilde{Z}_1'(\tau) = \tilde{Z}_1(\tau)A_{11}(\varepsilon) + \tilde{Z}_1(\tau-\varepsilon)B_{11}(\varepsilon) + \varepsilon \tilde{Z}_2(\tau)A_{21}(\varepsilon) + \varepsilon \tilde{Z}_2(\tau-\varepsilon)B_{21}(\varepsilon)$$

$$\tilde{Z}_2(\tau) = \tilde{Z}_1(\tau)A_{12}(\varepsilon) + \tilde{Z}_1(\tau-\varepsilon)B_{12}(\varepsilon) + \tilde{Z}_2(\tau-\varepsilon)[I + \varepsilon B_{22}(\varepsilon)]$$

$$\tilde{Z}_1(\tau) = 0 \text{ for } \tau < 0, \quad \tilde{Z}_1(0) = I, \quad \tilde{Z}_2(\tau) = 0 \quad \tau < 0.$$

The representation formula is now with $t_0 = 0$,

$$x_1(t) = \tilde{Z}_1(t)x_1(0) + \int_{-\varepsilon}^0 [\tilde{Z}_1(t-s-\varepsilon)B_{11}(\varepsilon) + \varepsilon \tilde{Z}_2(t-s-\varepsilon)B_{21}(\varepsilon)] \varphi_1(s) ds +$$

$$+ \int_{-\varepsilon}^0 [\tilde{Z}_1(t-s-\varepsilon)B_{12}(\varepsilon) + \tilde{Z}_2(t-s-\varepsilon)[I + \varepsilon B_{22}(\varepsilon)]] \varphi_2(s) ds$$

and our main problem is to study the behavior of \tilde{Z}_1 and \tilde{Z}_2 .

To have a system of the same form as the one for x_1, x_2

denote $Y_1(\bar{\varepsilon}) = \tilde{Z}_1^*(\bar{\varepsilon})$, $Y_2(\bar{\varepsilon}) = \varepsilon \tilde{Z}_2^*(\bar{\varepsilon})$ and we get
 for Y_1, Y_2 the problem

$$Y_1'(\bar{\varepsilon}) = A_{11}^*(\varepsilon) Y_1(\bar{\varepsilon}) + B_{11}^*(\varepsilon) Y_1(\bar{\varepsilon} - \varepsilon) + A_{21}^*(\varepsilon) Y_2(\bar{\varepsilon}) + B_{21}^*(\varepsilon) Y_2(\bar{\varepsilon} - \varepsilon)$$

$$Y_2'(\bar{\varepsilon}) = \varepsilon A_{12}^*(\varepsilon) Y_1(\bar{\varepsilon}) + \varepsilon B_{12}^*(\varepsilon) Y_1(\bar{\varepsilon} - \varepsilon) + (I + \varepsilon B_{22}^*(\varepsilon)) Y_2(\bar{\varepsilon} - \varepsilon)$$

$$Y_1(\bar{\varepsilon}) = 0 \quad \text{for } \bar{\varepsilon} < 0, \quad Y_1(0) = I, \quad Y_2(\bar{\varepsilon}) = 0 \quad \bar{\varepsilon} < 0.$$

To this problem we shall associate the reduced one

$$Y_1'(\bar{\varepsilon}) = [A_{11}^*(\varepsilon) + B_{11}^*(\varepsilon)] Y_1(\bar{\varepsilon}) + [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] Y_2(\bar{\varepsilon}) \quad (*)$$

$$Y_2'(\bar{\varepsilon}) = [A_{12}^*(\varepsilon) + B_{12}^*(\varepsilon)] Y_1(\bar{\varepsilon}) + B_{22}^*(\varepsilon) Y_2(\bar{\varepsilon})$$

$$Y_1(0) = I, \quad Y_2(0) = 0$$

Our main assumption is that the matrix of this reduced system

$$\begin{pmatrix} A_{11}^*(\varepsilon) + B_{11}^*(\varepsilon) & A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon) \\ A_{12}^*(\varepsilon) + B_{12}^*(\varepsilon) & B_{22}^*(\varepsilon) \end{pmatrix}$$

is Hurwitz, uniformly with respect to ε . A second essential assumption will be that $B_{22}(0)$ is Hurwitz.

Under these assumption we shall obtain exponential estimates for Y_1 and Y_2 and from here we shall deduce a stability result for the system we started with as well as the possibility of writing asymptotic expansions.

Remark that the main assumption is invariant with respect to transposition and it may be stated for the matrices in the given system.

3. A preliminary remark

Consider a system of the form

$$Y_1'(\zeta) = [A_{11}^*(\epsilon) + B_{11}^*(\epsilon)] Y_1(\zeta) + [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] Y_2(\zeta) + f(\zeta)$$

$$Y_2'(\zeta) = [A_{12}^*(\epsilon) + B_{12}^*(\epsilon)] Y_1(\zeta) + B_{22}^*(\epsilon) Y_2(\zeta)$$

$$Y_1(0) = I, \quad Y_2(0) = 0$$

Let Γ_{11} , Γ_{12} , Γ_{21} , Γ_{22} be the blocks of the fundamental matrix. We may write the variation of constants formula

$$\begin{pmatrix} Y_1(\zeta) \\ Y_2(\zeta) \end{pmatrix} = \begin{pmatrix} \Gamma_{11}(\zeta) & \Gamma_{12}(\zeta) \\ \Gamma_{21}(\zeta) & \Gamma_{22}(\zeta) \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} +$$

$$+ \int_0^\zeta \begin{pmatrix} \Gamma_{11}(\zeta-\sigma) & \Gamma_{12}(\zeta-\sigma) \\ \Gamma_{21}(\zeta-\sigma) & \Gamma_{22}(\zeta-\sigma) \end{pmatrix} \begin{pmatrix} f(\sigma) \\ 0 \end{pmatrix} d\sigma$$

hence

$$Y_1(\zeta) = \Gamma_{11}(\zeta) + \int_0^\zeta \Gamma_{11}(\zeta-\sigma) f(\sigma) d\sigma$$

and we see that Y_1 may be represented by using only the first block of the fundamental matrix. The matrix function Y_1 is also solution of the integro-differential equation obtained by representing Y_2 from the second equation and replacing into the first one.

$$Y_1'(\zeta) = [A_{11}^*(\epsilon) + B_{11}^*(\epsilon)] Y_1(\zeta) + \int_0^\zeta [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(\zeta-\sigma)} [A_{12}^*(\epsilon) + B_{12}^*(\epsilon)] Y_1(\sigma) d\sigma + f(\zeta)$$

Now we are in position to describe the idea of our proof.

Rewrite the first equation of system (*) in the form

$$\begin{aligned}
Y_1'(\zeta) = & [A_{11}^*(\epsilon) + B_{11}^*(\epsilon)] Y_1(\zeta) + \\
& + \int_0^{\zeta} [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(\zeta-\sigma)} [A_{12}^*(\epsilon) + B_{12}^*(\epsilon)] Y_1(\sigma) d\sigma + \\
& + B_{11}^*(\epsilon) [Y_1(\zeta-\epsilon) - Y_1(\zeta)] + A_{21}^*(\epsilon) Y_2(\zeta) + B_{21}^*(\epsilon) Y_2(\zeta-\epsilon) - \\
& - \int_0^{\zeta} [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(\zeta-\sigma)} [A_{12}^*(\epsilon) + B_{12}^*(\epsilon)] Y_1(\sigma) d\sigma
\end{aligned}$$

and use the above representation formula to get the integral equation

$$\begin{aligned}
Y_1(\zeta) = & \Gamma_{11}(\zeta) + \int_0^{\zeta} \Gamma_{11}(\zeta-\sigma) B_{11}^*(\epsilon) [Y_1(\sigma-\epsilon) - Y_1(\sigma)] d\sigma + \\
& + \int_0^{\zeta} \Gamma_{11}(\zeta-\sigma) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] Y_2(\sigma) d\sigma + \\
& + \int_0^{\zeta} \Gamma_{11}(\zeta-\sigma) B_{21}^*(\epsilon) [Y_2(\sigma-\epsilon) - Y_2(\sigma)] d\sigma - \\
& - \int_0^{\zeta} \left[\int_{\sigma}^{\zeta} \Gamma_{11}(\zeta-\lambda) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(\lambda-\sigma)} ds \right] [A_{12}^*(\epsilon) + \\
& + B_{12}^*(\epsilon)] Y_1(\sigma) d\sigma
\end{aligned}$$

where in the last term we have performed the change of the order of integration.

We use next the representation formula for Y_2

$$\begin{aligned}
Y_2(\sigma) = & \epsilon \sum_{j=1}^l [I + \epsilon B_{22}^*(\epsilon)]^{j-1} [A_{12}^*(\epsilon) Y_1(\sigma - (j-1)\epsilon) + \\
& + B_{12}^*(\epsilon) Y_1(\sigma - j\epsilon)]
\end{aligned}$$

for $(l-1)\epsilon \leq \sigma < l\epsilon$, replace it into the equation for Y_1 and use the integral equation in order to get good estimates for Y_1 .

4. The main estimates

A. Consider first

$$\begin{aligned}
 & \int_0^{\bar{t}} \Gamma_{11}(\bar{t}-\sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] Y_2(\sigma) d\sigma = \\
 & = \varepsilon \sum_{\ell=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} \Gamma_{11}(\bar{t}-\sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] \sum_{j=1}^{\ell} [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} \\
 & \cdot [A_{12}^*(\varepsilon) Y_1(\bar{t} - (j-1)\varepsilon) + B_{12}^*(\varepsilon) Y_1(\bar{t} - j\varepsilon)] d\sigma + \\
 & + \varepsilon \int_{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor \varepsilon}^{\bar{t}} \Gamma_{11}(\bar{t}-\sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor + 1} [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} \\
 & \cdot [A_{12}^*(\varepsilon) Y_1(\bar{t} - (j-1)\varepsilon) + B_{12}^*(\varepsilon) Y_1(\bar{t} - j\varepsilon)] d\sigma
 \end{aligned}$$

Change the order of summation and write further

$$\begin{aligned}
 & \int_0^{\bar{t}} \Gamma_{11}(\bar{t}-\sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] Y_2(\sigma) d\sigma = \\
 & = \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor + 1} \int_{(j-1)\varepsilon}^{\bar{t}} \Gamma_{11}(\bar{t}-\sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} \\
 & \cdot [A_{12}^*(\varepsilon) Y_1(\bar{t} - (j-1)\varepsilon) + B_{12}^*(\varepsilon) Y_1(\bar{t} - j\varepsilon)] d\sigma = \\
 & = \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor + 1} \int_0^{\bar{t} - (j-1)\varepsilon} \Gamma_{11}(\bar{t} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)]^{j-1} [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} \\
 & \cdot [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds = \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor + 1} \left(\sum_{k=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor - j + 1} \int_{(k-1)\varepsilon}^{k\varepsilon} \Gamma_{11}(\bar{t} - s - (j-1)\varepsilon) \cdot \right. \\
 & \cdot [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \\
 & + \int_{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor - j + 1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor} \Gamma_{11}(\bar{t} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(s) + \\
 & + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds \Big) = \varepsilon \sum_{k=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor} \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor - (k-1)} \int_{(k-1)\varepsilon}^{k\varepsilon} \Gamma_{11}(\bar{t} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + \\
 & + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \\
 & + \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor + 1} \int_{\lfloor \frac{\bar{t}}{\varepsilon} \rfloor - j + 1}^{\bar{t} - (j-1)\varepsilon} \Gamma_{11}(\bar{t} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(s) + \\
 & + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds
 \end{aligned}$$

Consider the first sum and write

$$\begin{aligned} & \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \sum_{j=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor - (k-1)} \int_0^{\tau - s - (j-1)\varepsilon} \Gamma_{11}(\tau - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(\lambda) + \\ & + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds = \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\tau - s - \theta} \Gamma_{11}(\tau - s - \theta) [A_{21}^*(\varepsilon) + \\ & + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds + \\ & + \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \left\{ \varepsilon \sum_{j=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor - (k-1)} \int_0^{\tau - s - (j-1)\varepsilon} \Gamma_{11}(\tau - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} \right. \\ & \left. - \int_0^{\tau - s - \theta} \Gamma_{11}(\tau - s - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(s) + \right. \\ & \left. + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds \right\} \end{aligned}$$

Concentrate again on the first sum and write

$$\begin{aligned} & \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \left[\int_0^{\tau - s - \theta} \Gamma_{11}(\tau - s - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \right. \\ & \left. [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds = \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\tau - \lambda} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - s - \theta) [A_{21}^*(\varepsilon) + \right. \\ & \left. + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds + \right. \\ & \left. + \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \left[\int_0^{\tau - \lambda} \Gamma_{11}(\tau - s - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(s) + \right. \right. \\ & \left. \left. + B_{12}^*(\varepsilon) Y_1(s-\varepsilon)] ds = \int_0^{\tau} \int_0^{\tau - \lambda} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - \lambda - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \right. \right. \\ & \left. \left. [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] d\lambda + \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - \lambda - \theta) [A_{21}^*(\varepsilon) + \right. \right. \\ & \left. \left. + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] d\lambda = \right. \\ & = \int_0^{\tau} \int_0^{\tau - \lambda} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - \lambda - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(\lambda) + \\ & \left. + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] d\lambda + \int_0^{\tau} \int_0^{\tau - \lambda} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - \lambda - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \right. \\ & \left. [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] d\lambda + \sum_{k=1}^{\lfloor \frac{\tau}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\tau - \lambda} \Gamma_{11}(\tau - \lambda - \theta) [A_{21}^*(\varepsilon) + \right. \end{aligned}$$

$$\begin{aligned}
& + B_{21}^*(\varepsilon) e^{B_{22}^*(\varepsilon)\theta} d\theta] [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] ds = \\
& = \int_0^{\bar{\tau}} \left[\int_0^{\bar{\tau}} \Gamma_{11}(\bar{\tau} - \sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)(\sigma - \lambda)} d\sigma \right] [A_{12}^*(\varepsilon) Y_1(\lambda) + \\
& + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] ds + \int_0^{\bar{\tau}} \int_0^{\bar{\tau}} \Gamma_{11}(\bar{\tau} - \sigma) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)(\sigma - \lambda)} d\sigma \\
& \cdot [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] ds + \sum_{k=1}^{\lfloor \frac{\bar{\tau}}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\bar{\tau} - (k-1)\varepsilon} \Gamma_{11}(\bar{\tau} - \sigma) [A_{21}^*(\varepsilon) + \\
& + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)(\sigma - \lambda)} d\sigma] [A_{12}^*(\varepsilon) Y_1(\lambda) + B_{12}^*(\varepsilon) Y_1(\lambda - \varepsilon)] ds
\end{aligned}$$

We may thus write

$$\begin{aligned}
Y_1(\bar{\tau}) &= \Gamma_{11}(\bar{\tau}) + \int_0^{\bar{\tau}} \Gamma_{11}(\bar{\tau} - \sigma) B_{11}^*(\varepsilon) [Y_1(\sigma - \varepsilon) - Y_1(\sigma)] d\sigma + \\
& + \int_0^{\bar{\tau}} \Gamma_{11}(\bar{\tau} - \sigma) B_{21}^*(\varepsilon) [Y_2(\sigma - \varepsilon) - Y_2(\sigma)] d\sigma + \\
& + \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{\tau}}{\varepsilon} \rfloor + 1} \int_{(j-1)\varepsilon}^{\bar{\tau} - (j-1)\varepsilon} \Gamma_{11}(\bar{\tau} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} [A_{12}^*(\varepsilon) Y_1(\lambda) + \\
& + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \sum_{k=1}^{\lfloor \frac{\bar{\tau}}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \left\{ \varepsilon \sum_{j=1}^{\lfloor \frac{\bar{\tau}}{\varepsilon} \rfloor - (k-1)} \Gamma_{11}(\bar{\tau} - s - (j-1)\varepsilon) [A_{21}^*(\varepsilon) + \right. \\
& + B_{21}^*(\varepsilon)] [I + \varepsilon B_{22}^*(\varepsilon)]^{j-1} - \int_0^{\bar{\tau} - s - \theta} \Gamma_{11}(\bar{\tau} - s - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \left. \right\} \\
& \cdot [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \int_0^{\bar{\tau}} \left[\int_0^{\bar{\tau} - \lambda - \theta} \Gamma_{11}(\bar{\tau} - \lambda - \theta) [A_{21}^*(\varepsilon) + \right. \\
& + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \left. \right] [A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \\
& + \sum_{k=1}^{\lfloor \frac{\bar{\tau}}{\varepsilon} \rfloor} \int_{(k-1)\varepsilon}^{k\varepsilon} \int_0^{\bar{\tau} - (k-1)\varepsilon} \Gamma_{11}(\bar{\tau} - s - \theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta [A_{12}^*(\varepsilon) Y_1(s) + \\
& + B_{12}^*(\varepsilon) Y_1(s - \varepsilon)] ds + \int_0^{\bar{\tau}} \left[\int_0^{\bar{\tau}} \Gamma_{11}(\bar{\tau} - s) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)(\sigma - \lambda)} d\sigma \right. \\
& \cdot B_{12}^*(\varepsilon) [Y_1(\sigma - \varepsilon) - Y_1(\sigma)] d\sigma
\end{aligned}$$

B. We may write

$$\begin{aligned}
 & \int_{(k-1)\epsilon}^{\lfloor \frac{\tau}{\epsilon} \rfloor \epsilon} \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)\theta} d\theta = \\
 & = \epsilon \sum_{j=1}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)} \int_{(j-1)\epsilon}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)\epsilon} \Gamma_{11}(\tau-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(j-1)\epsilon} + \\
 & + \sum_{j=1}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)} \int_{(j-1)\epsilon}^{j\epsilon} \left\{ \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)\theta} - \Gamma_{11}(\tau-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + \right. \\
 & \left. + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)(j-1)\epsilon} \right\} d\theta
 \end{aligned}$$

Denote $\phi(\theta) = \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)\theta}$; we shall have

$$\begin{aligned}
 & \epsilon \sum_{j=1}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)} \int_{(j-1)\epsilon}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)\epsilon} \Gamma_{11}(\tau-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] [I + \epsilon B_{22}^*(\epsilon)]^{j-1} - \\
 & - \int_{(k-1)\epsilon}^{\lfloor \frac{\tau}{\epsilon} \rfloor \epsilon} \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] e^{B_{22}^*(\epsilon)\theta} d\theta = \\
 & = \epsilon \sum_{j=1}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)} \int_{(j-1)\epsilon}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)\epsilon} \Gamma_{11}(\tau-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] [I + \epsilon B_{22}^*(\epsilon)]^{j-1} - \\
 & - e^{B_{22}^*(\epsilon)(j-1)\epsilon} + \sum_{j=1}^{\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)} \int_{(j-1)\epsilon}^{j\epsilon} [\phi((j-1)\epsilon) - \phi(\theta)] d\theta
 \end{aligned}$$

We have $\phi((j-1)\epsilon) - \phi(\theta) = \int_{\theta}^{(j-1)\epsilon} [-(j-1)\epsilon - \theta] \phi'(\theta + \nu((j-1)\epsilon - \theta)) d\nu$

Since the matrix defining Γ_{11}^0 and $B_{22}^*(\epsilon)$ are uniform.

Hurwitz we deduce that $\phi'(\theta + \nu((j-1)\epsilon - \theta))$ is estimated by

$$c e^{-\alpha(\tau-s - (1-\nu)\theta - \nu(j-1)\epsilon)} = c e^{-\alpha((1-\nu)\theta + \nu(j-1)\epsilon)} = c e^{-\alpha(\tau-s)}$$

hence the whole sum is estimated by

$$c \epsilon^2 (\lfloor \frac{\tau}{\epsilon} \rfloor - (k-1)) e^{-\alpha(\tau-s)} \leq c \epsilon (\tau-s) e^{-\alpha(\tau-s)}$$

(since we perform the estimate in the integral from $(k-1)\epsilon$ to $k\epsilon$) and if we replace α by a smaller constant we get an estimate of the form $c \epsilon e^{-\alpha(\tau-s)}$. Remark further that $e^{B_{22}^*(\epsilon)(j-1)\epsilon}$

is the value at $(j-1)\epsilon$ of the solution of the Cauchy problem

$$X' = B_{22}^*(\epsilon) X, \quad X(0) = I$$

while $[I + \epsilon B_{22}^*(\epsilon)]^{j-1}$ is the solution of the problem

$$X_{t+\epsilon} = [I + \epsilon B_{22}^*(\epsilon)] X_t, \quad X_1 = I \quad \text{which is the}$$

Euler approximation for the former.

Proceeding in the usual way to estimate the approximation error, denote by

$$R_l = e^{B_{22}^*(\epsilon) l \epsilon} - [I + \epsilon B_{22}^*(\epsilon)]^l \quad ; \text{ we then have}$$

$$R_{l+1} - [I + \epsilon B_{22}^*(\epsilon)] R_l = e^{B_{22}^*(\epsilon)(l+1)\epsilon} - [I + \epsilon B_{22}^*(\epsilon)]^{l+1} - [I + \epsilon B_{22}^*(\epsilon)] e^{B_{22}^*(\epsilon) l \epsilon} + [I + \epsilon B_{22}^*(\epsilon)]^{l+1} = e^{B_{22}^*(\epsilon) l \epsilon} [e^{B_{22}^*(\epsilon) \epsilon} - [I + \epsilon B_{22}^*(\epsilon)] \epsilon]$$

where $|\phi_l| \leq c e^{-\alpha l \epsilon} \epsilon^2$. We have

$$R_l = \sum_{m=1}^l (I + \epsilon B_{22}^*(\epsilon))^{m-1} \phi_{l-m}$$

Since the spectral radius of $I + \epsilon B_{22}^*(\epsilon)$ is less than $1 - \alpha \epsilon$,

we have the estimate

$$|R_l| \leq c \epsilon^2 \sum_{m=1}^l e^{-\alpha(l-m)\epsilon} (1 - \alpha \epsilon)^{m-1} =$$

$$= c \epsilon^2 e^{-\alpha l \epsilon} e^{\alpha \epsilon} \sum_{m=1}^l [e^{\alpha \epsilon} (1 - \alpha \epsilon)]^{m-1}$$

Since $e^{\alpha \epsilon} (1 - \alpha \epsilon) < 1$ we have in any case $|R_l| \leq c \epsilon^2 l e^{-\alpha l \epsilon}$

and by replacing α with a smaller one we may write

$$|R_l| \leq c \epsilon e^{-\alpha l \epsilon}, \text{ hence}$$

$$|e^{B_{22}^*(\epsilon)(j-1)\epsilon} - [I + \epsilon B_{22}^*(\epsilon)]^{j-1}| \leq c \epsilon e^{-\alpha(j-1)\epsilon}$$

We deduce that

$$\left| \epsilon \sum_{j=1}^{[\frac{z}{\epsilon}] - (k-1)} \prod_{11} (z-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] \left([I + \epsilon B_{22}^*(\epsilon)]^{j-1} - e^{B_{22}^*(\epsilon)(j-1)\epsilon} \right) \right| \leq \epsilon^2 c \sum_{j=1}^{[\frac{z}{\epsilon}] - (k-1)} e^{-\alpha(z-\lambda)} e^{\alpha(j-1)\epsilon} e^{-\alpha(j-1)\epsilon} = c \epsilon \left(\epsilon \left[\frac{z}{\epsilon} \right] - (k-1) \right) e^{-\alpha(z-\lambda)} \leq c \epsilon (z-s) e^{-\alpha(z-\lambda)}$$

and replacing again α by a smaller one and c by a larger one we have again an estimate of the form $c \epsilon e^{-\alpha(z-\lambda)}$.

We deduce in this way that

$$\left| \sum_{k=1}^{[\frac{z}{\epsilon}]} \int_{(k-1)\epsilon}^{k\epsilon} \left\{ \epsilon \sum_{j=1}^{[\frac{z}{\epsilon}] - (k-1)} \prod_{11} (z-s-(j-1)\epsilon) [A_{21}^*(\epsilon) + B_{21}^*(\epsilon)] [I + \epsilon B_{22}^*(\epsilon)]^{j-1} \right\} \right|$$

$$\begin{aligned} & \left(\left[\frac{\xi}{\varepsilon} \right] - (k-1) \right) \varepsilon \\ & - \int_0^{\tau} \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \left\{ [A_{12}^*(\varepsilon)Y_1(s) + \right. \\ & \left. + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)] ds \right\} \leq c \varepsilon \int_0^{\tau} e^{-\alpha(\tau-s)} |A_{12}^*(\varepsilon)Y_1(s) + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \end{aligned}$$

This is the form of estimate we are looking for in our formula for $Y_1(\tau)$.

We have also

$$\begin{aligned} & \left| \varepsilon \sum_{j=1}^{\left[\frac{\tau}{\varepsilon} \right] + 1} \int_{\left(\left[\frac{\xi}{\varepsilon} \right] - (j-1) \right) \varepsilon}^{\tau - (j-1) \varepsilon} \Gamma_{11}(\tau-s-(j-1)\varepsilon) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] \left[I + \varepsilon B_{22}^*(\varepsilon) \right]^{j-1} \cdot \\ & \cdot [A_{12}^*(\varepsilon)Y_1(s) + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)] ds \right| \leq c \varepsilon \sum_{j=1}^{\left[\frac{\tau}{\varepsilon} \right] + 1} \int_0^{\tau} e^{-\alpha(\tau-s)} \left[e^{\alpha\varepsilon} (1 - \right. \\ & \left. - \alpha\varepsilon) \right]^{j-1} |A_{12}^*(\varepsilon)Y_1(s) + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \leq c \varepsilon \int_0^{\tau} e^{-\alpha(\tau-s)} |A_{12}^*(\varepsilon)Y_1(s) + \\ & + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \\ & \left| \sum_{k=1}^{\left[\frac{\xi}{\varepsilon} \right]} \int_{(k-1)\varepsilon}^{k\varepsilon} \left[\int_{\tau-\lambda}^{\left(\left[\frac{\xi}{\varepsilon} \right] - (k-1) \right) \varepsilon} \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \right] [A_{12}^*(\varepsilon)Y_1(s) + \\ & + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)] ds \right| \leq c \sum_{k=1}^{\left[\frac{\xi}{\varepsilon} \right]} \int_{(k-1)\varepsilon}^{k\varepsilon} \left[\int_{\tau-\lambda}^{\left(\left[\frac{\xi}{\varepsilon} \right] - (k-1) \right) \varepsilon} e^{-\alpha(\tau-\lambda)} d\theta \right] \cdot \\ & |A_{12}^*(\varepsilon)Y_1(s) + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \leq c \varepsilon \int_0^{\tau} e^{-\alpha(\tau-s)} |A_{12}^*(\varepsilon)Y_1(s) + \\ & + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \end{aligned}$$

Finally

$$\begin{aligned} & \left| \int_0^{\tau} \left[\int_0^{\tau-\lambda} \Gamma_{11}(\tau-s-\theta) [A_{21}^*(\varepsilon) + B_{21}^*(\varepsilon)] e^{B_{22}^*(\varepsilon)\theta} d\theta \right] [A_{12}^*(\varepsilon)Y_1(s) + \right. \\ & \left. + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)] ds \right| \leq c \int_0^{\tau} e^{-\alpha(\tau-s)} |A_{12}^*(\varepsilon)Y_1(s) + B_{12}^*(\varepsilon)Y_1(s-\varepsilon)| ds \end{aligned}$$

C. Write now

$$\int_0^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [Y_2(\sigma-\varepsilon) - Y_2(\sigma)] d\sigma =$$

$$= \sum_{\ell=1}^{[\frac{\tau}{\varepsilon}]} \int_{(\ell-1)\varepsilon}^{\ell\varepsilon} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [Y_2(\sigma-\varepsilon) - Y_2(\sigma)] d\sigma +$$

$$+ \int_{[\frac{\tau}{\varepsilon}]\varepsilon}^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [Y_2(\sigma-\varepsilon) - Y_2(\sigma)] d\sigma$$

For $l \geq 2$ we have

$$Y_2(\tau-\varepsilon) - Y_2(\tau) = -\varepsilon [A_{12}^*(\varepsilon) Y_1(\tau) + B_{12}^*(\varepsilon) Y_1(\tau-\varepsilon)] -$$

$$-\varepsilon^2 \sum_{i=2}^l B_{22}^*(\varepsilon) (I + \varepsilon B_{22}^*(\varepsilon))^{i-2} [A_{12}^*(\varepsilon) Y_1(\tau - (i-1)\varepsilon) +$$

$$+ B_{12}^*(\varepsilon) Y_1(\tau - i\varepsilon)]$$

A similar expression can be replaced into the integral from

$[\frac{\tau}{\varepsilon}]\varepsilon$ to τ with $l = [\frac{\tau}{\varepsilon}] + 1$. We have further

$$\sum_{\ell=2}^{[\frac{\tau}{\varepsilon}]} \sum_{i=2}^{\ell} \int_{(i-1)\varepsilon}^{\ell\varepsilon} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) B_{22}^*(\varepsilon) (I + \varepsilon B_{22}^*(\varepsilon))^{i-2} [A_{12}^*(\varepsilon) Y_1(\tau - (i-1)\varepsilon) +$$

$$+ B_{12}^*(\varepsilon) Y_1(\tau - i\varepsilon)] d\sigma = \sum_{i=2}^{[\frac{\tau}{\varepsilon}]} \sum_{\ell=i}^{[\frac{\tau}{\varepsilon}]} \int_{(i-1)\varepsilon}^{\ell\varepsilon} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) B_{22}^*(\varepsilon) [I +$$

$$+ \varepsilon B_{22}^*(\varepsilon)]^{i-2} [A_{12}^*(\varepsilon) Y_1(\tau - (i-1)\varepsilon) + B_{12}^*(\varepsilon) Y_1(\tau - i\varepsilon)] d\sigma =$$

$$= \sum_{i=2}^{[\frac{\tau}{\varepsilon}]} \int_{(i-1)\varepsilon}^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) B_{22}^*(\varepsilon) (I + \varepsilon B_{22}^*(\varepsilon))^{i-2} [A_{12}^*(\varepsilon) Y_1(\tau - (i-1)\varepsilon) +$$

$$+ B_{12}^*(\varepsilon) Y_1(\tau - i\varepsilon)] d\sigma$$

We have thus

$$\int_0^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [Y_2(\sigma-\varepsilon) - Y_2(\sigma)] d\sigma =$$

$$= -\varepsilon \int_0^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [A_{12}^*(\varepsilon) Y_1(\sigma) + B_{12}^*(\varepsilon) Y_1(\sigma-\varepsilon)] d\sigma -$$

$$- \varepsilon^2 \sum_{i=2}^{[\frac{\tau}{\varepsilon}]+1} \int_{(i-1)\varepsilon}^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) B_{22}^*(\varepsilon) (I + \varepsilon B_{22}^*(\varepsilon))^{i-2} [A_{12}^*(\varepsilon) Y_1(\tau - (i-1)\varepsilon) +$$

$$+ B_{12}^*(\varepsilon) Y_1(\tau - i\varepsilon)] d\sigma =$$

$$= -\varepsilon \int_{[\frac{\tau}{\varepsilon}]\varepsilon}^{\tau} \Gamma_{11}(\tau-\sigma) B_{21}^*(\varepsilon) [A_{12}^*(\varepsilon) Y_1(\sigma) + B_{12}^*(\varepsilon) Y_1(\sigma-\varepsilon)] d\sigma -$$

$$-\varepsilon^2 \sum_{i=2}^{\lfloor \frac{b}{\varepsilon} \rfloor + 1} \int_0^{\tau - (i-1)\varepsilon} \Gamma_{11}(\tau - s - (i-1)\varepsilon) B_{21}^*(\varepsilon) B_{22}^*(\varepsilon) (I + \varepsilon B_{22}^*(\varepsilon))^{i-2},$$

$$\left[A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon) \right] ds$$

The first integral is estimated by

$$c\varepsilon \int_0^{\tau} e^{-\alpha(\tau - \sigma)} \left| A_{12}^*(\varepsilon) Y_1(\sigma) + B_{12}^*(\varepsilon) Y_1(\sigma - \varepsilon) \right| d\sigma \quad \text{and the sum by}$$

$$c\varepsilon^2 \sum_{i=2}^{\lfloor \frac{b}{\varepsilon} \rfloor + 1} \int_0^{\tau - (i-1)\varepsilon} e^{-\alpha'(\tau - (i-1)\varepsilon - \lambda)} (1 - \alpha\varepsilon)^{i-2} \left| A_{12}^*(\varepsilon) Y_1(s) + \right.$$

$$\left. + B_{12}^*(\varepsilon) Y_1(s - \varepsilon) \right| ds$$

where $\alpha' < \alpha$ and then by

$$c\varepsilon^2 \int_0^{\tau} e^{-\alpha'(z-\lambda)} \left| A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon) \right| ds \sum_{i=0}^{\infty} (e^{-\alpha'\varepsilon})^i \leq$$

$$\leq c\varepsilon \int_0^{\tau} e^{-\alpha'(z-\lambda)} \left| A_{12}^*(\varepsilon) Y_1(s) + B_{12}^*(\varepsilon) Y_1(s - \varepsilon) \right| ds$$

D. We have now to consider the terms of the form

$$\int_0^{\tau} \Gamma_{11}(\tau - \sigma) B(\varepsilon) [Y_1(\sigma - \varepsilon) - Y_1(\sigma)] d\sigma = \int_0^{\tau - \varepsilon} \Gamma_{11}(\tau - \lambda - \varepsilon) B(\varepsilon) Y_1(s) ds -$$

$$- \int_0^{\tau - \varepsilon} \Gamma_{11}(\tau - s) B(\varepsilon) Y_1(s) ds = \int_0^{\tau} \left[\Gamma_{11}(\tau - s - \varepsilon) - \Gamma_{11}(\tau - s) \right] B(\varepsilon) Y_1(s) ds -$$

$$- \int_{\tau - \varepsilon}^{\tau} \Gamma_{11}(\tau - s - \varepsilon) B(\varepsilon) Y_1(s) ds$$

Hence

$$\left| \int_0^{\tau} \Gamma_{11}(\tau - \sigma) B(\varepsilon) [Y_1(\sigma - \varepsilon) - Y_1(\sigma)] d\sigma \right| \leq c\varepsilon \int_0^{\tau} e^{-\alpha(z-\lambda)} |Y_1(s)| ds +$$

$$+ c \int_{\tau - \varepsilon}^{\tau} e^{-\alpha(z-\lambda)} |Y_1(s)| ds$$

We obtain in this way a inequality of the form

$$|Y_1(\tau)| \leq c e^{-\hat{\alpha}\tau} + c\varepsilon \int_0^{\tau} e^{-\hat{\alpha}(z-\lambda)} |Y_1(s)| ds + c \int_{\tau - \varepsilon}^{\tau} e^{-\hat{\alpha}(z-\lambda)} |Y_1(s)| ds +$$

$$+ \varepsilon c \int_0^{\tau} e^{-\hat{\alpha}(z-\lambda)} (|Y_1(s)| + |Y_1(s - \varepsilon)|) ds + c \int_{\tau - \varepsilon}^{\tau} e^{-\hat{\alpha}(z-\lambda)} (|Y_1(s)| +$$

$$+ |Y_1(s-\varepsilon)| ds$$

Denote $\mu(T, \varepsilon) = \sup_{0 \leq \tau \leq T} e^{\frac{\alpha}{2}\tau} |Y_1(\tau)|$

$$e^{\frac{\alpha}{2}\tau} |Y_1(\tau)| \leq c + c\varepsilon \mu(T, \varepsilon) \int_0^\tau e^{-\frac{\alpha}{2}(\tau-s)} ds + c \mu(T, \varepsilon) \varepsilon$$

It follows that

$$\mu(T, \varepsilon) \leq c + c\varepsilon \mu(T, \varepsilon) \quad \text{hence} \quad \mu(T, \varepsilon) \leq c, \quad |Y_1(\tau)| \leq ce^{-\frac{\alpha}{2}\tau}$$

and we have proved the exponential decay of Y_1 .

E. We have $[\frac{t}{\varepsilon}] + 1$

$$Y_2(\tau) = \varepsilon \sum_{j=1}^{[\frac{t}{\varepsilon}] + 1} (I + \varepsilon B_{22}^*(\varepsilon))^{j-1} [A_{12}^*(\varepsilon) Y_1(\tau - (j-1)\varepsilon) +$$

$$+ B_{12}^*(\varepsilon) Y_1(\tau - j\varepsilon)]$$

It follows that

$$|Y_2(\tau)| \leq ce^{-\frac{\alpha}{2}\tau} \varepsilon \sum_{j=1}^{\infty} (1 - \alpha\varepsilon)^{j-1} e^{\frac{\alpha}{2}(j-1)\varepsilon}$$

Since $\frac{\alpha}{2}$ may be considered smaller than α we deduce the

final estimate $|Y_2(\tau)| \leq ce^{-\frac{\alpha}{2}\tau}$.

5. The stability theorem

Remember that we had

$$x_1(t) = Y_1^*(t)x_1(0) + \int_{-\varepsilon}^0 [Y_1^*(t-s-\varepsilon)B_{11}(\varepsilon) + Y_2^*(t-s-\varepsilon)B_{21}(\varepsilon)] \varphi_1(s) ds +$$

$$+ \int_{-\varepsilon}^0 [Y_1^*(t-s-\varepsilon)B_{12}(\varepsilon) + \frac{1}{\varepsilon} Y_2^*(t-s-\varepsilon)[I + \varepsilon B_{22}(\varepsilon)]] \varphi_2(s) ds$$

It follows that

$$|x_1(t)| \leq ce^{-\tilde{\alpha}t} |x_1(0)| + c\varepsilon e^{-\tilde{\alpha}t} \sup |\varphi_1| + ce^{-\tilde{\alpha}t} \sup |\varphi_2|$$

We write now the representation formula for x_2

$$x_2(t) = [I + \varepsilon B_{22}(\varepsilon)]^{[\frac{t}{\varepsilon}] + 1} \varphi_2(t - [\frac{t}{\varepsilon}]\varepsilon - \varepsilon) +$$

$$+ \varepsilon \sum_{j=0}^{[\frac{t}{\varepsilon}]} [I + \varepsilon B_{22}(\varepsilon)]^j (A_{21}(\varepsilon)x_1(t-j\varepsilon) + B_{21}(\varepsilon)x_1(t-(j+1)\varepsilon))$$

and we get $|x_2(t)| \leq c(1-\alpha\varepsilon)^{\lfloor \frac{t}{\varepsilon} \rfloor + 1} \sup |\varphi_2| +$
 $+ ce^{-\tilde{\alpha}t} (|x_1(0)| + \varepsilon \sup |\varphi_1| + \sup |\varphi_2|)$

We state finally

Theorem 1. Assume the matrices

$$\begin{pmatrix} A_{11}(\varepsilon) + B_{11}(\varepsilon) & A_{12}(\varepsilon) + B_{12}(\varepsilon) \\ A_{21}(\varepsilon) + B_{21}(\varepsilon) & B_{22}(\varepsilon) \end{pmatrix}$$

and $B_{22}(\varepsilon)$ are Hurwitz, uniformly with respect to $\varepsilon \in [0, \varepsilon_0)$

Then for any solution of the system

$$\dot{x}_1(t) = A_{11}(\varepsilon)x_1(t) + A_{12}(\varepsilon)x_2(t) + B_{11}(\varepsilon)x_1(t-\varepsilon) + B_{12}(\varepsilon)x_2(t-\varepsilon)$$

$$x_2(t) = \varepsilon A_{21}(\varepsilon)x_1(t) + B_{21}(\varepsilon)x_1(t-\varepsilon) + [I + \varepsilon B_{22}(\varepsilon)]x_2(t-\varepsilon)$$

we have the exponential estimates

$$|x_1(t)| \leq c e^{-\tilde{\alpha}t} (|x_1(0)| + \varepsilon \sup |\varphi_1| + \sup |\varphi_2|)$$

$$|x_2(t)| \leq c e^{-\tilde{\alpha}t} (|x_1(0)| + \varepsilon \sup |\varphi_1| + \sup |\varphi_2|)$$

where φ_1 and φ_2 are the corresponding initial functions.

6 Asymptotic expansions

Let $x_1(t, \varepsilon)$, $x_2(t, \varepsilon)$ ^{be} a solution of the given system and define \tilde{x}_1 , \tilde{x}_2 from the problem

$$\dot{x}_1(t) = [A_{11}(\varepsilon) + B_{11}(\varepsilon)]x_1(t) + [A_{12}(\varepsilon) + B_{12}(\varepsilon)]x_2(t) \quad (**)$$

$$\dot{x}_2(t) = [A_{21}(\varepsilon) + B_{21}(\varepsilon)]x_1(t) + B_{22}(\varepsilon)x_2(t)$$

with initial conditions $x_1(0) = x_1^0$, $x_2(0) = \varphi_2(0)$

Define \tilde{x}_2 from the problem

$$x_2(t) = \varepsilon A_{21}(\varepsilon)\tilde{x}_1(t) + \varepsilon B_{21}(\varepsilon)\tilde{x}_1(t-\varepsilon) + [I + \varepsilon B_{22}(\varepsilon)]x_2(t-\varepsilon)$$

$$x_2(t) = \varphi_2(t) \quad \text{for} \quad -\varepsilon \leq t < 0$$

Define now \hat{x}_1 and \hat{x}_2 by writing

$$x_1(t, \varepsilon) = \tilde{x}_1(t, \varepsilon) + \varepsilon \hat{x}_1(t, \varepsilon)$$

$$x_2(t, \varepsilon) = \tilde{x}_2(t, \varepsilon) + \hat{x}_2(t, \varepsilon)$$

For \hat{x}_1 and \hat{x}_2 we get the problem

$$\begin{aligned} \hat{x}_1'(t) = & A_{11}(\varepsilon)\hat{x}_1(t) + B_{11}(\varepsilon)x_1(t-\varepsilon) + A_{12}(\varepsilon)\hat{x}_2(t) + \\ & + B_{12}(\varepsilon)\hat{x}_2(t-\varepsilon) + \frac{1}{\varepsilon} \left\{ B_{11}(\varepsilon) [\tilde{x}_1(t-\varepsilon, \varepsilon) - \tilde{x}_1(t, \varepsilon)] + A_{12}(\varepsilon) [\check{x}_2(t, \varepsilon) - \right. \\ & \left. - \tilde{x}_2(t, \varepsilon)] + B_{12}(\varepsilon) [\check{x}_2(t-\varepsilon, \varepsilon) - \tilde{x}_2(t, \varepsilon)] \right\} \end{aligned}$$

$$\hat{x}_2(t) = \varepsilon A_{21}(\varepsilon)\hat{x}_1(t) + \varepsilon B_{21}(\varepsilon)\hat{x}_1(t-\varepsilon) + [I + \varepsilon B_{22}(\varepsilon)]\hat{x}_2(t-\varepsilon)$$

$$\hat{x}_1(0) = 0, \quad \hat{x}_1(t) = \frac{1}{\varepsilon} [\varphi_1(t) - \tilde{x}_1(t, \varepsilon)] \quad \text{for } -\varepsilon \leq t < 0$$

$$\hat{x}_2(t) = 0 \quad \text{for } t < 0$$

We have the representation formula

$$\begin{aligned} \hat{x}_1(t, \varepsilon) = & \frac{1}{\varepsilon} \int_{-\varepsilon}^0 [Y_1^*(t-s-\varepsilon)B_{11}(\varepsilon) + Y_2^*(t-s-\varepsilon)B_{21}(\varepsilon)] [\varphi_1(s) - \\ & - \tilde{x}_1(s, \varepsilon)] ds + \frac{1}{\varepsilon} \int_0^t Y_1^*(t-s) \left\{ B_{11}(\varepsilon) [\tilde{x}_1(s-\varepsilon, \varepsilon) - \tilde{x}_1(s, \varepsilon)] + \right. \\ & \left. + A_{12}(\varepsilon) [\check{x}_2(s, \varepsilon) - \tilde{x}_2(s, \varepsilon)] + B_{12}(\varepsilon) [\check{x}_2(s-\varepsilon, \varepsilon) - \tilde{x}_2(s, \varepsilon)] \right\} ds \end{aligned}$$

We have now to estimate the differences that appear in

the second term. We remark directly that $|\tilde{x}_1(s-\varepsilon, \varepsilon) - \tilde{x}_1(s, \varepsilon)| \leq c\varepsilon e^{-\alpha s}$.

Denote $v(s, \varepsilon) = \check{x}_2(s, \varepsilon) - \tilde{x}_2(s, \varepsilon)$; we have

$$\begin{aligned} v(s, \varepsilon) - (I + \varepsilon B_{22}(\varepsilon))v(s-\varepsilon, \varepsilon) = & \check{x}_2(s, \varepsilon) - \tilde{x}_2(s, \varepsilon) - (I + \varepsilon B_{22}(\varepsilon))\check{x}_2(s-\varepsilon, \varepsilon) + \\ & + [I + \varepsilon B_{22}(\varepsilon)]\tilde{x}_2(s-\varepsilon, \varepsilon) = \varepsilon A_{21}(\varepsilon)\tilde{x}_1(s, \varepsilon) + \varepsilon B_{21}(\varepsilon)\tilde{x}_1(s, \varepsilon) + \\ & + \varepsilon B_{22}(\varepsilon)\tilde{x}_2(s-\varepsilon, \varepsilon) - [\tilde{x}_2(s, \varepsilon) - \tilde{x}_2(s-\varepsilon, \varepsilon)] = \varepsilon \tilde{x}_2'(s, \varepsilon) + \\ & + \varepsilon B_{22}(\varepsilon) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \tilde{x}_2(s, \varepsilon)] - \varepsilon \int_0^{-1} \tilde{x}_2'(s+\theta\varepsilon, \varepsilon) d\theta = \\ = & \varepsilon B_{22}(\varepsilon) [\tilde{x}_2(s-\varepsilon, \varepsilon) - \tilde{x}_2(s, \varepsilon)] + \varepsilon \int_{-1}^0 [\tilde{x}_2'(s, \varepsilon) - \tilde{x}_2'(s+\varepsilon, \varepsilon)] d\theta \end{aligned}$$

It follows that

$$v(s, \varepsilon) = [I + \varepsilon B_{22}(\varepsilon)] v(s-\varepsilon, \varepsilon) + \varepsilon^2 \psi(s, \varepsilon)$$

with $|\psi(s, \varepsilon)| \leq c\varepsilon e^{-\alpha s}$, $v(s) = \varphi_2(s) - \tilde{x}_2(s, \varepsilon)$ for $-\varepsilon \leq s < 0$.

$$v(s, \epsilon) = \left[I + \epsilon B_{22}(\epsilon) \right]^{\left[\frac{t}{\epsilon} \right] + 1} v\left(s - \left[\frac{t}{\epsilon} \right] \epsilon - \epsilon\right) + \epsilon^2 \sum_{j=0}^{\left[\frac{t}{\epsilon} \right]} \left[I + \epsilon B_{22}(\epsilon) \right]^j \psi(s - j\epsilon, \epsilon)$$

We deduce the estimate

$$|v(s, \epsilon)| \leq (1 - \alpha\epsilon)^{\left[\frac{t}{\epsilon} \right] + 1} \sup_{-\epsilon \leq s < 0} |\varphi_2(s) - \tilde{x}_2(s, \epsilon)| +$$

$$+ c\epsilon^2 e^{-s\alpha} \sum_{j=0}^{\infty} (1 - \alpha\epsilon)^j (e^{\alpha\epsilon})^j$$

and finally $|v(s, \epsilon)| \leq \epsilon c e^{-\alpha t}$ if φ_2 is Lipschitz.

By using the estimates for Y_1 and Y_2 we see now that \hat{x}_1 is bounded by $c e^{-\alpha t}$. We have further

$$\begin{aligned} \hat{x}_2(t, \epsilon) &= \epsilon \sum_{j=0}^{\left[\frac{t}{\epsilon} \right]} \left[I + \epsilon B_{22}(\epsilon) \right]^j \left[A_{21}(\epsilon) \hat{x}_1(t - j\epsilon) + \right. \\ &+ B_{21}(\epsilon) \hat{x}_1(t - (j+1)\epsilon) \left. \right] = \epsilon \left[I + \epsilon B_{22}(\epsilon) \right]^{\left[\frac{t}{\epsilon} \right]} \frac{1}{\epsilon} \left[\varphi_1(t - \left[\frac{t}{\epsilon} \right] \epsilon - \epsilon) - \right. \\ &- \tilde{x}_1(t - \left[\frac{t}{\epsilon} \right] \epsilon - \epsilon) \left. \right] + \epsilon \sum_{j=0}^{\left[\frac{t}{\epsilon} \right] - 1} \left[I + \epsilon B_{22}(\epsilon) \right]^j \left[A_{21}(\epsilon) \hat{x}_1(t - j\epsilon) + \right. \\ &+ B_{21}(\epsilon) \hat{x}_1(t - (j+1)\epsilon) \left. \right] \end{aligned}$$

It follows that

$$|\hat{x}_2(t, \epsilon)| \leq c (1 - \alpha\epsilon)^{\left[\frac{t}{\epsilon} \right]} + c e^{-\alpha t} \leq c e^{-\alpha t}$$

We have proved

Theorem 2. Let $x_1(t, \epsilon)$, $x_2(t, \epsilon)$ be a solution of the problem

$$\dot{x}_1(t) = A_{11}(\epsilon)x_1(t) + A_{12}(\epsilon)x_2(t) + B_{11}(\epsilon)x_1(t - \epsilon) + B_{12}(\epsilon)x_2(t - \epsilon)$$

$$x_2(t) = \epsilon A_{21}(\epsilon)x_1(t) + \epsilon B_{21}(\epsilon)x_1(t - \epsilon) + \left[I + \epsilon B_{22}(\epsilon) \right] x_2(t - \epsilon)$$

$$x_1(t) = \varphi_1(t) \quad x_2(t) = \varphi_2(t) \quad -\epsilon \leq t < 0$$

$$x_1(0) = x_1^0 \quad \varphi_2 \text{ Lipschitz ; let } \tilde{x}_1(t, \epsilon), \tilde{x}_2(t, \epsilon) \text{ be}$$

a solution of the problem (**). Then, under the same assumptions as in theorem 1,

$$|x_1(t, \epsilon) - \tilde{x}_1(t, \epsilon)| + |x_2(t, \epsilon) - \tilde{x}_2(t, \epsilon)| \leq c e^{-\alpha t}$$

ASYMPTOTIC EXPANSIONS FOR COUPLED NONLINEAR
DIFFERENCE-DIFFERENTIAL AND DIFFERENCE EQUATIONS
WITH SMALL DELAYS

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In this paper we extend previous results obtained for linear systems to a class of nonlinear problems.

1. Nonlinear difference equations with small
step-size

1. Consider a difference equation

$$x(t) = f(t, x(t-\varepsilon), \varepsilon) \quad t \geq t_0$$

$$x(t) = \varphi(t) \quad \text{for } t_0 - \varepsilon \leq t < t_0$$

Assume $f: I \times \mathbb{R}^n \times [0, \varepsilon_0] \rightarrow \mathbb{R}^n$ where $I \subset \mathbb{R}$ contains

$[t_0 - \varepsilon, +\infty)$; assume f be C^1 with partial derivatives

$D_2 f$ and $D_3 f$ locally Lipschitz; for any compact $K \subset \mathbb{R}^n$

there exists $L_K > 0$ such that

$$|(D_j f)(t, x', \varepsilon') - (D_j f)(t, x'', \varepsilon'')| \leq L(|x' - x''| + |\varepsilon' - \varepsilon''|)$$

$$j=2,3 \quad \text{for } x' \in K, x'' \in K, \varepsilon' \in [0, \varepsilon_0], \varepsilon'' \in [0, \varepsilon_0].$$

Assume also that the equation $x = f(t, x, 0)$ admits a solution $\underline{x}(t)$ defined for all $t \in I$ and such that

$$|\sigma((D_2 f)(t, \underline{x}(t), 0))| \leq \hat{\rho} < 1 \quad \text{for all } t \in I;$$

$\hat{\rho}$ being independent on t ; moreover it is assumed that

$t \rightarrow (D_j f)(t, \underline{x}(t), 0)$, $j=1,2,3$ are uniformly bounded

on I .

Consider the problem

$$x(\tau) = f(\tau, x(\tau-1), 0) \quad \tau \geq 0, \quad x(\tau) = \psi(\tau) \quad \text{for } -1 \leq \tau < 0.$$

A general theorem on stability by the first approximation shows that if $\sup_{-1 \leq \tau < 0} |\psi(\tau) - \underline{x}(\tau)|$ is small enough then $|x(\tau; \psi, t) - \underline{x}(\tau)| \leq k \rho^{\hat{\rho}} \sup_{-1 \leq \tau < 0} |\psi(\tau) - \underline{x}(\tau)|$ with ρ and k independent on τ , t and ψ , $\hat{\rho} < \rho < 1$.

We shall call "domain of attraction" for $\underline{x}(t)$ the collection of functions $\psi: [-1, 0) \rightarrow \mathbb{R}^n$ such that the solution $x(\cdot; \psi, t)$ satisfies the inequality above.

2. Let $A: I \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ uniformly continuous and uniformly bounded with $|\sigma(A(t))| < \rho < 1$ for all $t \in I$; then it is known that for the fundamental matrix $C(t, s, \varepsilon)$ associated to $x(t) = A(t)x(t-\varepsilon)$ we have the estimate

$$|C(t, s, \varepsilon)| \leq c \rho^{\frac{t-s}{\varepsilon}}, \quad t \geq s, \quad s \in I, \quad 0 < \varepsilon < \varepsilon_0$$

$$\rho_A < \rho < 1.$$

Proposition 1. Let A be as above and let

$$\hat{A}: I \times [0, \hat{\varepsilon}] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \quad \text{have for each } \varepsilon \in [0, \hat{\varepsilon}]$$

the same properties as A and

$$|\hat{A}(t, \varepsilon) - A(t)| \leq \hat{c} e^{-\hat{\alpha}(t-t_0)/\varepsilon} \quad \text{for all } t \geq t_0.$$

Then there exists $0 < \rho_1 < 1$ and $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the fundamental matrix $\hat{C}(t, s, \varepsilon)$ associated to $x(t) = \hat{A}(t, \varepsilon)x(t-\varepsilon)$ satisfies $|\hat{C}(t, s, \varepsilon)| \leq c_1 \rho_1^{\frac{t-s}{\varepsilon}}$ for $t \geq s \geq t_0$, c depending on \hat{c} , $\hat{\alpha}$, ρ_A .

Proof. We have

$$\hat{C}(t, s, \varepsilon) = \hat{A}(t, \varepsilon) \hat{C}(t-\varepsilon, s, \varepsilon) \quad t \geq s$$

$$\hat{C}(t, s, \varepsilon) = 0, \quad t < s, \quad C(s, s, \varepsilon) = E$$

(E the identity matrix).

For $t \geq s + \epsilon$ \hat{C} is the solution of the same difference equation with the initial function equal to E for $t = s$ and to 0 for $s < t < s + \epsilon$.

In fact we have $\hat{C}(t, s, \epsilon) = 0$ for $t \neq s + k\epsilon$
 $\hat{C}(s + k\epsilon) = \hat{A}(s + k\epsilon, \epsilon) \hat{A}(s + (k-1)\epsilon, \epsilon) \dots \hat{A}(s + \epsilon, \epsilon)$.

We may write

$$\hat{C}(t, s, \epsilon) = A(t) \hat{C}(t - \epsilon, s, \epsilon) + [\hat{A}(t, \epsilon) - A(t)] \hat{C}(t - \epsilon, s, \epsilon)$$

and by using the variation of constants formula we get the representation

$$\hat{C}(t, s, \epsilon) = C(t, t - [\frac{t-s}{\epsilon}] \epsilon, \epsilon) \hat{C}(t - [\frac{t-s}{\epsilon}] \epsilon, s, \epsilon) + \sum_{j=0}^{[\frac{t-s}{\epsilon}] - 1} C(t, t - j\epsilon, \epsilon) [\hat{A}(t - j\epsilon, \epsilon) - A(t - j\epsilon)] \hat{C}(t - (j+1)\epsilon, s, \epsilon)$$

This representation leads to the estimate

$$|\hat{C}(t, s, \epsilon)| \leq c_1 \rho_A^{\frac{t-s}{\epsilon}} + c_2 \sum_{j=0}^{[\frac{t-s}{\epsilon}] - 1} \rho_A^j \int_A \rho^{\frac{t-t_0-j}{\epsilon}} |\hat{C}(t - (j+1)\epsilon, s, \epsilon)|$$

We already know that we may take $\tau = s + k\epsilon$;

$$|\hat{C}(s + k\epsilon, s, \epsilon)| \leq c_1 \rho_A^k + c_2 \sum_{j=0}^{k-1} \rho_A^j \int_A \rho^{\frac{t-t_0-j}{\epsilon}} |\hat{C}(s + (k-j-1)\epsilon, s, \epsilon)|$$

Let $u_k(s, \epsilon) = \rho_A^{-k} |\hat{C}(s + k\epsilon, s, \epsilon)|$.

Our inequality is now

$$u_k(s, \epsilon) \leq c_1 + c_2 \sum_{j=0}^{k-1} \rho_A^{\frac{t-t_0-j}{\epsilon}} \int_A u_{k-j-1}(s, \epsilon)$$

$$u_k(s, \epsilon) \leq c_1 + c_2 \rho_A^{\frac{s-t_0}{\epsilon}} \sum_{j=0}^{k-1} \rho_A^{kj} \int_A u_{k-j-1}(s, \epsilon)$$

$$u_k(s, \epsilon) \leq c_1 + c_2 \rho_A^{\frac{s-t_0}{\epsilon}} \sum_{\ell=0}^{k-1} \rho_A^\ell u_\ell(s, \epsilon)$$

and since $t = s + k\epsilon$

$$u_k(s, \epsilon) \leq c_1 + c_2 \rho_A^{\frac{s-t_0}{\epsilon}} \sum_{\ell=0}^{k-1} \rho_A^\ell u_\ell(s, \epsilon)$$

Let $v_k = c_2 \rho_A^{\frac{s-t_0}{\epsilon}} \sum_{\ell=0}^{k-1} \rho_A^\ell u_\ell(s, \epsilon)$; then $u_k \leq c_1 + v_k$ and

$$v_1 = c_2 \rho_A^{\frac{s-t_0}{\epsilon}} u_0(s, \epsilon) = c_2 \rho_A^{\frac{s-t_0}{\epsilon}}$$

$$v_{k+1} = c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \sum_{l=0}^k \hat{\rho}^l u_l(s, \varepsilon) = v_k + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k u_k(s, \varepsilon) \leq$$

$$\leq v_k + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k (c_1 + v_k) = (1 + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k) v_k + c_1 c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k$$

We deduce that $v_k \leq w_k$ where $w_1 = v_1$ and

$$w_{k+1} = (1 + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k) w_k + c_1 c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^k$$

We have

$$w_k = \left(\prod_{j=1}^{k-1} a_j \right) w_1 + \sum_{j=2}^{k-1} \left(\prod_{l=j}^{k-1} a_l \right) f_j + f_k$$

where we have denoted

$$a_j = (1 + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^j), \quad f_j = c_1 c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^{j-1}$$

(For $k=3$ we have indeed

$$w_3 = a_2 a_1 w_1 + a_2 f_2 + f_3 = a_2 (a_1 w_1 + f_2) + f_3 = a_2 w_2 + f_3$$

and the formula is true ; we have further

$$w_{k+1} = \left(\prod_{j=1}^k a_j \right) w_1 + \sum_{j=2}^k \left(\prod_{l=j}^k a_l \right) f_j + f_{k+1}$$

$$a_k w_k = \left(\prod_{j=1}^k a_j \right) w_1 + \sum_{j=2}^{k-1} \left(\prod_{l=j}^k a_l \right) f_j + a_k f_k =$$

$$= \left(\prod_{j=1}^k a_j \right) w_1 + \sum_{j=2}^k \left(\prod_{l=j}^k a_l \right) f_j = w_{k+1} - f_{k+1}$$

and the formula is again true).

We have then

$$|w_k| \leq \prod_{j=1}^{k-1} (1 + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^j) |w_1| + c_1 c_2 \sum_{j=2}^{k-1} \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^{j-1} \prod_{l=j}^{k-1} (1 +$$

$$+ c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^{l-1}) + c_1 c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^{k-1}$$

The product

$$\prod_{j=1}^{\infty} (1 + c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^j)$$

is convergent since the series

$$\sum c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^j \quad \text{is such}$$

and we have

$$|w_k| \leq \prod_{\infty}^{w_1} (1 + c_1 c_2 \prod_{\infty} \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \sum_{j=2}^{k-1} \hat{\rho}^{j-1} + c_1 c_2 \hat{\rho}^{\frac{\Lambda-t_0}{\varepsilon}} \hat{\rho}^{k-1})$$

Since $\hat{\rho} < 1$ we finally deduce that w_k is uniformly

bounded, hence so is v_k and also u_k . We deduce that

$$|\hat{C}(s+k\varepsilon, s, \varepsilon)| \leq c \hat{\rho}^k$$

and thus is the type of estimate we were looking for.

3. In the proposition above we have assumed that $A(t)$ and $\hat{A}(t, \varepsilon)$ differ by a term of boundary-layer type; we need a similar result for the case the matrices are uniformly close.

Proposition 2. Let $A_j: I \times [0, \tilde{\varepsilon}] \rightarrow \mathcal{L}(R^n, R^n)$, $j=1,2$ be uniformly bounded and such that

$$|A_1(t, \varepsilon) - A_2(t, \varepsilon)| \leq \beta \varepsilon \quad \text{for all } t \in I, 0 \leq \varepsilon \leq \tilde{\varepsilon}.$$

Assume that for the fundamental matrix $C_1(t, s, \varepsilon)$ associated to $x(t) = A_1(t, \varepsilon)x(t - \varepsilon)$ the estimate

$$|C_1(t, s, \varepsilon)| \leq c_1 \rho_1^{\frac{t-s}{\varepsilon}} \quad t \geq s, 0 < \rho_1 < 1$$

is true. Then there exists $\hat{\varepsilon} < \tilde{\varepsilon}$ such that for $0 < \varepsilon < \hat{\varepsilon}$ we have an estimate

$$|C_2(t, s, \varepsilon)| \leq c_2 \rho_2^{\frac{t-s}{\varepsilon}} \quad t \geq s \quad \text{for some } \rho_2 \in (0, 1)$$

and c_2 , independent on t, s, ε ; here $C_2(t, s, \varepsilon)$ is the fundamental matrix associated to $A_2(t, \varepsilon)$.

Proof. We proceed as in the preceding proposition and write

$$C_2(t, s, \varepsilon) = C_1(t, t - [\frac{t-s}{\varepsilon}] \varepsilon, \varepsilon) C_2(t - [\frac{t-s}{\varepsilon}] \varepsilon, s, \varepsilon) + \sum_{j=0}^{[\frac{t-s}{\varepsilon}] - 1} C_1(t, t - j\varepsilon, \varepsilon) [A_2(t - j\varepsilon, \varepsilon) - A_1(t - j\varepsilon, \varepsilon)] C_2(t - (j+1)\varepsilon, s, \varepsilon)$$

we get the estimate

$$|C_2(t, s, \varepsilon)| \leq c_1 \rho_1^{\frac{t-s}{\varepsilon}} + c_2 \beta \varepsilon \sum_{j=0}^{[\frac{t-s}{\varepsilon}] - 1} \rho_1^j |C_2(t - (j+1)\varepsilon, s, \varepsilon)|$$

since we may take $t = s + k\varepsilon$,

$$|C_2(s + k\varepsilon, s, \varepsilon)| \leq c_1 \rho_1^k + c_2 \beta \varepsilon \sum_{j=0}^{k-1} \rho_1^j |C_2(s + (k-j-1)\varepsilon, s, \varepsilon)|$$

Define again $u_k(s, \varepsilon) = \rho_1^{-k} |C_2(s + k\varepsilon, s, \varepsilon)|$ to get the inequality

$$u_k(s, \varepsilon) \leq c_1 + \frac{c_2}{\rho_1} \beta \varepsilon \sum_{j=0}^{k-1} u_j(s, \varepsilon)$$

Denote

$$v_k = \frac{c_2}{\rho_1} \beta \varepsilon \sum_{j=0}^{k-1} u_j(s, \varepsilon), \quad v_1 = \frac{c_2}{\rho_1} \beta \varepsilon \cdot u_0(s, \varepsilon)$$

$$v_{k+1} = \frac{c_2}{\beta_1} \beta \varepsilon \sum_{j=0}^k u_j(s, \varepsilon) = v_k + \frac{c_2}{\beta_1} \beta \varepsilon u_k(s, \varepsilon) \leq \\ \leq v_k + \frac{c_2}{\beta_1} \beta \varepsilon (c_1 + v_k) = (1 + \frac{c_2}{\beta_1} \beta \varepsilon) v_k + \frac{c_1 c_2}{\beta_1} \beta \varepsilon ;$$

$$v_k \leq w_k \quad \text{where} \quad w_1 = v_1$$

$$w_{k+1} = (1 + \frac{c_2}{\beta_1} \beta \varepsilon) w_k + \frac{c_1 c_2}{\beta_1} \beta \varepsilon .$$

We deduce an estimate of the form

$$v_k \leq -c_1 + (v_1 + c_1) (1 + \frac{c_2}{\beta_1} \beta \varepsilon)^{k-1}$$

$$u_k(s, \varepsilon) \leq u_1(s, \varepsilon) + (v_1 + c_1) (1 + \frac{c_2}{\beta_1} \beta \varepsilon)^{k-1} \\ |c_2(s+k\varepsilon, s, \varepsilon)| \leq \frac{1}{\beta_1} |c_2(s+\varepsilon, s, \varepsilon)| \beta_1^k + \frac{(v_1 + c_1)}{1 + \frac{c_2}{\beta_1} \beta \varepsilon} \left[(1 + \frac{c_2}{\beta_1} \beta \varepsilon) \beta_1 \right]^k ;$$

we take

$$\beta_2 = (1 + \frac{c_2}{\beta_1} \beta \varepsilon) \beta_1$$

and for ε small enough we still have $0 < \beta_2 < 1$, and finally

$$|c_2(s+k\varepsilon, s, \varepsilon)| \leq c_2 \beta_2^k$$

the proposition being thus proved.

4. Assume $\tilde{x}(t)$ be bounded , take an initial function φ on $[t_0 - \varepsilon, t_0]$ and construct the solution $x(t; \varphi, \varepsilon)$ of the equation $x(t) = f(t, x(t-\varepsilon), \varepsilon)$ on $t \geq t_0$ with initial function φ . Define $z(t, \varepsilon) = \frac{1}{\varepsilon} [x(t, \varphi, \varepsilon) - \tilde{x}(\frac{t-t_0}{\varepsilon}; \varepsilon, t)]$ where \tilde{x} satisfies

$$\tilde{x}(\tau) = f(t, \tilde{x}(\tau-1), 0) \quad , \quad \tilde{x}(\tau) = \varphi(t_0 + \varepsilon \tau) \quad \text{for} \quad -1 \leq \tau < 0$$

hence

$$\tilde{x}(\frac{t-t_0}{\varepsilon}; \varepsilon, t) = f(t, \tilde{x}(\frac{t-t_0}{\varepsilon}; \varepsilon, t), 0)$$

$$\tilde{x}(\frac{t-t_0}{\varepsilon}, \varepsilon, t) = \varphi(t) \quad t_0 - \varepsilon \leq t < t_0 ;$$

we deduce that

$$z(t, \varepsilon) = \frac{1}{\varepsilon} [f(t, x(t-\varepsilon, \varphi, \varepsilon), \varepsilon) - f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), 0) = \\ = \frac{1}{\varepsilon} [f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon) + \varepsilon z(t-\varepsilon, \varepsilon), \varepsilon) - f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), 0)] = \\ = \frac{1}{\varepsilon} [f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon) + \varepsilon z(t-\varepsilon, \varepsilon), \varepsilon) -$$

$$\begin{aligned}
& -f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon), \varepsilon) \Big] + \frac{1}{\varepsilon} \left[f(t, \tilde{x}(\frac{t-t}{\varepsilon} - 1; \varepsilon, t-\varepsilon), \varepsilon) - f \right. \\
& \left. -f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), 0) \right] = \frac{1}{\varepsilon} \left[f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon), \varepsilon) - f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1, \varepsilon, t), 0) \right] \\
& + \int_0^1 (D_2 f)(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon) + \theta z(t-\varepsilon, \varepsilon), \varepsilon) d\theta z(t-\varepsilon, \varepsilon) = \\
& = \tilde{A}(t, \varepsilon) z(t-\varepsilon) + g(t, \varepsilon) + G(t, z(t-\varepsilon, \varepsilon), \varepsilon) \\
& \tilde{A}(t, \varepsilon) = (D_2 f)(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), \varepsilon) \\
& g(t, \varepsilon) = \frac{1}{\varepsilon} \left[f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon), \varepsilon) - f(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), 0) \right] \\
& G(t, z, \varepsilon) = \int_0^1 \left[(D_2 f)(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t-\varepsilon) + \theta z, \varepsilon) - \right. \\
& \quad \left. - (D_2 f)(t, \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t), \varepsilon) \right] d\theta z \quad ;
\end{aligned}$$

we have

$$\begin{aligned}
& | \tilde{A}(t, \varepsilon) - (D_2 f)(t, \underline{x}(t), 0) | \leq | \tilde{A}(t, \varepsilon) - (D_2 f)(t, \underline{x}(t), \varepsilon) | + \\
& + | (D_2 f)(t, \underline{x}(t), \varepsilon) - (D_2 f)(t, \underline{x}(t), 0) | \leq L \left| \tilde{x}(\frac{t-t_0}{\varepsilon} - 1; \varepsilon, t) - \right. \\
& \left. - \underline{x}(t) \right| + L\varepsilon \leq L(k\rho^{\frac{k-t_0}{\varepsilon}} \sup_{-1 \leq \theta \leq 0} | \varphi(t_0 + \theta\varepsilon) - \underline{x}(t) | + \varepsilon)
\end{aligned}$$

and Propositions 1 and 2 give a good estimate for the fundamental matrix associated to $\tilde{A}(t, \varepsilon)$. We have next

$$\begin{aligned}
& \tilde{x}(z; \varepsilon, t) - \tilde{x}(z; \varepsilon, t-\varepsilon) = f(t, \tilde{x}(z-1; \varepsilon, t), 0) - f(t-\varepsilon, \tilde{x}(z-1; \varepsilon, t-\varepsilon), 0) = \\
& = \int_0^1 \left\{ (D_1 f)(t-\theta\varepsilon, \tilde{x}(z-1; \varepsilon, t-\varepsilon) + \theta(\tilde{x}(z-1; \varepsilon, t) - \tilde{x}(z-1; \varepsilon, t-\varepsilon)), 0) \varepsilon + \right. \\
& + (D_2 f)(t-\theta\varepsilon, \tilde{x}(z-1; \varepsilon, t-\varepsilon) + \theta(\tilde{x}(z-1; \varepsilon, t) - \tilde{x}(z-1; \varepsilon, t-\varepsilon)), 0) \left[\tilde{x}(z-1; \varepsilon, t) - \right. \\
& \left. \left. - \tilde{x}(z-1; \varepsilon, t-\varepsilon) \right] \right\} d\theta
\end{aligned}$$

If we denote $y(z, \varepsilon) = \tilde{x}(z; \varepsilon, t) - \tilde{x}(z; \varepsilon, t-\varepsilon)$, we can write $y(z, \varepsilon) = (D_2 f)(t-\varepsilon, \tilde{x}(z-1, \varepsilon, t-\varepsilon), 0)y(z-1, \varepsilon) + h(z, \varepsilon) + H(z, y(z-1, \varepsilon), \varepsilon)$ with $h(z, \varepsilon) \in \mathcal{L}\varepsilon$, $|H(z, y(z-1, \varepsilon), \varepsilon)| \leq |y|L(|y| + \varepsilon)$ and zero initial conditions.

By using Proposition 1 we have a good estimate for the fundamental matrix and the estimates for h and H allow us to deduce that $|y(z, \varepsilon)| \leq H\varepsilon$ uniformly with respect to z and t .

We deduce from here that $|g|$ is uniformly bounded. We have then the estimate

$$|G(t, z, \epsilon)| \leq L|z| \left(\epsilon |z| + |y\left(\frac{t-t_0}{\epsilon} - 1, \epsilon\right)| \right) \leq L\epsilon |z| (|z| + M)$$

and this estimate, together with the one for the fundamental matrix shows that $|z|$ is uniformly bounded, hence

$$|x(t, \varphi, \epsilon) - \tilde{x}\left(\frac{t-t_0}{\epsilon}; \epsilon, t\right)| \leq M\epsilon \quad \text{and finally}$$

$$|x(t, \varphi, \epsilon) - \tilde{x}(t)| \leq M\left(\epsilon + \rho \frac{t-t_0}{\epsilon}\right)$$

This inequality has been thus obtained for all initial functions in the "domain of attraction" of $\tilde{x}(t)$ with a complementary boundedness assumption allowing to use the local Lipschitz property of the derivatives.

2. Coupled difference-differential and difference equations.

The asymptotic expansions

1. Consider the system

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1(t), x_2(t), x_1(t-\epsilon), x_2(t-\epsilon), \epsilon) \\ x_2(t) &= f_2(t, x_1(t), x_1(t-\epsilon), x_2(t-\epsilon), \epsilon) \\ x_j(t) &= \varphi_j(t) \quad , \quad t_0 - \epsilon \leq t < t_0 \quad , \quad x_1(t_0) = \varphi_1(t_0) \end{aligned}$$

with $f_1: I \times \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_1 \times \mathcal{D}_2 \times [0, \epsilon_0] \rightarrow \mathbb{R}^{n_1}$

$$f_2: I \times \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_2 \times [0, \epsilon_0] \rightarrow \mathbb{R}^{n_2} \quad \mathcal{D}_j \subset \mathbb{R}^{n_j} \quad j=1,2$$

$$I \subset \mathbb{R} \quad , \quad [t_0 - \epsilon, \infty) \subset I \quad .$$

We assume f_j are C^1 and $D_x f_j$ are locally Lipschitz.

Assume that the equation $x_2 = f_2(t, x_1, x_1, x_2, 0)$ has a solution $\tilde{x}_2(t, x_1)$ defined for all $t \in I$ and $x_1 \in \mathcal{D}_1$. Consider the diffe-

rential equation $\dot{x}_1 = \tilde{f}(t, x_1)$

$$\tilde{f}(t, x_1) = f_1(t, x_1, \tilde{x}_2(t, x_1), x_1, \tilde{x}_2(t, x_1), 0)$$

and assume the solution with $x_1(t_0) = \varphi_1(t_0)$ denoted $\tilde{x}_1(\cdot)$, is defined on $[t_0, \infty)$, and the corresponding curve lies in

\mathcal{D}_1 together with a \tilde{y}_1 -neighborhood.

Let $\underline{x}_2(t) = \underline{x}_2(t, \tilde{x}_1(t))$ and assume

a) $\underline{x}_2(t)$, $t \geq t_0$ are in \mathcal{D}_2 together with a \mathcal{J}_2 -neighborhood

b) $t \rightarrow f_1(t, \tilde{x}_1(t), \underline{x}_2(t), \tilde{x}_1(t), \underline{x}_2(t), 0)$ and

$t \rightarrow (D_k f_j)(t, \tilde{x}_1(t), \underline{x}_2(t), \tilde{x}_1(t), \underline{x}_2(t), 0)$

are uniformly bounded;

c) there exists $\hat{\rho} \in (0, 1)$ such that

$$|\sigma(D_4 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t), \underline{x}_2(t), 0)| \leq \hat{\rho} < 1 \text{ for all } t \in I.$$

Let $\tilde{x}_2(\tau, t)$ be the solution of the "boundary layer" problem

$$x_2(\tau) = f_2(t, \tilde{x}_1(t), \tilde{x}_1(t), x_2(\tau-1), 0) \quad \tau \geq 0$$

$$x_2(\tau) = \psi_2(\tau) \quad -1 \leq \tau < 0.$$

From assumption c) and using a theorem on stability by the first approximation we know that if $\sup_{-1 \leq \tau < 0} |\psi_2(\tau) - \underline{x}_2(t)|$ is small enough then $|\tilde{x}_2(\tau, t) - \underline{x}_2(t)| \leq c \hat{\rho}^\tau \sup_{-1 \leq \theta < 0} |\psi_2(\theta) - \underline{x}_2(t)|$ $\tau \geq 0$. We shall define the "domain of attraction" for \underline{x}_2 as in the preceding section; it consists of all initial functions for which the above estimate holds.

Consider the solution $x_1(t, \varepsilon)$, $x_2(t, \varepsilon)$ of the given problem

and define $\hat{x}_1(t, \varepsilon)$, $\hat{x}_2(t, \varepsilon)$ from

$$x_1(t, \varepsilon) = \tilde{x}_1(t) + \varepsilon \hat{x}_1(t, \varepsilon)$$

$$x_2(t, \varepsilon) = \tilde{x}_2\left(\frac{t-t_0}{\varepsilon}, t\right) + \varepsilon \hat{x}_2(t, \varepsilon)$$

where \tilde{x}_2 has been constructed with $\psi_2(\tau) = \psi_2(t_0 + \varepsilon\tau)$.

We want to obtain uniform boundedness for \hat{x}_1, \hat{x}_2 thus proving that \tilde{x}_1, \tilde{x}_2 give a first approximation for the solution.

2. For \hat{x}_1, \hat{x}_2 we get the equations

$$\begin{aligned} \varepsilon \frac{dx_1}{dt}(t, \varepsilon) &= f_1(t, \tilde{x}_1(t) + \varepsilon \hat{x}_1(t, \varepsilon), \tilde{x}_2\left(\frac{t-t_0}{\varepsilon}, t\right) + \varepsilon \hat{x}_2(t, \varepsilon), \tilde{x}_1(t-\varepsilon) + \\ &+ \varepsilon \hat{x}_1(t-\varepsilon, \varepsilon), \tilde{x}_2\left(\frac{t-t_0}{\varepsilon}-1, t-\varepsilon\right) + \varepsilon \hat{x}_2(t-\varepsilon, \varepsilon), \varepsilon) - \\ &- f_1(t, \tilde{x}_1(t), \underline{x}_2(t), \tilde{x}_1(t), \underline{x}_2(t), 0) \end{aligned}$$

$$\begin{aligned} \epsilon \hat{x}_2(t) = & f_2(t, \tilde{x}_1(t) + \epsilon \hat{x}_1(t, \epsilon), \tilde{x}_1(t-\epsilon) + \epsilon \hat{x}_1(t-\epsilon, \epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon) + \\ & + \epsilon \hat{x}_2(t-\epsilon, \epsilon), \epsilon) - f_2(t, \tilde{x}_1(t), \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t), 0) \end{aligned}$$

These equations may be written in the form

$$\begin{aligned} \frac{d\hat{x}_1}{dt}(t, \epsilon) = & \hat{A}_{11}(t, \epsilon) \hat{x}_1(t, \epsilon) + \hat{A}_{12}(t, \epsilon) \hat{x}_2(t, \epsilon) + \hat{B}_{11}(t, \epsilon) \hat{x}_1(t-\epsilon, \epsilon) + \\ & + \hat{B}_{12}(t, \epsilon) \hat{x}_2(t-\epsilon, \epsilon) + g_1(t, \epsilon) + g_2(t, \hat{x}_1(t, \epsilon), \hat{x}_2(t, \epsilon), \hat{x}_1(t-\epsilon, \epsilon), \\ & \hat{x}_2(t-\epsilon, \epsilon), \epsilon) \end{aligned}$$

$$\begin{aligned} \hat{x}_2(t, \epsilon) = & \hat{A}_{21}(t, \epsilon) \hat{x}_1(t, \epsilon) + \hat{B}_{21}(t, \epsilon) \hat{x}_1(t-\epsilon, \epsilon) + \hat{B}_{22}(t, \epsilon) \hat{x}_2(t-\epsilon, \epsilon) + \\ & + g_2(t, \epsilon) + g_2(t, \hat{x}_1(t, \epsilon), \hat{x}_1(t-\epsilon, \epsilon), \hat{x}_2(t-\epsilon, \epsilon), \epsilon) \end{aligned}$$

where

$$\hat{A}_{11}(t, \epsilon) = (D_2 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

$$\hat{A}_{12}(t, \epsilon) = (D_3 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

$$\hat{B}_{11}(t, \epsilon) = (D_4 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

$$\hat{B}_{12}(t, \epsilon) = (D_5 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

the last three derivatives being evaluated at the same point as in (D₂f₁);

$$\hat{A}_{21}(t, \epsilon) = (D_2 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

$$\hat{B}_{21}(t, \epsilon) = (D_3 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

$$\hat{B}_{22}(t, \epsilon) = (D_4 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)$$

the last two derivatives being evaluated at the same point as in (D₂f₂);

We have further

$$\begin{aligned} g_1(t, x_1, x_2, y_1, y_2, \epsilon) = & \int_0^1 \{ [(D_2 f_1)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}, t) + \\ & + \theta \epsilon x_2, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon) + \theta \epsilon y_2, \epsilon) - \\ & - (D_2 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)] x_1 + \\ & + [(D_3 f_1)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}, t) + \theta \epsilon x_2, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon) + \\ & + \theta \epsilon y_2, \epsilon) - (D_3 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)] x_2 + \\ & + [(D_4 f_1)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}, t) + \theta \epsilon x_2, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon) + \\ & + \theta \epsilon y_2, \epsilon) - (D_4 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon} - 1, t-\epsilon), \epsilon)] y_1 + \end{aligned}$$

+ [(D₅f₁)(t, $\tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}, t) + \theta \epsilon x_2, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon) + \theta \epsilon y_2, \epsilon) - (D_5 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon)] y_2$;
 the derivatives $(D_5 f_1)$ being evaluated at the same points as $(D_2 f_1)$;

$$G_2(t, x_1, y_1, y_2, \epsilon) = \int_0^1 \{ [(D_2 f_2)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon) + \theta \epsilon y_2, \epsilon) - (D_2 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon)] x_1 + [(D_3 f_2)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_2(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon) + \theta \epsilon y_2, \epsilon) - (D_3 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon)] y_1 + [(D_4 f_2)(t, \tilde{x}_1(t) + \theta \epsilon x_1, \tilde{x}_1(t-\epsilon) + \theta \epsilon y_1, \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon) + \theta \epsilon y_2, \epsilon) - (D_4 f_2)(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon)] y_2 \} d\theta$$

$$g_1(t, \epsilon) = \frac{1}{\epsilon} [f_1(t, \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}, t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon) - f_1(t, \tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_1(t), \tilde{x}_2(t), 0)]$$

$$g_2(t, \epsilon) = \frac{1}{\epsilon} [f_2(t, \tilde{x}_1(t), \tilde{x}_1(t-\epsilon), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t-\epsilon), \epsilon) - f_2(t, \tilde{x}_1(t), \tilde{x}_1(t), \tilde{x}_2(\frac{t-t_0}{\epsilon}-1, t), 0)]$$

3. To get the desired estimates for \hat{x}_1, \hat{x}_2 we need first estimates for the fundamental matrix of the linear system defined by matrices $\hat{A}_{jk}, \hat{B}_{jk}$ and also estimates for g_j, G_j .

Proposition 3. Consider the system

$$\dot{x}_1(t) = A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_{11}(t)x_1(t-\epsilon) + B_{12}(t)x_2(t-\epsilon)$$

$$x_2(t) = A_{21}(t)x_1(t) + B_{21}(t)x_1(t-\epsilon) + B_{22}(t)x_2(t-\epsilon)$$

with $t \rightarrow A_{jk}(t), t \rightarrow B_{jk}(t)$ uniformly Lipschitz and uniformly bounded on I; assume $|\sigma(B_{22}(t))| \leq \rho < 1$ for all $t \in I$, and the linear system $\dot{x} = \tilde{A}(t)x, \tilde{A}(t) = A_{11}(t) + B_{11}(t) + [A_{12}(t) + B_{12}(t)] [E - B_{22}(t)]^{-1} [A_{21}(t) + B_{21}(t)]$ be exponentially stable. Consider also the "perturbed system"

$$\dot{x}_1(t) = \hat{A}_{11}(t, \epsilon)x_1(t) + \hat{A}_{12}(t, \epsilon)x_2(t) + \hat{B}_{11}(t, \epsilon)x_1(t-\epsilon) + \hat{B}_{12}(t, \epsilon)x_2(t-\epsilon)$$

$$x_2(t) = \hat{A}_{21}(t, \epsilon)x_1(t) + \hat{B}_{21}(t, \epsilon)x_1(t-\epsilon) + \hat{B}_{22}(t, \epsilon)x_2(t-\epsilon)$$

$$\text{with } |\hat{A}_{jk}(t, \epsilon) - A_{jk}(t)| \leq c \left(e^{-\frac{\alpha(t-t_0)}{\epsilon}} + \epsilon \right)$$

$$|\hat{B}_{jk}(t, \epsilon) - B_{jk}(t)| \leq c \left(e^{-\frac{\alpha(t-t_0)}{\epsilon}} + \epsilon \right)$$

for all $t \geq t_0$ and $\epsilon > 0$ small enough.

Then there exists $\hat{\epsilon} > 0$ such that for all $0 < \epsilon \leq \hat{\epsilon}$ the matrix solution $(Y_1(t, s, \epsilon), Y_2(t, s, \epsilon))$ of the perturbed system defined by the conditions $Y_1(t, s, \epsilon) = 0$, $Y_2(t, s, \epsilon) = 0$ for $s - \epsilon \leq t < s$, $Y_1(s, s, \epsilon) = E$, satisfies

$$|Y_1(t, s, \epsilon)| \leq c e^{-\lambda(t-s)}, \quad |Y_2(t, s, \epsilon)| \leq c e^{-\lambda(t-s)}$$

Proof. From the linear theory we know that for the matrices

\tilde{Y}_1, \tilde{Y}_2 associated to the unperturbed system we have estimates of the form $|\tilde{Y}_1(t, s, \epsilon)| \leq c e^{-\lambda(t-s)}$, $|\tilde{Y}_2(t, s, \epsilon)| \leq c e^{-\lambda(t-s)}$

and we want to show that these estimates remain true

(with the corresponding change in the constants) after the

perturbations. We may use the representation

$$Y_1(t, s, \epsilon) = \tilde{Y}_1(t, s, \epsilon) + \int_{\lambda}^t \tilde{Y}_1(t, \sigma, \epsilon) \left\{ [\hat{A}_{11}(\sigma, \epsilon) - A_{11}(\sigma)] Y_1(\sigma, s, \epsilon) + [\hat{A}_{12}(\sigma, \epsilon) - A_{12}(\sigma)] Y_2(\sigma, s, \epsilon) + [\hat{B}_{11}(\sigma, \epsilon) - B_{11}(\sigma)] Y_1(\sigma - \epsilon, s, \epsilon) + [\hat{B}_{12}(\sigma, \epsilon) - B_{12}(\sigma)] Y_2(\sigma - \epsilon, s, \epsilon) \right\} d\sigma + \int_{\lambda}^t \tilde{Y}_2(\sigma, t, \epsilon) \left\{ [\hat{A}_{21}(\sigma, \epsilon) - A_{21}(\sigma)] Y_1(\sigma, s, \epsilon) + [\hat{B}_{21}(\sigma, \epsilon) - B_{21}(\sigma)] Y_1(\sigma - \epsilon, s, \epsilon) + [\hat{B}_{22}(\sigma, \epsilon) - B_{22}(\sigma)] Y_2(\sigma - \epsilon, s, \epsilon) \right\} d\sigma$$

For Y_2 we shall use the representation

$$Y_2(t, s, \epsilon) = \sum_{j=0}^{\lfloor \frac{t-s}{\epsilon} \rfloor} \hat{C}_2(t, t-j\epsilon, \epsilon) \left\{ \hat{A}_{21}(t-j\epsilon, \epsilon) Y_1(t-j\epsilon, s, \epsilon) + \hat{B}_{21}(t-j\epsilon, \epsilon) Y_1(t-(j+1)\epsilon, s, \epsilon) \right\}$$

with \hat{C}_2 the fundamental matrix for $x_2(t) = \hat{B}_{22}(t, \epsilon)x_2(t-\epsilon)$,

since by Propositions 1 and 2 in the preceding section we

know that $|\hat{C}_2(t, s, \epsilon)| \leq c e^{-\frac{\alpha(t-s)}{\epsilon}}$.

Denote $\mu(T, \epsilon) = \sup_{t_0 - \epsilon \leq t \leq T} \left\{ e^{\frac{\tilde{\alpha}}{2}(t-t_0)} |Y_1(t, s, \epsilon)| \right\}$.

Then

$$|Y_2(t, s, \epsilon)| \leq c \sum_{j=0}^{\lfloor \frac{t-t_0}{\epsilon} \rfloor} e^{-\alpha j} e^{-\frac{\tilde{\alpha}}{2}(t-t_0-j\epsilon)} \mu(T, \epsilon) = c e^{-\frac{\tilde{\alpha}}{2}(t-t_0)} \mu(T, \epsilon) \sum_{j=0}^{\lfloor \frac{t-t_0}{\epsilon} \rfloor} e^{-(\alpha - \frac{\tilde{\alpha}}{2}\epsilon)j}$$

hence

$$|Y_2(t, s, \epsilon)| \leq c e^{-\frac{\tilde{\alpha}}{2}(t-t_0)} \mu(T, \epsilon)$$

Use this estimate in the representation formula for Y_1 to

get

$$e^{\frac{\tilde{\alpha}}{2}(t-t_0)} |Y_1(t, s, \epsilon)| \leq c e^{-\frac{\tilde{\alpha}}{2}(t-t_0)} + c \mu(T, \epsilon) \int_{t_0}^t e^{-\frac{\tilde{\alpha}}{2}(t-\sigma)} \left(e^{-\frac{\alpha(\sigma-t_0)}{\epsilon}} + \right.$$

$\left. + \epsilon \right) d\sigma$ and finally $\mu(T, \epsilon) \leq c + c \epsilon \mu(T, \epsilon)$ hence $\mu(T, \epsilon)$

is uniformly bounded and the desired estimates for Y_1 and

Y_2 follow directly. We shall apply this proposition to our

situation. We have

$$\begin{aligned} & \left| \hat{A}_{11}(t, \epsilon) - (D_2 f_1)(t, \tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_1(t), \tilde{x}_2(t), 0) \right| \leq \\ & \leq L \left[\left| x_2\left(\frac{t-t_0}{\epsilon}, t\right) - \tilde{x}_2(t) \right| + \left| \tilde{x}_1(t-\epsilon) - \tilde{x}_1(t) \right| + \left| \tilde{x}_2\left(\frac{t-t_0}{\epsilon} - 1, t-\epsilon\right) - \right. \right. \\ & \left. \left. - \tilde{x}_2(t) \right| + \epsilon \right] \leq M \left(e^{-\frac{\alpha(t-t_0)}{\epsilon}} + \epsilon \right) \end{aligned}$$

and similar estimates hold true for the other matrices in the linear part of the system.

We have also directly the uniform boundedness for g_1 and g_2 .

For G_1 and G_2 we have estimates of the form

$$|G_1(t, x_1, x_2, y_1, y_2, \epsilon)| \leq L \epsilon (|x_1| + |x_2| + |y_1| + |y_2|)^2$$

$$|G_2(t, x_1, y_1, y_2, \epsilon)| \leq L \epsilon (|x_1| + |y_1| + |y_2|)^2$$

Remark also that the initial function for \hat{x}_2 is zero,

while the one for \hat{x}_1 is $\frac{1}{\epsilon} [\varphi_1(t) - \tilde{x}_1(t)]$, $t_0 - \epsilon \leq t \leq t_0$.

This initial function is bounded if we assume φ_1 to be Lipschitz.

These estimates together with ones for the fundamental matrices give the uniform boundedness of \hat{x}_1, \hat{x}_2 hence the validity of the asymptotic expansion.

3. Stability

Consider again the system in the preceding section and \check{x}_1, \check{x}_2 a solution defined for $t \geq t_0$ by initial functions on $[t_0 - \varepsilon, t_0)$.

We linearize in the usual way the system around this solution.

Assume the linear part has good stability properties; then since we have the corresponding representation formulae we may proceed in the usual way to prove a result on stability by the first approximation. The stability properties for the linear approximation are deduced in the linear theory. The only problem that deserves some attention is concerned with the "domain of attraction".

4. Applications.

Consider the problem in hydraulics we started with [1], but

assuming nonlinear boundary conditions

$$\frac{\partial H}{\partial t} + \frac{c^2}{gf} \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial H}{\partial x} + \frac{1}{gf} \frac{\partial Q}{\partial t} = 0$$

$$H(0, t) = 0, \quad F \frac{d_2(t)}{dt} + \hat{q} = Q(l, t)$$

$$H(l, t) = z(t) + R' \frac{d_2(t)}{dt} \left| \frac{d_2(t)}{dt} \right|$$

Taking the nondimensional form, using Riemann invariants

and denoting $\varepsilon = \frac{2l\hat{q}}{cf}$, $\kappa = \frac{gcf}{c\hat{q}}$ we get the problem

$$y'(\tau) = -\frac{\kappa}{F} \left[w(\tau) + w(\tau - \varepsilon) + \frac{\hat{q}}{l\hat{q}} \right]$$

$$w(\tau) - w(\tau - \varepsilon) = -\frac{\varepsilon \kappa R' F^2}{2F^2} \left[w(\tau) + w(\tau - \varepsilon) + \frac{\hat{q}}{l\hat{q}} \right] \left| w(\tau) + w(\tau - \varepsilon) + \frac{\hat{q}}{l\hat{q}} \right| + \frac{1}{2} y(\tau)$$

Denote $\zeta = \frac{g\kappa R' F^2}{F^2}$ and write the difference equation in

standard form

$$w(z) = -w(z-\varepsilon) - \frac{q}{i|q|} - \frac{1}{\varepsilon} \operatorname{sign} \left[\frac{1}{R} y(z) - 2w(z-\varepsilon) + \frac{q}{i|q|} \right] \left[1 - \sqrt{1 + 2\varepsilon \zeta \left| \frac{1}{R} y(z) + 2w(z-\varepsilon) + \frac{q}{i|q|} \right|} \right]$$

We see that this equation has the form

$$w(z) = \phi(y(z), w(z-\varepsilon))$$

and in order to apply the general theory we have to study the derivative of ϕ with respect to the second argument. A direct computation shows that $-1 < (D_2 \phi)(y(z), w(z-\varepsilon)) \leq -1$

The reduced equation is
$$-\frac{\varepsilon \zeta}{2} \frac{F^2}{f^2} |y'(z)| + \frac{1}{R} y(z) = 0$$

and it gives $y(z) = 0$; for this y the general theory may be

applied if the solution of the difference equation $w(z) =$

$\phi(0, w(z-\varepsilon))$ stays at a positive, independent of ε , distance from $-\frac{q}{2i|q|}$. Denote $\hat{w}(z) = w(z) + \frac{q}{2i|q|}$; then $\hat{w}(z) = -\operatorname{sign} \hat{w}(z-\varepsilon) \varphi(|\hat{w}(z-\varepsilon)|)$ with $\varphi(\hat{w}) = \hat{w} + \frac{2}{\varepsilon \zeta} \left[1 - \sqrt{1 + 4|\zeta \hat{w}|} \right]$ and initial condition $\hat{w}(z_0) = \frac{q}{2i|q|}$

for $\zeta \leq 0$. A simple analysis, using the properties of φ shows that for $\varepsilon \leq 24$ the solution \hat{w} is piece-wise constant, does not change the sign and $|\hat{w}(z)| \geq \frac{1}{12} \cdot \frac{7 \cdot \sqrt{10}}{1 + \sqrt{10}}$. In this way the theory may be applied and we have for y and w an asymptotic expansion starting with $\tilde{y}(z) = 0$ and $\tilde{w}(z)$ computed from the reduced difference equation.

As a second example consider a problem in combined heat-electricity generation, described by Vl. Rasvan in [2]

by the model

$$T_c \frac{\partial^3 \tilde{z}_p}{\partial t^2} + \frac{\partial^3 \tilde{z}_w}{\partial \lambda} = 0, \quad \psi_c^2 T_c \frac{\partial^3 \tilde{z}_w}{\partial t^2} + \frac{\partial^3 \tilde{z}_p}{\partial \lambda} = 0$$

$$\tilde{z}_w(0, t) + \alpha_p \tilde{z}_p(0, t) = \alpha_p \pi_s(t)$$

$$\tilde{z}_w(1, t) - \alpha_s \tilde{z}_p(1, t) = 0$$

$$T_s \frac{d\tilde{s}}{dt} = \alpha \pi_1 + (4 - \alpha) \pi_2 - \nu g$$

$$T_1 \frac{d\pi_1}{dt} = \mu_1 - \pi_1$$

$$T_2 \frac{d\pi_2}{dt} = \pi_1 - \beta_1 \mu_2 \pi_2 - \beta_2 \xi_{05}(0,t)$$

Here μ_1 and μ_2 are control parameters; if they are constant, the model is linear and if they are constructed via linear feed-back the model is nonlinear.

By using Riemann invariants we get, taking nondimensional time

$$\tau = \frac{t}{T_a}$$

$$\frac{ds}{d\tau} = \alpha \pi_1 + (1-\alpha) \pi_2 = \eta_1$$

$$\frac{T_1}{T_a} \frac{d\pi_1}{d\tau} = \mu_1 - \pi_1$$

$$\frac{T_2}{T_a} \frac{d\pi_2}{d\tau} = \pi_1 - \left(\beta_1 \mu_2 + \frac{\beta_2 \alpha_p \psi_c}{1 + \alpha_p \psi_c} \right) \pi_2 + \frac{2 \beta_2 \alpha_p \psi_c}{1 + \alpha_p \psi_c} \eta_1 \left(\tau - \frac{2 \psi_c T_c}{T_a} \right)$$

$$\eta_1(\tau) = \frac{1 - \alpha_p \psi_c}{1 + \alpha_p \psi_c} \frac{1 - \psi_s \psi_c}{1 + \psi_s \psi_c} \eta_1 \left(\tau - \frac{2 \psi_c T_c}{T_a} \right) + \frac{\alpha_p}{1 + \alpha_p \psi_c} \pi_2(\tau)$$

We may apply the general theory if $(1 - \alpha_p \psi_c)(1 - \psi_s \psi_c) \geq 0$ and $\frac{T_c}{T_a}$ is "small" while ψ_c is "normal". The general theory leads to one of the reduced models suggested by Vl. Rasvan.

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